On Sierpiński's nonmeasurable set

by

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Abstract. We obtain some sufficient conditions under which the Lebesgue measurability of a plane set E can be guaranteed by the nature of its "slices" (intersections with lines L). In particular, we show in Theorem 2 that if $E \cap L$ is the union of at most two disjoint open intervals for each line L, except for lines whose inclinations θ form a subset of $[0, \pi)$ with linear measure zero, then E is measurable.

This note is motivated by a famous example due to Sierpiński [2, 4] of a nonmeasurable (with respect to Lebesgue measure) subset E of $\mathbf{R}^2$ whose intersection with any line consists of at most two points. The complement E^c of E is of course nonmeasurable, too, and it is clear that the intersection of E^c with any line is the union of at most three disjoint open intervals. Thus, despite the nonmeasurability of E^c, each slice is a linearly measurable, very simple open subset of $\mathbf{R}$. One is therefore led to wonder just how "simple" the slices of a set E in $\mathbf{R}^2$ must be in order to guarantee its measurability. Some answers to this question will be provided in this note.

We will examine sets E in $\mathbf{R}^2$ for which there is at least one direction in which every slice of E is an open set, when viewed as a subset of $\mathbf{R}$. In order to classify and compare these sets, it will be helpful to introduce some notation. Let λ and λ_2 denote the Lebesgue measures in $\mathbf{R}$ and $\mathbf{R}^2$ respectively, let Λ denote the collection of all λ-measurable subsets of $\mathbf{R}$, and let $\mathcal{O}_n$ be the collection of all open subsets of $\mathbf{R}$ which are the unions of at most n disjoint open intervals (we assume for convenience that the empty set belongs to $\mathcal{O}_n$). For each $\theta \in D := [0, \pi)$ we denote by $\mathcal{L}(\theta)$ the collection of all lines L in $\mathbf{R}^2$ with inclination θ. Finally, for any fixed θ and n, let

$$\mathcal{M}(\theta) = \{E \subseteq \mathbf{R}^2 \mid E \cap L \in \Lambda \text{ for each } L \in \mathcal{L}(\theta)\},$$

$$\mathcal{G}_n(\theta) = \{E \subseteq \mathbf{R}^2 \mid E \cap L \in \mathcal{O}_n \text{ for each } L \in \mathcal{L}(\theta)\}.$$

Clearly for fixed θ, $\mathcal{G}_1(\theta) \subseteq \mathcal{G}_2(\theta) \subseteq \ldots \subseteq \mathcal{M}(\theta)$, and the sets from $\mathcal{G}_n(\theta)$ are intuitively simpler than those from $\mathcal{M}(\theta)$. Further improvements can be obtained by

1980 *Mathematics Subject Classification* (1985 Revision): Primary 28A05.

reducing n or by taking intersections, such as $\mathscr{G}_n(\theta_1)\cap\mathscr{G}_m(\theta_2)$. In our notation, the Sierpiński example asserts basically that $\bigcap_{\theta\in D}\mathscr{G}_3(\theta)$ contains a nonmeasurable set. The results obtained in this note are stated in the following theorems. We do not give a proof of Theorem 5, since the referee has pointed out that it is a special case of a result due to Grande [3].

THEOREM 1. *If $\tilde{D}\subset D$ satisfies* $\operatorname{card}\tilde{D}<\mathfrak{c}$, *then* $\bigcap_{\theta\in D}\mathscr{G}_3(\theta)\cap\bigcap_{\tilde\theta\in\tilde D}\mathscr{G}_2(\tilde\theta)$ *contains a nonmeasurable set.*

THEOREM 2. *If $\lambda(D\setminus\tilde D)=0$, then every set $E\in\bigcap_{\tilde\theta\in\tilde D}\mathscr{G}_2(\tilde\theta)$ is measurable.*

THEOREM 3. *If $D\setminus\tilde D$ is finite, then every set $E\in\bigcap_{\theta\in D}\mathscr{G}_3(\theta)\cap\bigcap_{\tilde\theta\in\tilde D}\mathscr{G}_2(\tilde\theta)$ is open in $\mathbf{R}^2$.*

THEOREM 4. *$\mathscr{G}_1(\theta)$ contains a nonmeasurable set.*

THEOREM 5. *Every set $E\in\mathscr{G}_1(\theta)\cap\mathscr{M}(\tilde\theta)$ is measurable, provided $\theta\neq\tilde\theta$.*

Proof of Theorem 1. The proof is just a minor modification of that given in [2, pp. 142–144], and hence we give here only a sketch of the necessary changes in that proof. The reader is referred to [2] for the missing details. Our notation is that of [2] except for the symbols $\tilde\theta$ and $\tilde D$ (already defined), and L' (defined below).

The idea is to prove the existence of a nonmeasurable subset E of $\mathbf{R}^2$ having at most two points in common with any line and having at most one point in common with any line in the directions $\tilde\theta\in\tilde D$, where $\operatorname{card}\tilde D<\mathfrak{c}$; then E^c is the desired nonmeasurable set. Assume $\tilde D$ has been chosen. The first modification of the proof is to add a third condition (c) on the functions $p(\alpha)\in F$ defined in [2], namely:

(c) no two points in the range of $p(\alpha)$ are collinear in the directions $\tilde\theta\in\tilde D$.

The second modification is that the direction θ in which L intersects F_β should be chosen from the set $D\setminus\tilde D$ (which has cardinality $\mathfrak{c}$). Third, before choosing the points p_β from the subset $L'\subset L\cap F_\beta$ whose points are not collinear with pairs of points in $E=\operatorname{Range}q(\alpha)$, we must exclude those points which are collinear with other points in E in the directions $\tilde\theta\in\tilde D$. This can be done without exhausting L', because $\operatorname{card}L'=\mathfrak{c}$ and, for E at this stage of the proof, the set of excluded points has cardinality $\leqslant(\operatorname{card}E)(\operatorname{card}\tilde D)<\mathfrak{c}$. Finally, we can choose p_β from the remaining points in L' and finish the proof in a fashion analogous to [2].

Remarks. Under the assumption of the continuum hypothesis it is not difficult to use a similar method to prove that if $\lambda(\tilde D)=0$, then $\bigcap_{\theta\in D}\mathscr{G}_3(\theta)\cap\bigcap_{\tilde\theta\in\tilde D}\mathscr{G}_2(\tilde\theta)$ contains a nonmeasurable set.

In a recently published note [1] van Douwen proved the existence of a nonmeasurable subset A of $\mathbf{R}^2$ such that each vertical line in $\mathbf{R}^2$ contains at most one point of A. The proof of Theorem 1 shows that, in fact, nonmeasurable sets exist in $\mathbf{R}^2$ which have this intersection property for any arbitrary number of directions.

Proof of Theorem 2. From now on we shall use the brackets $\langle\ ,\ \rangle$ to denote ordered pairs, in order to distinguish them from open intervals. Let $E\in\bigcap_{\tilde\theta\in\tilde D}\mathscr{G}_2(\tilde\theta)$; without loss of generality we may assume that $0,\pi/2\in\tilde D$, i.e. that vertical and horizontal slices belong to $\mathscr{O}_2$. (In fact, there must exist directions $\tilde\theta_1,\tilde\theta_2\in\tilde D$ such that $\tilde\theta_1+\pi/2=\tilde\theta_2$. If not, then the set $\{\tilde\theta+\pi/2\,|\,\tilde\theta\in\tilde D\cap[0,\pi/2)\}\cup\{\tilde\theta-\pi/2\,|\,\tilde\theta\in\tilde D\cap[\pi/2,\pi)\}$ would belong to $D\setminus\tilde D$ and have measure $=\lambda(\tilde D)=\pi$.) Our goal is to write E as the countable union

$$E=\bigcup_{\substack{p,q\in\mathbf{Q}\\ p<q}}A_{p,q}$$

where each set $A_{p,q}$ is λ_2-measurable. For each $r\in\mathbf{R}$, let E^r denote the set

$$E^r=\{x\in\mathbf{R}\,|\,\langle x,r\rangle\in E\},$$

and for each $p,q\in\mathbf{Q}$, such that $p<q$, let $E^{p,q}$ denote the set

$$E^{p,q}=\{x\in\mathbf{R}\,|\,\{x\}\times[p,q]\subset E\}.$$

That is, $E^{p,q}$ is the set of all x for which E contains the vertical line segment with endpoints $\langle x,p\rangle$ and $\langle x,q\rangle$. Note that $E^{p,q}\subset E^p\cap E^q$ and that $E^p\cap E^q$ (if nonempty) is the disjoint union of a finite number of open intervals, since the same is true of E^p and E^q by the assumption that $0\in\tilde D$. Let us fix $p,q\in\mathbf{Q}$ with $p<q$ and assume for simplicity that $E^p\cap E^q$ is a single, bounded, open interval (otherwise our construction of $A_{p,q}$ can be done in pieces, taking one interval of $E^p\cap E^q$ at a time).

If $E^{p,q}$ is λ-measurable, then simply let $A_{p,q}$ be the λ_2-measurable set

$$A_{p,q}=E^{p,q}\times[p,q].$$

If $E^{p,q}$ is not λ-measurable, let us set $a_{p,q}=\inf E^{p,q}$, $b_{p,q}=\sup E^{p,q}$, and let $R_{p,q}$ be the rectangle

$$R_{p,q}=[a_{p,q},b_{p,q}]\times[p,q].$$

In this case we define $A_{p,q}$ to be the set

$$A_{p,q}=R_{p,q}\cap E.$$

We will show that $A_{p,q}$ is a λ_2-measurable subset of E by showing that $A_{p,q}=R_{p,q}$ λ_2-almost everywhere.

Since $E^{p,q}$ is nonmeasurable, its relative complement $F^{p,q}:=(a_{p,q},b_{p,q})\setminus E^{p,q}$ is also nonmeasurable, and hence $F^{p,q}$ must contain a point x_0 such that $[x_0,x_0+\varepsilon)\cap E^{p,q}$ and $(x_0-\varepsilon,x_0]\cap E^{p,q}$ are nonempty for any $\varepsilon>0$; otherwise $F^{p,q}$ could be written $F^{p,q}=\bigcup_{x\in F^{p,q}}I_x$, where each I_x is a closed-open or open-closed interval in $F^{p,q}$ containing x. But $\mathbf{R}$ is Lindelöf in the topologies generated by either type interval, and hence the cover $\bigcup_{x\in F^{p,q}}I_x$ could be reduced to a countable cover of measurable sets, thereby contradicting the nonmeasurability of $F^{p,q}$.

Now since $x_0\in E^p\cap E^q\setminus E^{p,q}$, it follows from the definition of $E^{p,q}$ that the line segment $\{x_0\}\times(p,q)$ must contain a point $\langle x_0,y_0\rangle\notin E$. This point clearly lies in the interior of $R_{p,q}$; let us use it as a polar origin to measure $A_{p,q}$ in polar coordinates and show that, as claimed, $\lambda_2(A_{p,q})=\lambda_2(R_{p,q})$.

Of course we must also show that $A_{p,q}$ is in fact λ_2-measurable. Recalling that $\lambda(D\setminus\tilde D)=0$, we can accomplish both tasks by showing that for any line $L_{\tilde\theta}$ with inclination $\tilde\theta\in\tilde D$ which passes through $\langle x_0,y_0\rangle$, we have $\lambda(L_{\tilde\theta}\cap A_{p,q})=\lambda(L_{\tilde\theta}\cap R_{p,q})$. In fact, except for the point $\langle x_0, y_0\rangle$, the two sets just mentioned coincide in the interior $\mathring{R}_{p,q}$ of $R_{p,q}$. To prove this, suppose by way of contradiction that for some $\tilde{\theta}\in\tilde{D}$ there exists a point $\langle x_\alpha, y_\alpha\rangle \neq \langle x_0, y_0\rangle$ such that $\langle x_\alpha, y_\alpha\rangle \in L_{\tilde{\theta}} \cap (\mathring{R}_{p,q}\backslash A_{p,q})$, or equivalently, such that $\langle x_\alpha, y_\alpha\rangle \in L_{\tilde{\theta}} \cap (\mathring{R}_{p,q}\backslash E)$ (see Fig. 1).

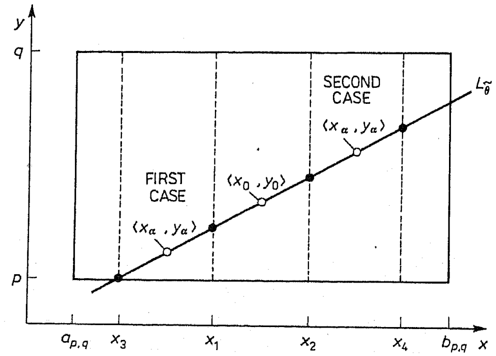

Fig. 1

There are two cases to consider. First, suppose that the ray from $\langle x_0, y_0\rangle$ through $\langle x_\alpha, y_\alpha\rangle$ intersects the bottom (as drawn) or top of the rectangle $R_{p,q}$. For the purposes of measure we will ignore the special case in which $L_{\tilde{\theta}}$ intersects a corner point of the rectangle, although this can be handled as well. Let us call this intersection point $\langle x_3, p\rangle$ and recall that since $x_3 \in (a_{p,q}, b_{p,q}) \subset E^p$, it follows that $\langle x_3, p\rangle \in E$ (points which belong to E are indicated in solid black in the figure). Moreover, since x_0 is both a left- and right-hand limit point of $E^{p,q}$, we can find points $x_1, x_2 \in E^{p,q}$ sufficiently close to x_0 such that $x_3 < x_\alpha < x_1 < x_0 < x_2$ and such that the segments $\{x_1\}\times[p, q]$ and $\{x_2\}\times[p, q]$ intersect $L_{\tilde{\theta}}$ at the points $\langle x_1, y_1\rangle$ and $\langle x_2, y_2\rangle$, respectively; by definition of $E^{p,q}$ these points also belong to E. But now five points lie on $L_{\tilde{\theta}}$ in the following order:

$$\langle x_3, p\rangle,\ \langle x_\alpha, y_\alpha\rangle,\ \langle x_1, y_1\rangle,\ \langle x_0, y_0\rangle,\ \langle x_2, y_2\rangle,$$

where the first, third, and fifth points belong to E and the second and fourth points do not. This clearly contradicts the fact that $E\cap L_{\tilde{\theta}} \in \mathcal{O}_2$; i.e., that $E \cap L_{\tilde{\theta}}$ is the union of at most two disjoint open intervals.

The second case to consider is the case in which the ray from $\langle x_0, y_0\rangle$ through $\langle x_\alpha, y_\alpha\rangle$ intersects a side (say the right side, as drawn) of the rectangle $R_{p,q}$. By a familiar argument, we can find points $\langle x_1, y_1\rangle, \langle x_2, y_2\rangle \in E\cap L_{\tilde{\theta}}$ such that $x_1 < x_0 < x_2 < x_\alpha$. Moreover, since $b_{p,q} = \sup E^{p,q}$, we can find $x_4 \in E^{p,q}$ with $x_\alpha < x_4 \leqslant b_{p,q}$, and a corresponding point $\langle x_4, y_4\rangle \in E\cap L_{\tilde{\theta}}$. Again we have five points

$$\langle x_1, y_1\rangle,\ \langle x_0, y_0\rangle,\ \langle x_2, y_2\rangle,\ \langle x_\alpha, y_\alpha\rangle,\ \langle x_4, y_4\rangle,$$

whose memberships alternate between E and its complement. By the same reasoning as before, this gives us the desired contradiction for the second case, and the claim that $A_{p,q} = R_{p,q}$ λ_2-almost everywhere is therefore proved.

Finally, it is clear that

$$\bigcup_{\substack{p,q\in\mathbf{Q}\\ p<q}} A_{p,q} \subseteq E.$$

To show the reverse inclusion, let $\langle x, y\rangle \in E$, and observe that, since $\pi/2 \in \tilde{D}$, there must exist rationals p and q with $p < y < q$ such that the segment $\{x\}\times[p, q]$ belongs to E. But then $\langle x, y\rangle \in A_{p,q}$, by either definition of $A_{p,q}$.

Proof of Theorem 3. Let $E \in \bigcap_{\theta\in D}\mathcal{G}_3(\theta) \cap \bigcap_{\tilde{\theta}\in\tilde{D}}\mathcal{G}_2(\tilde{\theta})$ and assume by way of contradiction that E is not open. By assumption, E contains a limit point of E^c; without loss of generality, we can assume this point to be the origin. Then by the line-intersection properties of E, the set E must contain a countable number of line segments s_n, each containing the origin and having a point (in polar coordinates) $\langle r_n, \theta_n\rangle \in E^c$ as one (deleted) endpoint, where $r_n \downarrow 0$, $\theta_n \in [0, 2\pi)$, and $\theta_n \neq \theta_m$ if $n \neq m$. Moreover, since $\theta_n \in [0, 2\pi)$, the sequence $\{\theta_n\}$ must have a limit point θ; let us assume for convenience that $\theta = 0$ and $\theta_n \downarrow 0$.

For each n, let us convert $\langle r_n, \theta_n\rangle$ to rectangular coordinates $\langle a_n, b_n\rangle$, and work with rectangular coordinates hereafter. Again by the nature of E, there must be a point $\langle c, 0\rangle$ with $c > 0$ and $[0, c)\times\{0\} \subseteq E$ (see Fig. 2). For the same reason, we can choose $a\in(0, c)$ and a point $\langle a, b\rangle$ with $b > 0$ such that $\{a\}\times[0, b) \subseteq E$. Since $r_n \downarrow 0$ and $\theta_n \downarrow 0$, it follows that every triangle with vertices $\langle 0, 0\rangle$, $\langle a', 0\rangle$, and $\langle a', b'\rangle$, where $a', b' > 0$, contains a tail of the sequence $\{\langle a_n, b_n\rangle\}$. Thus we can choose $\langle a_i, b_i\rangle$ such that $0 < a_i < a$ and $0 < b_i/a_i < b/a$, and choose $\langle a_j, b_j\rangle$ such that $0 < a_j < a_i$ and $0 < b_j < b_i$.

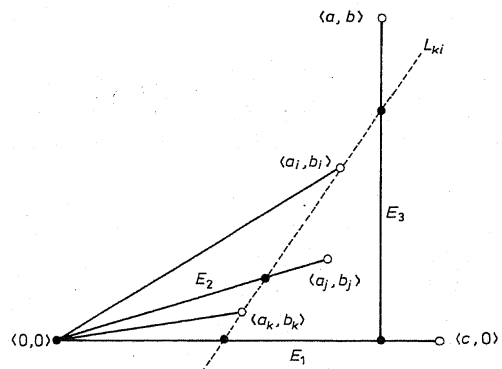

Fig. 2

Finally, we can choose $\langle a_k, b_k\rangle$ such that $0 < a_k < a_j$, $0 < b_k < b_j$, and such that the line L_{ki} through $\langle a_k, b_k\rangle$ and $\langle a_i, b_i\rangle$ intersects the following three subsets of E:

$$E_1 := [0, c)\times\{0\},$$

$$E_2 := \text{the segment from } \langle 0, 0\rangle \text{ to } \langle a_j, b_j\rangle,$$

$$E_3 := \{a\}\times[0, b).$$

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But now five points of L_{ki} are arranged in the following order:

$$E_1 \cap L_{ki},\ \langle a_k, b_k\rangle,\ E_2 \cap L_{ki},\ \langle a_i, b_i\rangle,\ E_3 \cap L_{ki}.$$

Since $\langle a_k, b_k\rangle, \langle a_i, b_i\rangle \in E^c$, this means that $E \cap L_{ki}$ cannot be the union of at most two disjoint open intervals (i.e., $E \cap L_{ki} \notin \mathcal{O}_2$). Furthermore, it should be clear that the choice of $\langle a_k, b_k\rangle$ can be made from an infinite number of points $\langle a_n, b_n\rangle$ such that $E \cap L_{ni} \notin \mathcal{O}_2$, where L_{ni} and L_{mi} have different inclinations if $m \neq n$. This contradicts the fact that $D\backslash\tilde{D}$ is finite.

Remarks. It is not difficult to apply a similar method and induction to prove the following generalization: If E is a subset of $\mathbf{R}^k$ for any $k \in \mathbf{N}$ such that $E \cap L$ is open in L for every line L in $\mathbf{R}^k$, and $E \cap L \in \mathcal{O}_2$ for every line L in $\mathbf{R}^k$ except for lines in a finite number of directions, then E is open in $\mathbf{R}^k$.

Note that the theorem does not hold if the number of directions in which $E \cap L \notin \mathcal{O}_2$ is infinite. For example, let B denote the open unit ball in $\mathbf{R}^2$, let S be the sequence of points $\langle 1/n, 1/n^2\rangle$ for $n \geqslant 2$, and let $E = B\backslash S$. Then $E \cap L \in \mathcal{O}_2$ except for the lines L_{mn} determined by pairs of points in S (observe that $\operatorname{card}\{L_{mn}\} = \aleph_0$ and that $E \cap L_{mn} \in \mathcal{O}_3$ for each L_{mn}). In this case, E is neither open nor closed.

Examples of sets satisfying the hypothesis of the theorem suggest an affirmative answer to the following open question suggested by the referee: Can any set satisfying the hypothesis of Theorem 3 be written as a finite union of convex sets?

Proof of Theorem 4. Let F be a (linearly) nonmeasurable subset of $I := (0, 1)$. Then $E = F \times I$ belongs to $\mathcal{G}_1(\pi/2)$, and by Fubini's theorem, E is clearly nonmeasurable.

The author is indebted to Professor C. D. Aliprantis for a careful reading of an early draft, and especially to Professor Yuri Abramovich for his advice and encouragement. Thanks are also due to the referee for several corrections and improvements.

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Received 10 April 1990;
in revised form 22 October 1990

Les fonctions continues et les fonctions intégrables au sens de Riemann comme sous-espaces de $\mathscr{L}^1$

par

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Abstract. Let $\mathscr{R}$ (resp. $\mathscr{C}$) be the subspace of $\mathscr{L}^1([0, 1])$ consisting of those elements having a representative which is Riemann-integrable (resp. continuous). We prove that $\mathscr{R}$ and $\mathscr{C}$ are homeomorphic to the countable product Σ^∞, where $\Sigma = \{(x_n) \in l^2 : \sum_{n=1}^{\infty} (nx_n)^2 < \infty\}$. We give topological characterizations of the pair $(\mathscr{R}, \mathscr{C})$ and of the difference $\mathscr{R}\backslash\mathscr{C}$.

1. Introduction. Soit $\mathscr{L}^1$ l'espace des classes de fonctions réelles intégrables sur $[0, 1]$ avec la norme usuelle $\|f\| = \int_0^1 |f(t)|\,dt$. Soit $\mathscr{C}$ (resp. $\mathscr{R}$) le sous-espace de $\mathscr{L}^1$ formé des éléments ayant un représentant continu (resp. intégrable au sens de Riemann). Nous nous proposons ici de caractériser topologiquement les espaces $\mathscr{R}$ et $\mathscr{C}$, leur différence $\mathscr{R}\backslash\mathscr{C}$ et le couple $(\mathscr{R}, \mathscr{C})$.

Soit Σ le sous-espace de l'espace de Hilbert l^2 formé des suites (x_n) telles que $\sum_{n=1}^{\infty} (nx_n)^2 < \infty$, et soit Σ^∞ le produit d'une infinité dénombrable de copies de Σ.

1.1. THÉORÈME. *$\mathscr{R}$ et $\mathscr{C}$ sont homéomorphes à Σ^∞.*

Pour formuler nos caractérisations de $\mathscr{R}\backslash\mathscr{C}$ et de $(\mathscr{R}, \mathscr{C})$, nous avons besoin de quelques définitions. Si f et g sont deux fonctions de Y dans X, et si $\mathscr{U}$ est un recouvrement ouvert de X, nous dirons que f est *$\mathscr{U}$-proche* de g si, pour tout y dans Y, il y un élément de $\mathscr{U}$ contenant à la fois $f(y)$ et $g(y)$. Un sous-ensemble F d'un rétracte absolu de voisinage X est appelé un *Z-ensemble* dans X s'il est fermé et si, pour tout recouvrement ouvert $\mathscr{U}$ de X, il existe une fonction continue f de X dans X, $\mathscr{U}$-proche de l'identité, et telle que $f(X) \subset X\backslash F$; si, de plus, il est toujours possible de choisir la fonction f de façon que $\overline{f(X)} \cap F = \varnothing$, alors F est appelé un *Z-ensemble au sens fort* dans X. Une fonction $f\colon Y \to X$ est appelée un *Z-plongement* si c'est un plongement et si $f(Y)$ est un Z-ensemble dans X.

Par un *couple* (X, X'), nous entendons un espace X et un sous-espace X' de X. Soient $\mathscr{K}_1$ et $\mathscr{K}_2$ deux classes d'espaces métrisables et séparables. Un couple (X, X') où X est un rétracte absolu de voisinage est dit *$(\mathscr{K}_1, \mathscr{K}_2)$-universel* si, pour tout couple (C_1, C_2) où C_1 appartient à $\mathscr{K}_1$ et C_2 à $\mathscr{K}_2$, toute fonction continue f de C_1 dans X et tout recouvrement ouvert $\mathscr{U}$ de X, il existe un Z-plongement g de C_1 dans X qui est