On the average of inner and outer measures

by

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Abstract. Let (X, Σ, μ) be a measure space, and write $\theta(A) = \frac{1}{2}(\mu^* A + \mu_* A)$ for any $A \subseteq X$. C. Carathéodory showed that θ is an outer measure; let ν be the corresponding measure. I give a complete description (Theorem 2) of the circumstances in which ν can fail to be equal to μ, and show that these cannot arise from "ordinary" measure spaces.

1. Introduction. Let (X, Σ, μ) be a measure space. Write μ^*, μ_* for the associated outer and inner measures on X, given by

$$\mu^*(A) = \min\{\mu E\colon A \subseteq E \in \Sigma\}, \quad \mu_*(A) = \max\{\mu E\colon A \supseteq E \in \Sigma\};$$

set

$$\theta(A) = \tfrac{1}{2}(\mu^* A + \mu_* A)$$

for every $A \subseteq X$. Then θ is an outer measure on X ([1], § 600–603). Let ν be the measure on X defined from θ by Carathéodory's method; write T for the domain of ν. Then ν is an extension of μ. The question arises: when is ν a proper extension of μ? Carathéodory seems to have left this open even when μ is Lebesgue measure. For this case, J. C. Oxtoby (private communication to A. H. Stone) showed that $\nu = \mu$ if the continuum hypothesis is true. Here I describe the ways in which ν can be different from μ (§§ 2–4) and show (in ZFC) that this never occurs if μ is a Radon measure (§ 11).

2. Theorem. *Let X, Σ, μ, μ^*, μ_*, θ, T and ν be as in* § 1. *Then the following are equivalent*:

(a) $\nu \neq \mu$;

(b) either (i) (X, Σ, μ) *is not complete* (*that is to say, there is a set $A \subseteq X$ such that* $\mu^* A = 0$ but $A \notin \Sigma$),
  or (ii) *there is a set $A \subseteq X$ such that $A \cap E \in \Sigma$ whenever $E \in \Sigma$ and $\mu E < \infty$, but* $A \notin \Sigma$,

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or (iii) *there are sets* $D, D' \subseteq X$ *such that*

$$D \cap D' = \emptyset, \quad D \cup D' = E \in \Sigma, \quad \mu_* D = \mu_* D' = 0 < \mu E < \infty,$$

$$\mathscr{P}(D) = \{D \cap F\colon F \in \Sigma\}, \quad \mathscr{P}(D') = \{D' \cap F\colon F \in \Sigma\}.$$

Proof. (a)$\Rightarrow$(b). Assume that (a) is true, but that (b-i) and (b-ii) are both false; I have to show that (b-iii) is true. Because ν is a proper extension of μ, there must be a set $D_0 \in T \setminus \Sigma$; because (b-ii) is false, there must be an $E_0 \in \Sigma$ such that $\mu E_0 < \infty$ and $D_0 \cap E_0 \notin \Sigma$. Let $E_1, E_2 \in \Sigma$ be such that $E_1 \subseteq D_0 \cap E_0 \subseteq E_2$ and

$$\mu E_1 = \mu_*(D_0 \cap E_0), \quad \mu E_2 = \mu^*(D_0 \cap E_0).$$

Because (b-i) is false, $\mu^*((D_0 \cap E_0) \setminus E_1) > 0$ and $\mu E_2 > \mu E_1$. Set

$$E = E_2 \setminus E_1, \quad D = (D_0 \cap E_0) \setminus E_1, \quad D' = E \setminus D.$$

Then $D \cap D' = \emptyset$, $D \cup D' = E$ and $\mu_* D = \mu_* D' = 0 < \mu E < \infty$. Also $D, D' \in T$.

Let A be any subset of D. Consider $B = A \cup D'$. Then

$$\theta(B) = \theta(B \cap D) + \theta(B \setminus D)$$

because $D \in T$. But let us seek to calculate the relevant values of μ^*, μ_*. We have

$$\mu^* B = \mu^* D' = \mu E = \delta \quad \text{say}, \quad \mu^*(B \cap D) = \mu^* A,$$

$$\mu_*(B \cap D) = 0, \quad \mu^*(B \setminus D) = \delta, \quad \mu_*(B \setminus D) = 0.$$

So we get

$$\tfrac{1}{2}(\delta + \mu_* B) = \tfrac{1}{2}(\mu^* A + 0) + \tfrac{1}{2}(\delta + 0),$$

and $\mu^* A = \mu_* B$. Let $F_1, F_2 \in \Sigma$ be such that $F_1 \subseteq B$, $F_2 \supseteq A$ and $\mu F_1 = \mu_* B = \mu^* A = \mu F_2$. Then $F_1 \setminus F_2 \subseteq D'$, so $\mu(F_1 \setminus F_2) = 0$; consequently $\mu(F_2 \setminus F_1) = 0$; because (b-i) is false, it follows that $A \setminus F_1$ and $F = F_1 \cup A$ belong to Σ, and we see that $A = D \cap F$. As A is arbitrary, $\mathscr{P}(D) = \{D \cap F\colon F \in \Sigma\}$. Of course the same argument applies to D', so all the clauses of (b-iii) are satisfied by D, D'.

(b-i)$\Rightarrow$(a). If $\mu^* A = 0$ then $\theta A = 0$ and $A \in T$; so if also $A \notin \Sigma$ then $\nu \neq \mu$.

(b-ii)$\Rightarrow$(a). If $A \cap E \in \Sigma$ whenever $\mu E < \infty$, then for any $B \subseteq X$ with $\theta B < \infty$ we have an $E \in \Sigma$ such that $B \subseteq E$ and $\mu E = \mu^* B < \infty$; in which case

$$\theta B = \theta(B \cap (A \cap E)) + \theta(B \setminus (A \cap E)) = \theta(B \cap A) + \theta(B \setminus A),$$

and $A \in T$. So if $A \notin \Sigma$ then $\nu \neq \mu$.

(b-iii)$\Rightarrow$(a). If D, D' and E are as specified in (b-iii), then of course $\mu^* D = \mu E > \mu_* D$, so $D \notin \Sigma$. On the other hand, D does belong to T. To see this, take any $B \subseteq X$. Let $H \in \Sigma$ be such that $B \cap E \subseteq H$ and $\mu H = \mu^*(B \cap E)$. Let $F, F' \in \Sigma$ be such that

$$D \cap B = D \cap F, \quad D' \cap B = D' \cap F';$$

we may suppose that $F \cup F' \subseteq E \cap H$, so that $F \cap F' \subseteq B$. Now

$$\begin{aligned}\theta(B \cap D) + \theta(B \cap D') &= \tfrac{1}{2}(\mu^*(B \cap D) + \mu^*(B \cap D')) \leqslant \tfrac{1}{2}(\mu F + \mu F') \\ &= \tfrac{1}{2}(\mu(F \cap F') + \mu(F \cup F')) \leqslant \tfrac{1}{2}(\mu_*(B \cap E) + \mu H) \\ &= \theta(B \cap E).\end{aligned}$$

So

$$\theta(B \cap D) + \theta(B \setminus D) \leqslant \theta(B \cap D) + \theta(B \cap D') + \theta(B \setminus E) \leqslant \theta(B \cap E) + \theta(B \setminus E) = \theta(B)$$

(because $E \in \Sigma \subseteq T$). As B is arbitrary, $D \in T$ and $\nu \neq \mu$.

3. Remark. The conditions (b-i) and (b-ii) of Theorem 2 are straightforward; they are the two ways in which μ can fail to be the measure defined from the outer measure μ^*. If (following Carathéodory) we restrict attention to the case in which μ is derived from a regular outer masure, or if (for instance) we are interested only in complete σ-finite measure spaces, then neither of these will occur. The rest of this paper will accordingly be devoted to the phenomenon of (b-iii). This can be elaborated upon in the following manner. Let (X, Σ, μ) be any measure space. For any subset A of X, write Σ_A for $\{A \cap F\colon F \in \Sigma\}$, and μ_A for $\mu^* \restriction \Sigma_A$; then (A, Σ_A, μ_A) is a measure space; write $\mathfrak{A}(\mu_A)$ for the measure algebra $\Sigma_A / (\Sigma_A \cap \mathscr{N}_\mu)$, where $\mathscr{N}_\mu = \{B\colon \mu^* B = 0\}$. If D, D' are subsets of X such that $E = D \cup D' \in \Sigma$ and $\mu_* D = \mu_* D' = 0$, then we have an isomorphism $\phi\colon \mathfrak{A}(\mu_D) \to \mathfrak{A}(\mu_{D'})$ given by the formula

$$\phi(D \cap F)^\bullet = (D' \cap F)^\bullet \quad \forall F \in \Sigma,$$

where $B^\bullet \in \mathfrak{A}(\mu_D)$ is the equivalence class of $B \in \Sigma_D$. Moreover, if (X, Σ, μ) is complete and $\mu E < \infty$, then Σ_E is precisely

$$\{A\colon A \subseteq E,\ A \cap D \in \Sigma_D,\ A \cap D' \in \Sigma_{D'},\ \phi(A \cap D)^\bullet = (A \cap D')^\bullet\}.$$

Accordingly, the following construction is a canonical method of constructing examples in which (b-iii) of Theorem 2 is true.

4. Proposition. *Let* $(X, \mathscr{P}(X), \nu)$ *be a measure space with* $0 < \nu X < \infty$. *Suppose that* $D \subseteq X$ *is such that* $\mathfrak{A}(\nu_D)$ *is isomorphic, as measure algebra, to* $\mathfrak{A}(\nu_{D'})$, *where* $D' = X \setminus D$; *let* $\phi\colon \mathfrak{A}(\nu_D) \to \mathfrak{A}(\nu_{D'})$ *be a measure-preserving isomorphism. Set*

$$\Sigma = \{F\colon F \subseteq X,\ \phi(F \cap D)^\bullet = (F \cap D')^\bullet\}$$

and $\mu = \nu \restriction \Sigma$. *Then* (X, Σ, μ) *is a complete totally finite measure space. Set* $\theta = \nu$, $T = \mathscr{P}(X)$; *then* $X, \Sigma, \mu, \mu^*, \mu_*, \theta, T$ *and* ν *are as in* §1, *with* $\nu \neq \mu$.

5. Remarks. To put flesh on these ideas we need examples satisfying the conditions of Proposition 4. The simplest case is when $X = \{x, y\}$, $\nu\{x\} = \nu\{y\} = \frac{1}{2}$ and $D = \{x\}$. The corresponding phenomenon in the language of Theorem 2 is when there is a doubleton set $E = \{x, y\} \in \Sigma$ such that $\mu E > 0$ but $\{x\} \notin \Sigma$; then $\{x\} \in T$ with $\nu\{x\} = \frac{1}{2}\mu E$. It is relatively consistent with ZFC to suppose that this is the only way in which (b-iii) of Theorem 2 can arise. For it is consistent to suppose that whenever $(Y, \mathscr{P}(Y), \lambda)$ is a measure space with $0 < \lambda Y < \infty$, then there is a $y \in Y$ such that $\lambda\{y\} > 0$ ([7], §28). In this case, if D, D' and E are as in (b-iii) of Theorem 2, there is an $x \in D$ with $\nu\{x\} > 0$; there is an $F \in \Sigma$ with $D \cap F = \{x\}$; now $\nu(D' \cap F) > 0$, so there is a $y \in D' \cap F$ with $\nu\{y\} > 0$; there is an $F' \in \Sigma$ with $D' \cap F' = \{y\}$; but as $\nu(D \cap F \cap F')$ must now be greater than 0, $F \cap F' \cap E$ must be exactly $\{x, y\}$, and $\mu\{x, y\} > 0$, while $\{x\} \notin \Sigma$ because $\mu_* D = 0$.

It seems likely that it is also consistent to suppose that there are measure spaces $(Y, \mathscr{P}(Y), \lambda)$ with $0 < \lambda Y < \infty$ but $\lambda\{y\} = 0$ for every $y \in Y$. However, these are necessarily extraordinary in various ways. For instance, if λ has an atom, then there is a two-valued-measurable cardinal $\varkappa \leqslant \#(Y)$ ([7], §27); if λ does not have an atom, then there is an atomlessly-measurable cardinal strictly less than the Maharam type of $(Y, \mathscr{P}(Y), \lambda)$ ([6], Theorem 2.6). These ideas suffice to prove the following.

6. Corollary. *In Theorem 2, if* (b-iii) *is true, then*

either *there is a doubleton set* $E = \{x, y\} \in \Sigma$ *such that* $\mu E > 0$ *and* $\{x\} \notin \Sigma$;

or μ *has an atom* $E \in \Sigma$ *such that* $\#(E) \geqslant \varkappa$ *for some two-valued-measurable cardinal* $\varkappa$;

or *there is an* $E \in \Sigma$ *such that* $\mu E < \infty$ *and the Maharam type of* (E, Σ_E, μ_E) *is greater than* $\varkappa$ *for some atomlessly-measurable cardinal* $\varkappa$.

7. Remarks. This is already more than enough to ensure that $\nu = \mu$ if $X = \mathbf{R}$ and μ is Lebesgue measure. But we can extend this to all Radon measures and many perfect measures, using some "well-known" facts about atomlessly-measurable cardinals. In [8], Kunen showed that if there is any atomlessly-measurable cardinal $\varkappa$ then there is a set $A \subseteq \mathbf{R}$ such that $\#(A) < \varkappa$ and $\tilde{\mu}^* A > 0$, where $\tilde{\mu}$ is Lebesgue measure on $\mathbf{R}$. Later, Solovay showed that if there is a probability space $(Y, \mathscr{P}(Y), \lambda)$ with Maharam type greater than ω, then there is a set $A \subseteq \mathbf{R}$ such that $\#(A) = \omega_1$ and $\tilde{\mu}^* A > 0$. The results of [6] show that the existence of any atomlessly-measurable cardinal is enough for Solovay's argument to work. Because neither Kunen's nor Solovay's ideas are readily accessible in print (so far as I am aware), I give a proof of a lemma which essentially covers Solovay's argument, in a form due to K. Prikry, and may be of independent interest. I repeat that this is not original.

8. Lemma. *If* $\varkappa$ *is an atomlessly-measurable cardinal, then for every cardinal* $\varkappa' < \varkappa$ *there is a set* $A \subseteq [0, 1]$ *such that* $\#(A) = \varkappa'$ *and* $\tilde{\mu}^* B > 0$ *for every uncountable* $B \subseteq A$.

Proof. Let λ be an atomless $\varkappa$-additive probability defined on $\mathscr{P}(\varkappa)$. Theorem 2.6 of [6] shows that the Maharam type of $(C, \mathscr{P}(C), \lambda_C)$ is at least $\varkappa^+$ for every $C \subseteq \varkappa$ with $\lambda(C) > 0$; so from 3.13 (a) and 2.21 of [5] we see that there is a function $f\colon \varkappa \to [0, 1]^{\varkappa^+}$ which is inverse-measure-preserving for λ and the usual measure of $[0, 1]^{\varkappa^+}$. For $\xi < \varkappa$, set

$$A_\xi = \{f(\xi)(\eta)\colon \eta < \varkappa'\} \subseteq [0, 1].$$

Suppose, if possible, that for every $\xi < \varkappa$ there is a set $J_\xi \subseteq \varkappa'$ such that $\#(J_\xi) = \omega_1$ but $E_\xi = f(\xi)[J_\xi]$ is Lebesgue negligible. Fix an enumeration $\langle U_m \rangle_{m \in \mathbf{N}}$ of a countable base for the topology of $[0, 1]$, and for each $\xi < \varkappa$, $n \in \mathbf{N}$ choose a relatively open set $G_{n\xi} \subseteq [0, 1]$ such that $E_\xi \subseteq G_{n\xi}$ and $\tilde{\mu}(G_{n\xi}) \leqslant 2^{-n}$. For $m, n \in \mathbf{N}$ set

$$D_{nm} = \{\xi\colon U_m \subseteq G_{n\xi}\}.$$

For each $\alpha < \varkappa^+$, set $f_\alpha(\xi) = f(\xi)(\alpha)$ for $\xi < \varkappa$; then the real variables f_α are all stochastically independent. Consequently, there is for each $\xi < \varkappa$ an $\alpha(\xi) \in J_\xi$ such that $f_{\alpha(\xi)}$ is stochastically independent from the countable family $\{D_{nm}\colon n, m \in \mathbf{N}\} \subseteq \mathscr{P}(\varkappa)$. Because $\varkappa' < \varkappa$ and λ is $\varkappa$-additive, there is a $\gamma < \varkappa'$ such that $B = \{\xi\colon \alpha(\xi) = \gamma\}$ has $\lambda(B) > 0$. Take $n \in \mathbf{N}$ such that $\lambda(B) > 2^{-n}$, and examine

$$C = \bigcup_{m \in \mathbf{N}} (D_{nm} \cap f_\gamma^{-1}[U_m]).$$

Because f_γ is independent from all the D_{nm}, and is inverse-measure-preserving for λ and $\tilde{\mu}$, $\lambda C = (\lambda \times \tilde{\mu})(C')$ where

$$C' = \bigcup_{m \in \mathbf{N}} (D_{nm} \times U_m) \subseteq \varkappa \times [0, 1].$$

But, for each $\xi < \varkappa$, the vertical section $C'[\{\xi\}]$ is just $G_{n\xi}$, so

$$(\lambda \times \tilde{\mu})(C') = \int \tilde{\mu}(G_{n\xi}) \lambda(d\xi) \leqslant 2^{-n}.$$

There must therefore be a $\xi \in B \setminus C$. But in this case $f_\gamma(\xi) \in E_\xi$, because $\gamma = \alpha(\xi) \in J_\xi$, while $f_\gamma(\xi) \notin G_{n\xi}$, because there is no m such that $f_\gamma(\xi) \in U_m \subseteq G_{n\xi}$; contrary to the choice of $G_{n\xi}$.

So take some $\xi < \varkappa$ such that $\tilde{\mu}^*(f(\xi)[J]) > 0$ for every uncountable $J \subseteq \varkappa'$. Evidently $f(\xi) \upharpoonright \varkappa'$ is countable-to-one, so A_ξ must have cardinal $\varkappa'$ (passing over the trivial case of countable $\varkappa'$), and will serve for A.

Remark. I do not know whether, under the hypothesis of this lemma, there is always a set $A \subseteq \mathbf{R}$ with $\#(A) = \varkappa$ and no uncountable subset of A Lebesgue negligible.

9. Proposition. *If there is an atomlessly-measurable cardinal, and* (X, Σ, μ) *is an atomless, perfect,* σ*-finite measure space with* $\mu(X) > 0$*, then* μ *is not* ω_2*-additive.*

Proof. (Recall that a σ-finite measure space (X, Σ, μ) is called *perfect* if for every measurable function $f\colon X \to \mathbf{R}$ there is a Borel subset H of $\mathbf{R}$ such that $H \subseteq f[X]$ and $\mu(X \setminus f^{-1}[H]) = 0$; see [10], Lemma 2.) Because (X, Σ, μ) is atomless and σ-finite, there is a function $f\colon X \to [0, \mu X[$ which is inverse-measure-preserving for μ and $\tilde{\mu}_{[0, \mu X[}$. By Lemma 8, with $\varkappa = \omega_1$, there is a set $A \subseteq [0, \mu X[$ with $\#(A) = \omega_1$ and $\tilde{\mu}^* A > 0$; in which case $\mathscr{E} = \{f^{-1}[\{a\}]\colon a \in A\}$ is a disjoint family in Σ with $\#(\mathscr{E}) = \omega_1$ but with $\sum_{E \in \mathscr{E}} \mu E = 0 < \tilde{\mu}^* A = \mu^*(\bigcup \mathscr{E})$ ([3], Lemma 1E), so that μ cannot be ω_2-additive.

10. Theorem. *Suppose, in* §1, *that* (X, Σ, μ) *is an atomless complete perfect* σ*-finite measure space. Then* $\nu = \mu$.

Proof. Suppose, if possible, otherwise. Because (X, Σ, μ) is complete and σ-finite, (b-i) and (b-ii) of Theorem 2 are false; take D, D' and E from (b-iii). Then μ_D and $\mu_{D'}$ are atomless, totally finite, non-zero measures with domains $\mathscr{P}(D)$, $\mathscr{P}(D')$ respectively, so their additivities $\varkappa$, $\varkappa'$ are atomlessly-measurable cardinals; suppose that $\varkappa \leqslant \varkappa'$. In this case μ_E must be $\varkappa$-additive, because it is complete and

$$\{A\colon \mu_E(A) = 0\} = \{A\colon A \subseteq E,\ \mu_D(A \cap D) = \mu_{D'}(A \cap D') = 0\}.$$

But (E, Σ_E, μ_E) is a perfect measure space, and $\varkappa \geqslant \omega_2$ ([7], §27), so this contradicts Proposition 9 above.

11. THEOREM. *Suppose, in* §1, *that there is a topology* $\mathfrak{T}$ *on* X *such that* $(X, \mathfrak{T}, \Sigma, \mu)$ *is a Radon measure space. Then* $\nu = \mu$.

Proof. (For the general theory of Radon measure spaces, see [2].) The argument follows that of Theorem 10. Because I take Radon measure spaces to be complete and locally determined, (b-i) and (b-ii) are both disallowed. If we take D, D' and E from (b-iii), we can be sure that μ_E is atomless (because in a Radon measure space every atom is concentrated at a point), while also (E, Σ_E, μ_E) is perfect ([10], Theorem 10); so we reach the same contradiction as before.

12. It is perhaps worth remarking here that a plausible route to Theorem 11 is blocked.

PROPOSITION. *If it is relatively consistent with* ZFC *to suppose that there is a two-valued-measurable cardinal, then it is relatively consistent to suppose that there is a Radon measure space* $(X, \mathfrak{T}, \Sigma, \mu)$ *with a set* $D \subseteq X$ *such that* $\mu^* D > 0$, $\mu\{x\} = 0$ *for every* $x \in D$ and $\Sigma_D = \mathscr{P}(D)$.

Proof. (a) The first step is to show that we can have an atomlessly-measurable cardinal $\varkappa$, with a $\varkappa$-additive probability λ defined on $\mathscr{P}(\varkappa)$, such that the Maharam type of $(\varkappa, \mathscr{P}(\varkappa), \lambda)$ is $2^\varkappa$. For Solovay's theorem ([12], or [7], §34) shows that if $\varkappa$ is two-valued-measurable and we add $\varkappa' = 2^\varkappa$ random reals, then $\varkappa$ becomes atomlessly-measurable, with a $\varkappa$-additive probability λ on $\mathscr{P}(\varkappa)$ for which there is a family $\langle F_\xi\rangle_{\xi<\varkappa'}$ of stochastically independent subsets of $\varkappa$, all of λ-measure $\frac{1}{2}$. Also, of course, $2^\varkappa$ is now $\mathfrak{c}$.

(b) This shows in fact that the Maharam type of $(A, \mathscr{P}(A), \lambda_A)$ is at least $2^\varkappa = \mathfrak{c}$ for every non-negligible set $A \subseteq \varkappa$; but since the Maharam type of $(\varkappa, \mathscr{P}(\varkappa), \lambda)$ is surely no greater than $2^\varkappa$, $\mathfrak{A}(\lambda)$ must be homogeneous and isomorphic to the measure algebra of the usual Radon measure μ on $X = \{0, 1\}^{\mathfrak{c}}$. Consequently there is a stochastically independent family $\langle E_\xi\rangle_{\xi<\mathfrak{c}}$ of sets of λ-measure $\frac{1}{2}$ which generates the whole algebra $\mathfrak{A}(\lambda)$. Let $\langle B_\xi\rangle_{\xi<\mathfrak{c}}$ enumerate $\mathcal{N}_\lambda$. Define $f\colon \varkappa \to X$ by setting

$$f(\alpha)(\xi) = \begin{cases} 1 & \text{if } \alpha \in B_\xi \cup E_\xi, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is inverse-measure-preserving for λ and μ ([5], Prop. 1.18). Set $D = f[\varkappa]$. Then $1 \geqslant \mu^* D \geqslant \lambda\varkappa = 1$, and $\mu\{x\} = 0$ for every $x \in D$. If A is any subset of D, there is an $F \in \Sigma = \mathrm{dom}(\mu)$ such that $f^{-1}[A]^\bullet = f^{-1}[F]^\bullet$ in $\mathfrak{A}(\lambda)$, i.e. $B = f^{-1}[A] \triangle f^{-1}[F] \in \mathcal{N}_\lambda$. Take any infinite subset I of $\mathfrak{c}$ such that $B \subseteq B_\xi$ for every $\xi \in I$. Then

$$N = \{x\colon x \in X,\ x(\xi) = 1\ \forall \xi \in I\}$$

is μ-negligible and

$$N \supseteq f[B] = f[f^{-1}[A \triangle F]] = D \cap (A \triangle F) = A \triangle (D \cap F).$$

So $A \triangle (D \cap F) \in \Sigma$ and $A \in \Sigma_D$, as required.

Remark. $(D, \mathfrak{T}_D, \mathscr{P}(D), \mu_D)$ is now an atomless quasi-Radon probability space, if $\mathfrak{T}_D$ is the topology on D induced by that of X; this clears up a question left open in [4].

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