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A converse to a theorem of K. Kuratowski on parametrizations of compacta on the Cantor set

by

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Abstract. A classical result of K. Kuratowski asserts that if a perfect compactum X is a countable union of finite-dimensional compacta then almost every continuous map of the Cantor set onto X (in the sense of the Baire category in the function space) is finite-to-one. We prove the converse to this theorem and some of its refinements.

1. Introduction. Our terminology is explained in Sec. 2. Given a compactum X, we shall consider the space of all parametrizations of X on the Cantor set $\mathbf{2}^\infty$, i.e., continuous mappings of $\mathbf{2}^\infty$ onto X, equipped with the topology of uniform convergence, and the phrase "almost every parametrization" will refer to the Baire category in the function space.

This note is related to the following results in dimension theory (cf. [Ku2; §45, II], [Na; VI.4], see also Sec. 5.1):

1.1. THEOREM (Kuratowski [Ku1]). *If a compactum X without isolated points is a countable union of finite-dimensional compacta then almost every parametrization of X on the Cantor set is finite-to-one.*

1.2. THEOREM (Hurewicz [Hu]). *A compactum X without isolated points is a countable union of finite-dimensional subsets if, and only if, X has a finite-to-one parametrization on the Cantor set.*

In fact, one easily checks that the existence of a finite-to-one parametrization of a compactum X on $\mathbf{2}^\infty$ implies that finite-to-one mappings are dense in the space of all parametrizations of X on $\mathbf{2}^\infty$. The main result of this note is the following theorem:

1.3. THEOREM. *Let X be a compactum which can not be covered by countably many finite-dimensional compacta. Then almost every parametrization of X on the Cantor set has a perfect fibre.*

It follows that a perfect compactum X is a countable union of finite-dimensional compacta if, and only if, a typical, in the sense of Baire category, parametrization of

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X on $\mathbf{2}^\infty$ is finite-to-one. There are natural examples of compacta which are countable unions of finite-dimensional subsets, but can not be covered by countably many finite-dimensional compacta (cf. Sec. 2.5). For each such compactum X without isolated points, finite-to-one parametrizations of X on $\mathbf{2}^\infty$ form a dense, but meagre set in the space of all parametrizations of X on $\mathbf{2}^\infty$.

These facts are, in some sense, complementary to the results of Elżbieta Pol [PE]. She discovered similar phenomena, while investigating the Baire category of the spaces of embeddings of compacta into the universal countable-dimensional Nagata space [Na; VI.6]. We should also quote here a classical paper by S. Saks [Sa] which, although concerned with quite different topics, contains certain general ideas related to this note.

In Section 4 we discuss some refinements of Theorem 1.3. In particular, we exhibit a kind of extreme behaviour of typical parametrizations of perfect compacta on the Cantor set: either a typical parametrization of X on $\mathbf{2}^\infty$ is finite-to-one (see Kuratowski's Theorem 1.1), or for almost every parametrization $f\colon \mathbf{2}^\infty \to X$ the fibres $f^{-1}(x)$ represent all topological types of nonempty compacta in $\mathbf{2}^\infty$ (cf. Theorem 4.1 and Sec. 5.4). Other extreme properties of typical parametrizations are recorded in Corollary 4.2.

2. Terminology, some background and auxiliary facts. Our terminology follows Kuratowski [Ku2]. We shall consider only separable metrizable spaces and by a compactum we mean a compact space. The symbol $\operatorname{diam} E$ stands for the diameter of the set E in a metric space.

2.1. *The Cantor set* $\mathbf{2}^\infty$. Let $\mathbf{2}^\infty$ be the space of infinite zero-one sequences $(j_1, j_2, \ldots)$ with the topology of pointwise convergence. We denote by $\mathbf{2}^n$ the set of zero-one sequences $(j_1, \ldots, j_n)$ of length n, and $\mathbf{2}^{<\omega} = \bigcup_{n=1}^\infty \mathbf{2}^n$.

If $\mathbf{a} = (j_1, \ldots, j_n) \in \mathbf{2}^n$ and $j \in \mathbf{2}$, we set $\mathbf{a}^\frown j = (j_1, \ldots, j_n, j) \in \mathbf{2}^{n+1}$. For $t = (j_1, j_2, \ldots) \in \mathbf{2}^\infty$ and natural n, let $t|n = (j_1, \ldots, j_n) \in \mathbf{2}^n$ and, for each $\mathbf{a} \in \mathbf{2}^n$, we define

$$B(\mathbf{a}) = \{t \in \mathbf{2}^\infty\colon\ t|n = \mathbf{a}\}. \tag{1}$$

The collection $\{B(\mathbf{a})\colon\ \mathbf{a} \in \mathbf{2}^{<\omega}\}$ is the standard dyadic base for $\mathbf{2}^\infty$: each $B(\mathbf{a})$ is split into two disjoint open compacta $B(\mathbf{a}^\frown 0)$ and $B(\mathbf{a}^\frown 1)$.

2.2. *Parametrizations of compacta on the Cantor set.* We shall call continuous mappings from $\mathbf{2}^\infty$ onto a compactum X *parametrizations of X on the Cantor set* (cf. [Ku2; §45, II]). Given a compactum X, we shall consider the space $S(\mathbf{2}^\infty, X)$ of all parametrizations of X on $\mathbf{2}^\infty$ with the topology of uniform convergence. Speaking about $S(\mathbf{2}^\infty, X)$ we shall consider this space with the supremum metric on X; each such metric on $S(\mathbf{2}^\infty, X)$ is complete (cf. [Ku2; §44, V]).

2.3. *Analytic sets and residuality.* A space M is *analytic* if M is a continuous image of the irrationals. Analytic sets in a space E are open modulo meagre sets in E (cf. [Ku2; §11, VII]). Therefore, an analytic set M in a complete space E is residual (i.e., $E \setminus M$ is meagre) if, and only if, M intersects each set of type $\mathbf{G}_\delta$, dense in some nonempty open set in E.

2.4. *The kernel of a space with respect to a σ-ideal of subsets.* Let E be a separable metrizable space and let $\mathscr{I}$ be a nontrivial σ-ideal of sets in E. There exists a nonempty closed set Z in E such that $E \setminus Z \in \mathscr{I}$ and no nonempty relatively open subset of Z is in $\mathscr{I}$ (cf. [Ku2; §24, II], [St]).

2.5. *Countable-dimensional and strongly countable-dimensional spaces.* A space E is *countable-dimensional* (*strongly countable-dimensional*) if E can be covered by countably many finite-dimensional sets (respectively, closed sets). Examples of compacta which are countable-dimensional but not strongly countable-dimensional can be found in [Sm], or [E-P]. The compactum X of this kind described in [E-P; Example 1.12] contains finite-dimensional compacta $K_1, K_2, \ldots$ such that $X \setminus \bigcup_{i=1}^\infty K_i$ is zero-dimensional.

Let us recall the following fact (a special case of Sklyarenko's theorem [Sk], cf. [Na; VI.3.E]): *if $f\colon \mathbf{2}^\infty \to X$ is a parametrization and $E \subset X$ is not countable-dimensional, then the set $\{x \in E\colon f^{-1}(x)$ is perfect$\}$ is not countable-dimensional.*

2.6. Hemmingsen's Lemma [He], [En; 1.6.10] (cf. [Ku2; §45, IV, Th. 10]). *Let S be a compact set of dimension $\geqslant n$ in a space X. There are open subsets $G_1, \ldots, G_{n+1}$ of X such that $G_i \cap S \neq \emptyset$, $S \subset G_1 \cup \ldots \cup G_{n+1}$ and, for arbitrary closed sets $L_1, \ldots, L_{n+1}$ in X, if $L_i \subset G_i$ and $S \subset L_1 \cup \ldots \cup L_{n+1}$, then $S \cap L_1 \cap \ldots \cap L_{n+1} \neq \emptyset$.*

2.7. *A universal compactum in $\mathbf{2}^\infty \times \mathbf{2}^\infty$.* We shall need the following fact, easily verified by standard arguments. For the sake of completeness, a proof is given.

Lemma. *There exists a compactum $S \subset \mathbf{2}^\infty \times \mathbf{2}^\infty$, each vertical section of S being nonempty, such that for every continuous surjection $q\colon \mathbf{2}^\infty \to \mathbf{2}^\infty$ there exists an embedding $h\colon \mathbf{2}^\infty \to \mathbf{2}^\infty$ with*

$$S \cap (h(\mathbf{2}^\infty) \times \mathbf{2}^\infty) = \{(h \circ q(t), h(t))\colon\ t \in \mathbf{2}^\infty\}.$$

Proof. Let us write $\mathbf{2}^\infty = C \times C$, C being a Cantor set. Let $S(C, C)$ be the space of continuous surjections of C onto itself, endowed with the topology of uniform convergence, let Q be a countable dense set in C, and let $P = C \setminus Q$. The set P being homeomorphic to the irrationals, there exists a continuous surjection $p\colon P \to S(C, C)$. We define a closed subset T of $(P \times C) \times (P \times C)$ by

$$T = \{((x, p(x)(y)), (x, y))\colon\ x \in P,\ y \in C\},$$

and let S be the closure of T in $\mathbf{2}^\infty \times \mathbf{2}^\infty$. Notice that $S \setminus T \subset (Q \times C) \times (Q \times C)$, and therefore, the vertical sections of S and T coincide at each point of $P \times C$. The projection of S onto the first coordinate contains $P \times C$, and hence it is equal to $\mathbf{2}^\infty$.

Let $q\colon \mathbf{2}^\infty \to \mathbf{2}^\infty$ be an arbitrary continuous surjection, and let $q = p(x)$, for some $x \in P$. Let $h\colon C \to \mathbf{2}^\infty$ be an embedding defined by $h(y) = (x, y)$. Then

$$S \cap (h(C) \times \mathbf{2}^\infty) = T \cap (h(C) \times \mathbf{2}^\infty) = \{((x, q(y)), (x, y))\colon\ y \in C\}.$$

Identifying C with $\mathbf{2}^\infty$, we complete the proof.

3. Proof of the theorem. Let X be a compactum which can not be covered by countably many finite-dimensional compacta and let $Z \subset X$ be a nonempty compactum all of whose relatively open nonempty subsets are infinite-dimensional (cf. Sec. 2.4).

We shall prove that for a typical mapping in the space $S(2^\infty, X)$ of parametrizations of X on 2^∞ (see Sec. 2.2), there exists $z \in Z$ with the perfect fibre $f^{-1}(z)$.

The main element of the proof is a branching process described in Step (I), and Step (II) is a straightforward inductive application of this process. In the next section we shall refine this reasoning and, bearing this application in mind, we shall consider in Step (I) a function d, not necessary for the present proof.

Step (I). Let $f\colon 2^\infty \to X$ be a parametrization. Assume that we are given a finite collection $\mathscr{E}$ of pairwise disjoint closed-and-open sets in 2^∞, a point $x \in Z \cap \bigcap \{f(E)\colon E \in \mathscr{E}\}$ with $f^{-1}(x) \subset \bigcup \mathscr{E}$, and an $\varepsilon > 0$. Assume, in addition, that to each $E \in \mathscr{E}$ a natural number $d(E) \geqslant 0$ is assigned, at least one of them being positive.

Then there exist a finite collection $\mathscr{F}$ of pairwise disjoint closed-and-open sets in 2^∞ of diameter $\leqslant \varepsilon$, and a nonempty open set $\mathscr{U}$ in the ε-ball about f in the space $S(2^\infty, X)$, such that:

(A) $\mathscr{F}$ refines $\mathscr{E}$ and each $E \in \mathscr{E}$ contains exactly $d(E)$ elements of $\mathscr{F}$,

(B) if $u \in \mathscr{U}$ then $u(2^\infty \setminus \bigcup \mathscr{E}) \cap u(\bigcup \mathscr{F}) = \emptyset$ and $\operatorname{diam} u(\bigcup \mathscr{F}) < \varepsilon$,

(C) for each $u \in \mathscr{U}$ there exists $y \in Z \cap \bigcap \{u(F)\colon F \in \mathscr{F}\}$ with $u^{-1}(y) \subset \bigcup \mathscr{F}$.

Proof of (I). Let $\mathscr{E} = \{E_1, \ldots, E_p\}$, set $d(i) = d(E_i)$, and

$$m = d(1) + \ldots + d(p) > 0.$$

For each i we choose a closed-and-open set A_i such that $f^{-1}(x) \cap E_i \subset A_i \subset E_i$, $\operatorname{diam} f(A_i) < \varepsilon/3$ and $f(A_i) \cap f(2^\infty \setminus \bigcup \mathscr{E}) = \emptyset$. For the collection $\mathscr{A} = \{A_1, \ldots, A_p\}$ we have:

$$f(2^\infty \setminus \bigcup \mathscr{E}) \cap f(\bigcup \mathscr{A}) = \emptyset, \tag{1}$$

$$\operatorname{diam} f(\bigcup \mathscr{A}) < \varepsilon, \tag{2}$$

and there exists an open set W in X with

$$x \in W \quad \text{and} \quad f(2^\infty \setminus \bigcup \mathscr{A}) \cap W = \emptyset. \tag{3}$$

Split each A_i into $d(i)+1$ pairwise disjoint closed-and-open sets $D(i)$, $C_1(i), \ldots, C_{d(i)}(i)$, where $\operatorname{diam} C_l(i) \leqslant \varepsilon$, and let (see (A))

$$\mathscr{F} = \{C_l(i)\colon i = 1, \ldots, p,\ l = 1, \ldots, d(i)\}, \tag{4}$$

$$D = D(1) \cup \ldots \cup D(p). \tag{5}$$

Choose a compactum $S \subset Z \cap W$ of dimension $\geqslant m-1$, with $W \setminus S \neq \emptyset$, and let $G_1, \ldots, G_m$ be open sets described in Hemmingsen's Lemma 2.6. Let $K_i \subset G_i$ be nonempty compacta such that, for some open set V in X,

$$S \subset V \subset K_1 \cup \ldots \cup K_m \subset W \quad \text{and} \quad W \setminus V \neq \emptyset. \tag{6}$$

Let e be any bijection of $\{(l, i)\colon i = 1, \ldots, p,\ l = 1, \ldots, d(i)\}$ onto $\{1, \ldots, m\}$, and consider a parametrization $g\colon 2^\infty \to X$ such that g coincides with f on $2^\infty \setminus \bigcup \mathscr{A}$, $g(C_l(i)) = K_{e(l,i)}$, and $g(D) = f(\bigcup \mathscr{A}) \setminus V$. For each $t \in 2^\infty$, either $g(t) = f(t)$, or both $g(t)$, $f(t)$ are in $f(\bigcup \mathscr{A})$, and therefore by (2), g is in the ε-ball about f in $S(2^\infty, X)$. Since f and g coincide on $2^\infty \setminus \bigcup \mathscr{A}$ and $f(\bigcup \mathscr{A}) = g(\bigcup \mathscr{A})$, from (1) we obtain

$$g(2^\infty \setminus \bigcup \mathscr{E}) \cap g(\bigcup \mathscr{F}) = \emptyset. \tag{7}$$

We also have $g(2^\infty \setminus \bigcup \mathscr{F}) = f(2^\infty \setminus \bigcup \mathscr{A}) \cup [f(\bigcup \mathscr{A}) \setminus V]$, so by (3) and (6),

$$g(2^\infty \setminus \bigcup \mathscr{F}) \cap S = \emptyset. \tag{8}$$

Finally,

$$g(C_l(i)) \subset G_{e(l,i)}, \quad i = 1, \ldots, p,\ l = 1, \ldots, d(i), \tag{9}$$

and, by (2) and (6),

$$\operatorname{diam} g(\bigcup \mathscr{F}) < \varepsilon. \tag{10}$$

Let $\mathscr{U}$ be a neighbourhood of g in $S(2^\infty, X)$ contained in the ε-ball centred at f, and small enough that for each $u \in \mathscr{U}$ all relations (7)–(10) hold with g replaced by u.

Let $u \in \mathscr{U}$. Property (B) is guaranteed by (7) and (10). Let $u(C_l(i)) = L_{e(l,i)}$. Then $L_k \subset G_k$, by (9), and since $u(2^\infty) = X$, and $u(2^\infty \setminus \bigcup \mathscr{F}) \cap S = \emptyset$, by (8), we have $S \subset L_1 \cup \ldots \cup L_m$. Therefore (see Sec. 2.6) there exists

$$y \in S \cap L_1 \cap \ldots \cap L_m \subset Z \cap \bigcap \{u(F)\colon F \in \mathscr{F}\},$$

and, by (8), $u^{-1}(y) \subset \bigcup \mathscr{F}$, which gives (C).

Step (II). One readily checks that parametrizations with a perfect fibre form an analytic set in $S(2^\infty, X)$. Therefore, by Sec. 2.3, we have to show that if $\mathscr{G}$ is a $\mathbf{G}_\delta$-set dense in some ε_0-ball about $g_0 \in \mathscr{G}$ in $S(2^\infty, X)$, then there exist $f \in \mathscr{G}$ and $z \in Z$ such that the fibre $f^{-1}(z)$ is perfect.

Let $\mathscr{G} = \mathscr{G}_0 \cap \mathscr{G}_1 \cap \ldots$, where $\mathscr{G}_0 = S(2^\infty, X) \supset \mathscr{G}_1 \supset \ldots$ are open sets. Starting from the triple $(g_0, \mathscr{F}_0, \varepsilon_0)$, with $\mathscr{F}_0 = \{2^\infty\}$, one can define inductively, using Step (I) (the function d being constantly equal to 2), triples $(g_1, \mathscr{F}_1, \varepsilon_1)$, $(g_2, \mathscr{F}_2, \varepsilon_2), \ldots$, where $g_i \in \mathscr{G}$, $\mathscr{F}_i$ are finite families of pairwise disjoint closed-and-open sets in 2^∞, and the sequence $\varepsilon_0 > \varepsilon_1 > \ldots$ converges to 0, such that the following conditions hold:

(11) denoting by $\mathscr{B}_i$ the closed ε_i-ball centred at g_i, we have $\mathscr{B}_{i+1} \subset \mathscr{B}_i \subset \mathscr{G}_i$, for $i = 0, 1, \ldots$,

(12) $\mathscr{F}_{i+1}$ refines $\mathscr{F}_i$, the diameter of each element of $\mathscr{F}_{i+1}$ and of $g_{i+1}(\bigcup \mathscr{F}_{i+1})$ is less than ε_i,

(13) $2\varepsilon_{i+1}$ is less than the distance between $g_{i+1}(2^\infty \setminus \bigcup \mathscr{F}_i)$ and $g_{i+1}(\bigcup \mathscr{F}_{i+1})$,

(14) there exist $x_i \in Z \cap \bigcap \{g_i(F)\colon F \in \mathscr{F}_i\}$ with $g_i^{-1}(x_i) \subset \bigcup \mathscr{F}_i$,

(15) each member of $\mathscr{F}_i$ contains exactly two members of $\mathscr{F}_{i+1}$.

Let (see (11))

(16) $$f \in \bigcap_{i=0}^{\infty} \mathcal{B}_i \subset \mathcal{G} \quad \text{and} \quad F = \bigcap_{i=0}^{\infty} \left(\bigcup \mathcal{F}_i\right).$$

The compactum F is perfect, by (12) and (15). We shall check that

(17) $$F = f^{-1}(z) \quad \text{for some } z \in Z.$$

Since the diameters of $g_i(\bigcup \mathcal{F}_i)$ tend to zero, by (12), so do the diameters of $f(\bigcup \mathcal{F}_i)$, and hence $f(F) = \{z\}$, $z \in Z$ being the limit of the sequence $x_1, x_2, \ldots$ in (14). Let $t \notin F$ and let i be such that $t \notin \bigcup \mathcal{F}_i$. By (13), the distance between $g_{i+1}(t)$ and $g_{i+1}(F)$ is greater than $2\varepsilon_{i+1}$ and since f is in the ε_{i+1}-ball about g_{i+1}, we get $f(t) \notin f(F)$, i.e., $f(t) \neq z$. This proves (17) and completes the proof of Theorem 1.3.

4. A refinement of the theorem. The arguments given in Section 3 can be refined to yield the following theorem.

4.1. THEOREM. *Let X be a compactum and let M be an analytic set in X which can not be covered by countably many finite-dimensional compacta. Then almost every parametrization $f\colon \mathbf{2}^\infty \to X$ has the following property: for each continuous surjection $q\colon \mathbf{2}^\infty \to \mathbf{2}^\infty$ there are homeomorphic embeddings $v\colon \mathbf{2}^\infty \to \mathbf{2}^\infty$ and $w\colon \mathbf{2}^\infty \to M$ such that $v(q^{-1}(t)) = f^{-1}(w(t))$ for all $t \in \mathbf{2}^\infty$.*

For each parametrization $k\colon \mathbf{2}^\infty \to K$ the set $P(k)$ of points x in K with the perfect fibre $k^{-1}(x)$ is of type $\mathbf{F}_{\sigma\delta}$, while the set $U(k)$ of points in K with uncountable fibre is analytic (cf. [Br2], [Ku2; § 39, VII]) and there is a continuous surjection $k\colon \mathbf{2}^\infty \to \mathbf{2}^\infty$ such that $P(k)$ is not of type $\mathbf{G}_{\delta\sigma}$ (cf. [Br1]) and $U(k)$ is not Borel (cf. [M-S], [Ku2; § 39, VII]). Therefore, Theorem 4.1 yields the following corollary.

4.2. COROLLARY. *Let X be a compactum which can not be covered by countably many finite-dimensional compacta. Then for almost every parametrization $f\colon \mathbf{2}^\infty \to X$ the set $\{x \in X: f^{-1}(x)$ is perfect$\}$ is not of type $\mathbf{G}_{\delta\sigma}$ (being of type $\mathbf{F}_{\sigma\delta}$) and the set $\{x \in X: f^{-1}(x)$ is uncountable$\}$ is not Borel (being analytic).*

4.3. Proof Theorem 4.1. (A) By Sec. 2.7, it is enough to prove that if $S \subset \mathbf{2}^\infty \times \mathbf{2}^\infty$ is a compact set with all vertical sections nonempty, i.e.,

(1) $$S_t = S \cap (\{t\} \times \mathbf{2}^\infty) \neq \emptyset, \quad \text{for } t \in \mathbf{2}^\infty,$$

then for almost every parametrization $f\colon \mathbf{2}^\infty \to X$ there are homeomorphic embeddings $v\colon S \to \mathbf{2}^\infty$ and $w\colon \mathbf{2}^\infty \to M$ such that

(2) $$v(S_t) = f^{-1}(w(t)), \quad \text{for } t \in \mathbf{2}^\infty.$$

Since, as one readily checks (cf. 5.6), parametrizations with this property form an analytic set, we have to verify that each $\mathbf{G}_\delta$-set $\mathcal{G}$ of second category in the space of parametrizations $S(\mathbf{2}^\infty, X)$ contains such a mapping (cf. 2.3).

Let $\mathcal{G} = \mathcal{G}_0 \cap \mathcal{G}_1 \cap \ldots$ be a $\mathbf{G}_\delta$-set, dense in some ε_0-ball about $g_0 \in \mathcal{G}$, where the sets $\mathcal{G}_0 = S(\mathbf{2}^\infty, X) \supset \mathcal{G}_1 \supset \ldots$ are open.

(B) We shall use the notation introduced in Sec. 2.1. For each $\mathbf{a} \in \mathbf{2}^n$, let (see Sec. 2.1(1))

$$\mathcal{S}(\mathbf{a}) = \{B(\mathbf{b})\colon \mathbf{b} \in \mathbf{2}^n \text{ and } S \cap (B(\mathbf{a}) \times B(\mathbf{b})) \neq \emptyset\}.$$

Given $\mathbf{a} \in \mathbf{2}^n$, $B \in \mathcal{S}(\mathbf{a})$ and $j \in \mathbf{2}$, we let

$$d_{\mathbf{a}^\frown j}(B) = \operatorname{card}\{B' \in \mathcal{S}(\mathbf{a}^\frown j)\colon B' \subset B\} \in \{0, 1, 2\};$$

by (1), $\sum_{B \in \mathcal{S}(\mathbf{a})} d_{\mathbf{a}^\frown j}(B) > 0$. Notice that for each $t \in \mathbf{2}^\infty$,

$$S_t = \{t\} \times \bigcap_{i=1}^{\infty} \bigcup \mathcal{S}(t|i).$$

Fix a closed subset P of the irrationals ω^∞ and a continuous mapping $p\colon P \to M$ such that if $N \subset P$ is a relatively open nonempty set, then $p(N)$ can not be covered by countably many finite-dimensional compacta.

To this end, we start with an arbitrary continuous surjection $k\colon \omega^\infty \to M$, next we consider the kernel P in ω^∞ with respect to the σ-ideal of sets N such that $k(N)$ can be covered by countably many finite-dimensional compacta (see Sec. 2.4), and p is the restriction of k to P.

(C) We shall define inductively the following objects: parametrizations

$$g_1, \ldots, g_n\colon \mathbf{2}^\infty \to X,$$

positive numbers

$$\varepsilon_1 > \varepsilon_2 > \ldots, \quad \varepsilon_i \to 0,$$

families

$$\mathcal{U}(n) = \{U(\mathbf{a})\colon \mathbf{a} \in \mathbf{2}^n\}$$

of pairwise disjoint open sets in X with $\operatorname{diam} U(\mathbf{a}) \leqslant 1/n$ for $\mathbf{a} \in \mathbf{2}^n$, families

$$\mathcal{N}(n) = \{N(\mathbf{a})\colon \mathbf{a} \in \mathbf{2}^n\}$$

of pairwise disjoint closed-and-open sets in P with $\operatorname{diam} N(\mathbf{a}) \leqslant 1/n$ for $\mathbf{a} \in \mathbf{2}^n$, finite families

$$\mathcal{F}(\mathbf{a}), \quad \mathbf{a} \in \mathbf{2}^n,$$

of pairwise disjoint closed-and-open sets in $\mathbf{2}^\infty$, and bijections

$$\sigma(\mathbf{a})\colon \mathcal{S}(\mathbf{a}) \to \mathcal{F}(\mathbf{a}), \quad \text{for } \mathbf{a} \in \mathbf{2}^n,$$

such that the following conditions are met:

(3) $\overline{U(\mathbf{a}^\frown j)} \subset U(\mathbf{a})$, $N(\mathbf{a}^\frown j) \subset N(\mathbf{a})$, for $\mathbf{a} \in \mathbf{2}^{<\omega}$,

(4) $\overline{p(N(\mathbf{a}))} \subset U(\mathbf{a})$, $g_n(\bigcup \mathcal{F}(\mathbf{a})) \subset U(\mathbf{a})$, for $\mathbf{a} \in \mathbf{2}^{<\omega}$,

(5) if $B' \in \mathcal{S}(\mathbf{a})$, $B'' \in \mathcal{S}(\mathbf{a}^\frown j)$ and $B'' \subset B'$ then $\sigma(\mathbf{a}^\frown j)(B'') \subset \sigma(\mathbf{a})(B')$,

(6) for each $t \in 2^\infty$ the sequence of the triples $(g_1, \mathscr{F}(t|1), \varepsilon_1), (g_2, \mathscr{F}(t|2), \varepsilon_2), \ldots$ has properties (11)–(13) formulated in Step (II), Sec. 3 (with $\mathscr{F}_i$ replaced by $\mathscr{F}(t|i)$),

(7) there are points $x_{\mathbf{a}}$, $\mathbf{a} \in 2^n$, such that $x_{\mathbf{a}} \in \overline{p(N(\mathbf{a}))} \cap \bigcap \{g_n(F)\colon F \in \mathscr{F}(\mathbf{a})\}$ and $g_n^{-1}(x_{\mathbf{a}}) \subset \bigcup \mathscr{F}(\mathbf{a})$ (cf. (14), Step (II), Sec. 3),

(8) for $\mathbf{a} \in 2^n$, if $C \in \mathscr{F}(\mathbf{a})$ and $C = \sigma(\mathbf{a})(B)$, where $B \in \mathscr{S}(\mathbf{a})$, then C contains exactly $d_{\mathbf{a}^\frown j}(B)$ elements of $\mathscr{F}(\mathbf{a}^\frown j)$ (see (B), cf. Sec. 3(15)).

(D) Let us postpone to the next section the construction of the objects described in (C), and let us check that, once we have them, the statement formulated in (A) follows easily. We shall appeal to the reasoning in Step (II), Sec. 3.

Let $f \in \mathscr{G}$ be the limit of the sequence $g_1, g_2, \ldots$ (cf. Sec. 3(16)). For each $t \in 2^\infty$ let

(9) $$F_t = \bigcap_{i=1}^{\infty} \bigcup \mathscr{F}(t|i).$$

Property (6) allows one to repeat the arguments justifying (17) in Sec. 3, which, combined with (4) and (7), gives us

(10) $$F_t = f^{-1}(x_t), \quad \text{where} \quad \{x_t\} = \bigcap_{i=1}^{\infty} U(t|i) \subset M.$$

Let a homeomorphic embedding $w\colon 2^\infty \to M$ be defined by (see (10) and (3))

(11) $$w(t) = x_t.$$

Fix $(t, s) \in S$. For each n there exists a unique $B_n(t, s) \in \mathscr{S}(t|n)$ containing s; let $F_n(t, s) = \sigma(t|n)(B_n(t, s))$ be the corresponding element of $\mathscr{F}(t|n)$. The decreasing sequence $F_1(t, s) \supset F_2(t, s) \supset \ldots$ (see (5)) of closed-and-open sets in 2^∞ with diameters converging to 0 determines a unique point $v(t, s)$ of F_t (see (9)). Thus a homeomorphic embedding $v\colon S \to 2^\infty$ is defined such that (see (B) and (9))

$$v(S_t) = F_t.$$

This, together with (10) and (11), shows that $v(S_t) = f^{-1}(w(t))$.

(E) The construction of the objects in (C) requires only some modifications of the proof of Step (I) in Sec. 3. Suppose that for some n we are given a parametrization g_n, $\varepsilon > 0$, collections $\mathscr{U}(n)$, $\mathscr{N}(n)$, and collections $\mathscr{F}(\mathbf{a})$, $\mathbf{a} \in 2^n$, together with bijections $\sigma(\mathbf{a})\colon \mathscr{S}(\mathbf{a}) \to \mathscr{F}(\mathbf{a})$, described in (C) (we start with the parametrization g_0 and ε_0 introduced at the end of (A), and we let $\mathscr{U}(0) = \{X\}$, $\mathscr{N}(0) = \{P\}$, $\mathscr{F}(\emptyset) = \{2^\infty\}$, $\mathscr{S}(\emptyset) = \{2^\infty\}$, $\sigma(\emptyset)(2^\infty) = X$).

We shall proceed for each $\mathbf{a} \in 2^n$ separately, so let us fix some $\mathbf{a} \in 2^n$ and set $f = g_n$, $\varepsilon = \varepsilon_n$, $\mathscr{E} = \mathscr{F}(\mathbf{a})$, $N = N(\mathbf{a})$, $U = U(\mathbf{a})$, $x = x_{\mathbf{a}}$ (see (7)), $\sigma = \sigma(\mathbf{a})\colon \mathscr{S}(\mathbf{a}) \to \mathscr{E}$.

Let $\mathscr{E} = \{E_1, \ldots, E_p\}$. For each $j \in 2$ and $i = 1, \ldots, p$, define $d_j(i) \in \{0, 1, 2\}$ in the following way: pick $B \in \mathscr{S}(\mathbf{a})$ with $\sigma(\mathbf{a})(B) = E_i$ and set $d_j(i) = d_{\mathbf{a}^\frown j}(B)$ (see (B)). By (B), $m(j) = d_j(1) + \ldots + d_j(p) > 0$, for $j \in 2$.

Let $\mathscr{A} = \{A_1, \ldots, A_p\}$ and $W \subset f(\bigcup \mathscr{A}) \subset \mathscr{U}$ be as in the proof of Step (I) (see (1)–(3), Sec. 3). Since $x \in \overline{p(N)} \cap W$, one can choose closed-and-open nonempty subsets N_0, N_1 of N such that $\overline{p(N_0)} \cap \overline{p(N_1)} = \emptyset$, $\overline{p(N_j)} \subset W$, and $\operatorname{diam} N_j \leqslant 1/(n+1)$, $\operatorname{diam} p(N_j) \leqslant 1/(2(n+1))$. Let U_0, U_1 be disjoint open sets of diameter $\leqslant 1/(n+1)$ with $\overline{p(N_j)} \subset U_j$ and $\bar{U}_j \subset W$. We split each A_i into $d_0(i) + d_1(i) + 1$ pairwise disjoint closed-and-open sets

$$D(i), \; C_{jl}(i), \quad l = 1, \ldots, d_j(i), \; j = 0, 1,$$

with $\operatorname{diam} C_{jl}(i) < \varepsilon$, and we set (see (4), (5), Sec. 3)

$$\mathscr{F}(j) = \{C_{jl}(i)\colon i = 1, \ldots, p, \; l = 1, \ldots, d_j(i)\}, \quad D = D(1) \cup \ldots \cup D(p).$$

If $E_i = \sigma(\mathbf{a})(B)$, $B \in \mathscr{S}(\mathbf{a})$, the collections $\mathscr{S}_{ji} = \{B' \in \mathscr{S}(\mathbf{a}^\frown j)\colon B' \subset B\}$ and $\mathscr{F}_{ji} = \{C \in \mathscr{F}(j)\colon C \subset E_i\}$ have the same cardinality $d_j(i)$ and therefore, for $j \in 2$, there exists a bijection $\sigma(\mathbf{a}^\frown j)\colon \mathscr{S}(\mathbf{a}^\frown j) \to \mathscr{F}(j)$ which maps each $\mathscr{S}_{ji}$ onto $\mathscr{F}_{ji}$.

Let $S_j \subset \overline{p(N_j)}$ be compacta with $\dim S_j \geqslant m(j) - 1$ and let the sets G_{jl}, K_{jl}, V_j, $l \leqslant m(j)$, $K_{jl} \subset U_j$, be chosen as in the proof of Step (I), for each S_j separately (cf. Sec. 3(6)). A parametrization $g\colon 2^\infty \to X$ is defined following Step (I), where $g(D) = f(\bigcup \mathscr{A}) \backslash (V_0 \cup V_1)$ and $g|2^\infty \backslash \bigcup \mathscr{A} = f|2^\infty \backslash \bigcup \mathscr{A}$, so that (7)–(10) in Sec. 3 are satisfied with $\mathscr{F}$ replaced by each $\mathscr{F}(j)$. We set $U(\mathbf{a}^\frown j) = U_j$, $N(\mathbf{a}^\frown j) = N_j$, $\mathscr{F}(\mathbf{a}^\frown j) = \mathscr{F}(j)$, $g_{\mathbf{a}} = g$. The parametrizations $g_{\mathbf{a}}$, defined for each $\mathbf{a} \in 2^n$ separately, coincide with g_n outside $\bigcup \mathscr{F}(\mathbf{a})$ and $g_{\mathbf{a}}(\bigcup \mathscr{F}(\mathbf{a})) \subset U(\mathbf{a})$. Therefore, if $\bar{g}\colon 2^\infty \to X$ is identical with g_n outside the union $\bigcup \{\bigcup \mathscr{F}(\mathbf{a})\colon \mathbf{a} \in 2^n\}$, and $\bar{g}$ coincides with $g_{\mathbf{a}}$ on each $\bigcup \mathscr{F}(\mathbf{a})$, the parametrization $\bar{g}$ has all properties of the parametrizations $g_{\mathbf{a}}$ we were interested in. The distance between $\bar{g}$ and g_n is less than ε_n. Let $\mathscr{B}_n$ be the closed ε_n-ball about g_n in $S(2^\infty, X)$ and let $\mathscr{U} \subset \mathscr{B}_n$ be a neighbourhood of $\bar{g}$, small enough that each parametrization $u \in \mathscr{U}$ keeps the properties of $\bar{g}$ which are essential for us (cf. Step (I), next to (10)). Let $g_{n+1} \in \mathscr{U} \cap \mathscr{G}$. The argument following (10) in Sec. 3 shows that for each $\mathbf{b} = \mathbf{a}^\frown j$, $\mathbf{a} \in 2^n$, $j \in 2$, there exists

$$x_{\mathbf{b}} \in \overline{p(N(\mathbf{b}))} \cap \bigcap \{g_{n+1}(F)\colon F \in \mathscr{F}(\mathbf{b})\}, \quad \text{with} \quad g_{n+1}^{-1}(x_{\mathbf{b}}) \subset \bigcup \mathscr{F}(\mathbf{b}).$$

Finally, the distance $\delta(\mathbf{a}^\frown j)$ between $g_{n+1}(2^\infty \backslash \bigcup \mathscr{F}(\mathbf{a}))$ and $g_{n+1}(\bigcup \mathscr{F}(\mathbf{a}^\frown j))$ is positive for each $\mathbf{a} \in 2^n$, $j \in 2$ (see (7), Sec. 3) and let $\varepsilon_{n+1} > 0$ be small enough that $2\varepsilon_{n+1} < \min\{\delta(\mathbf{b})\colon \mathbf{b} \in 2^{n+1}\}$ and the closed ε_{n+1}-ball $\mathscr{B}_{n+1}$ about g_{n+1} is contained in $\mathscr{B}_n \cap \mathscr{G}_{n+1}$. This completes the inductive construction, ending the proof of Theorem 4.1.

5. Comments

5.1. The theorem of Kuratowski provides in fact more information than stated in Theorem 1.1 (see [Ku2; §45, II]):

THEOREM (Kuratowski). *Let $K_1, K_2, \ldots$ be a sequence of perfect finite-dimensional compacta in a compactum X. Then, for almost every parametrization $f\colon 2^\infty \to X$, the order of f at each point of K_i is at most $\dim K_i + 1$, $i = 1, 2, \ldots$*

Hence, if Z is a perfect subcompactum of a compactum X, for a typical parametrization $f\colon 2^\infty \to X$ the points in Z of order one form a dense G_δ-set in Z. In particular, a typical parametrization f of a perfect compactum X on 2^∞ is irreducible, i.e., $f(F) \neq X$ for any proper compactum F in 2^∞.

Consider the countable-dimensional, but not strongly countable-dimensional compactum X mentioned in Sec. 2.5. By the theorem of Kuratowski, for almost all parametrizations $f\colon 2^\infty \to X$ the set $U(f) = \{x \in X\colon f^{-1}(x) \text{ is uncountable}\}$ is disjoint from $\bigcup_{i=1}^{\infty} K_i$, hence zero-dimensional. Nevertheless, typically, $U(f)$ is not Borel in X, by Corollary 4.2.

5.2. A classical theorem of Lusin [Lu; Ch. IV], the theorem of Kuratowski, and Theorem 4.1 (cf. also Purves [Pu]) provide the following

PROPOSITION. *Let M be a Borel set in a compactum X. Then M can be covered by countably many finite-dimensional compacta if, for a typical parametrization $f\colon 2^\infty \to X$, whenever $B \subset 2^\infty$ is a Borel set, so is the intersection $f(B) \cap M$.*

5.3. Let $\mathcal{J}$ be the σ-ideal of subsets of the Hilbert cube I^∞ which can be covered by countably many finite-dimensional compacta, and let $\mathcal{J}^*$ be the σ-ideal of sets $E \subset I^\infty$ such that for almost every parametrization $f\colon 2^\infty \to I^\infty$ all but countably many fibres $f^{-1}(x)$ with $x \in E$ are countable. By Kuratowski's theorem in 5.1, $\mathcal{J} \subset \mathcal{J}^*$, and Theorem 4.1 shows that all analytic sets in $\mathcal{J}^*$ are in fact in $\mathcal{J}$. Under the Continuum Hypothesis, any uncountable Lusin set L in I^∞ (i.e., L intersects each meagre subset of I^∞ in an at most countable set [Ku2; §40, VII]) belongs to $\mathcal{J}^*$ (see Sec. 5.1), but $L \notin \mathcal{J}$ (the members of $\mathcal{J}$ being meagre in I^∞). The sets in $\mathcal{J}^*$ are countable-dimensional (see the theorem at the end of Sec. 2.5). It seems that more information about the σ-ideal $\mathcal{J}^*$ would be of some interest.

5.4. Let X be a compactum all of whose nonempty open sets are infinite-dimensional. Given a parametrization $f\colon 2^\infty \to X$ and a compact set $T \subset 2^\infty$ we let

$$P(f, T) = \{x \in X\colon f^{-1}(x) \text{ is homeomorphic to } T\}.$$

By a theorem of Ryll-Nardzewski [RN], the sets $P(f, T)$ are Borel. Theorem 4.1 shows that, for a typical parametrization $f\colon 2^\infty \to X$, each point in X is a point of condensation of every set $P(f, T)$ with nonempty T. As was observed in Sec. 5.1, typically $P(f, \{t\})$ is a dense G_δ-set in X.

5.5. Bruckner and Garg [B-G] proved that for a typical continuous mapping $f\colon I \to I$ of the unit interval onto itself, the fibres $f^{-1}(x)$ are of one of the following three types: a perfect set, a union of a perfect set and a singleton, a singleton, and all but countably many fibres of f are perfect.

For $n \geq 2$, a typical Peano function $f\colon I \to I^n$, i.e., a continuous mapping of the unit interval onto the n-cube, is finite-to-one. This can be easily checked by standard arguments (more spcifically, a typical Peano function $f\colon I \to I^n$ is of order ≤ 5 if $n = 2$, and of order $\leq n+2$ if $n \geq 3$).

A typical parametrization $f\colon I \to I^\infty$ of the Hilbert cube on the unit interval has the universal property described in Theorem 4.1. In fact, only a slight adaptation is needed in the proof of Theorem 4.1 to pass from parametrizations of I^∞ on 2^∞ to parametrizations on I. In particular, for a typical Peano function $f\colon I \to I^\infty$ for the Hilbert cube, all topological types of nonempty compacta in 2^∞ appear as the fibres $f^{-1}(x)$.

5.6. For the reader's convenience we provide an argument justifying analyticity of the set of parametrizations defined by (2) in Sec. 4.3.

Let $B_1, B_2, \ldots$ be a base of closed-and-open sets in 2^∞. Using the notation preceding (2) consider the set

$$L = \bigcup_i \{(f, v, w, t)\colon \big(v(S_i) \cap B_i \neq \emptyset \text{ and } w(t) \notin f(B_i)\big) \text{ or } \big(v(S_i) \cap B_i = \emptyset \text{ and } w(t) \in f(B_i)\big)\}.$$

The set L is of type F_σ in the product of appropriate complete function spaces with the Cantor set 2^∞ and so is the projection H of L parallel to the compact coordinate 2^∞. The complement of H is therefore a G_δ-set and its projection on the first coordinate is the set defined by (2).

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