A note on continuous linear mappings between function spaces

by

G. D. Spiliopoulos (Athens)

Abstract. Let $0: C_0(X) \to C_0(Y)$ be a linear continuous function. If $0$ is an order preserving homeomorphism (an isometry from $C_0^c(X)$ to $C_0^c(Y)$), then the Tikhonov spaces $X$ and $Y$ are homeomorphic. This generalizes the well known theorem of Nagata that if $C_0(X)$ and $C_0(Y)$ are topologically isomorphic than $X$ and $Y$ are homeomorphic. If $0$ is $1$-1 and $Y$ has caliber $(\tau, \lambda)$ (resp. is pseudocompact) then $X$ has caliber $(\tau, \lambda)$ (resp. is pseudocompact), proving in this way that if $L_0(X)$ has caliber $(\tau, \lambda)$ then so does $X$. Related results for $L$-embedded spaces are obtained.

Introduction. Everywhere below, $X, Y$ and $Z$ stand for infinite Tikhonov topological spaces, $C_0(X)$ is the space of all continuous real-valued functions on $X$ endowed with the topology of pointwise convergence, $C_0^c(X) = \{ f \in C_0(X) : \text{f bounded} \}$ and $C_0^c(Y) = \{ f \in C_0(Y) : \text{f bounded} \}$ endowed with the topology of uniform convergence. It is clear that the family of sets $V(x; G) = \{ f \in C_0(X) : f(x) \in G \}$ where $G$ is open in $R$, is an open subspace of $C_0(X)$.

A space $X$ has caliber $(\tau, \lambda)$, where $\tau, \lambda$ are infinite cardinals, if for every family $y$ of non-empty open subsets of $X$ such that $|y| = \tau$, there exists a subfamily $y' \subset y$ with $|y'| = \lambda$ and $|y| = \lambda$.

We denote by $N_\tau$ the family of open basic neighbourhoods of $x$ and by $L_0(X)$ the unit function on $C_0(X)$. For $A \subset X$ and $f \in C_0(X)$, we write $f[A]$ for the restriction of $f$ on $A$, $\text{supp} f = \{ x \in X : f(x) \neq 0 \}$ for the support of $f$, and $\bar{A}$ for the closure of $A$ in $X$.

Let $e: X \to C_0(\bar{C}_0(X))$ such that $e(x) = \delta_x$, where $\delta_x(f) = f(x)$ for $f \in C_0(X)$. The set of all finite linear combinations $s = a_1 \delta_{x_1} + \ldots + a_n \delta_{x_n}$ is denoted by $L_0(X)$. It is known (16) that $L_0(X)$ is the dual space of $C_0(X)$, while $C_0(X)$ is the dual space of $L_0(X)$. If $\theta$ is a continuous linear function from $C_0(X)$ to $C_0(Y)$, then the induced function $\theta^* \in L_0(Y)$ is defined by $\theta^*(y) = \theta(\bar{y})$ for $\bar{y} \in L_0(Y)$ and is also linear and continuous. For $y \in Y$, $\theta^*(y)(f) = \theta(f(y))$ for every $f \in L_0(X)$. Suppose that for $y \in Y$, $\theta^*(y) = a_1 \delta_{x_1} + \ldots + a_n \delta_{x_n}$, where $a_1, \ldots, a_n \neq 0$. The determining set of $y$ in $X$ with
respect to $\theta$ is defined to be the set $D(y, \theta) = \{x_1, \ldots, x_n\}$. When $\theta$ is clear from the context, we write $D(y)$ instead of $D(y, \theta)$. Obviously, for $f \in C_\mathcal{P}(X)$, if $f(x) = 0$ for every $x \in D(y)$, then $\theta(f)(y) = 0$. If $f(x) = 0$ for every $x \in D(y)$, then $\theta(f)(y) = 0$. Thus, if $f \in D(y)$ and $\theta(f)(y) = 0$ for $f \notin C_\mathcal{P}(X)$.

We claim that for $x \in X$, we have $D(y)$ if and only if for every $g \in \mathcal{P}$, there is an $f \in C_\mathcal{P}(X)$ such that $f \leq g$ and $\theta(f)(y) = 0$. Indeed, let $x \notin D(y)$. For $g \in \mathcal{P}$, such that $g \leq x$, and $f \in C_\mathcal{P}(X)$ such that $supp f \leq g$ and $\theta(f)(y) = 0$. Conversely, for $x \in D(y)$, any function $f \in C_\mathcal{P}(X)$ such that $supp f \leq g$, and $f \in D(y)$, satisfies $\theta(f)(y) = 0$. But this implies that $D(y)$ is in the case the set $H(y, \theta)$ defined in [2].

**Lemma 1.** Let $\theta: C(X) \to C(Y)$ be a linear continuous function and $x \in X$. Then the following assertions are true for $y \in Y$.

(a) Let $x_0 \in D(y)$ and $f, g \in C_\mathcal{P}(X)$ such that $f \mid D(y) \cap \{x_0\} = g \mid D(y) \cap \{x_0\}$. Then $f(x_0) - g(x_0)$ if and only if $\theta(f)(y) = \theta(g)(y)$.

(b) If $y \in A$, then $D(y) \subseteq \bigcup \{D(y): y \in A\}$.

(c) If $\theta$ is a homomorphism, then $D(y) \subseteq \{y\}$ if and only if $D(y) = \{x_0\}$.

**Proof.** (a) Immediate consequence of the definition of $D(y)$.

(b) Let $x \in D(y) \cap \bigcup \{D(y): y \in A\}$. Consider an $f \in C_\mathcal{P}(X)$ such that $f(x) = 1$ and $f(\bigcup \{D(y): y \in A\}) = 0$. Then $\theta(f)(y) = 0$ and $\theta(f)(A) = 0$, contradicting the hypothesis.

(c) Let $D(y) = \{x_0\}$. If $x_0 \notin D(y)$, there exists an $f \in C_\mathcal{P}(X)$ such that $f \mid D(y) = 0$ and $f(x_0) = 1$. Hence, $\theta(f)(y) = 0$, implying that $f \mid D(y)$ is a contradiction. Let $f_0 \in C_\mathcal{P}(X)$ such that $supp f_0 \subseteq C_\mathcal{P}(X)$ and $\theta(f_0)(y) = 0$. Then $f_0(x_0) = 0$. Let also $f_0 \in C_\mathcal{P}(X)$ such that $supp f_0 \subseteq C_\mathcal{P}(X)$ and $\theta(f_0)(y) = 0$. For a suitable $f \in C_\mathcal{P}(X)$, we obtain $\theta(f)(y) = 0$. Thus $\theta(f_0)(y) = 0$. Hence, $\{f_0, f_0 \mid D(y) = \{x_0\} \neq 0$, a contradiction.

**Proposition 1.** Let $\theta: C(X) \to C(Y)$ be a linear homomorphism such that $\theta(f)(y) = 0$, if and only if $f \in C_\mathcal{P}(X)$. Then the spaces $X$ and $Y$ are homomorphic.

**Proof.** Let $x_0 \in X$. Suppose that $D(x_0) = \{y_1, \ldots, y_n\}$, $n \geq 2$. According to Lemma 1(c), $D(y) \cap \{x_0\} = \emptyset$ for every $i = 1, \ldots, n$. Let $f = \bigcup \{f_i \mid D(y) \subseteq \{x_0\}\}$. For every $x \in X$, let $G_x \subseteq \mathcal{P}$ satisfying the conditions: $x_0 \notin G_x$ and $G_x \cap G_x = \emptyset$ if $x \neq x$. Let $f \in C_\mathcal{P}(X)$ such that $f(x) = 0$, $supp f \subseteq G_x$ and $f(x) = 1$ for every $x \in X$. Then it is immediate from Lemma 1(a) and the hypothesis that $\theta(f)(y) = 0$, if $x \notin D(y)$. Thus, the continuous function $f = \sum_i f_i$ satisfies $\theta(f)(y) = 0$, for every $i = 1, \ldots, n$.

We now select, for every $i = 1, \ldots, n$, $V_i \subseteq \mathcal{P}$ such that $V_i \cap Y = \emptyset$ if $i \neq i$, and $V_i \in C_\mathcal{P}(Y)$ such that $V_i \ni \theta(y) = \theta(y)(y) = 0$. Obviously, $\theta^{-1}(y)(x_0) > 0$ in view of Lemma 1(a). Hence, the function $h \in \mathcal{P}$, such that $h(x_0) = \sum_i h_i$ satisfies $h(x_0) = \theta(f)(y)$, for every $y \in Y$. However, we have $\theta^{-1}(h)(x_0) = \sum_i \theta^{-1}(h_i)(x_0) > 0$, a contradiction. Thus, $D(y) = \{y_0\}$ and according to Lemma 1(c), $D(y) = \{y_0\}$. We may now define a function $t: Y \to X$ such that $t(y) = x$ if $D(y) = \{x\}$. Lemma 1 implies that $t$ is a homeomorphism.

A linear function $\theta: C(X) \to C(Y)$ is a lattice homomorphism, provided that $\theta(max (f, g)) = max \{\theta(f), \theta(g)\}$. Obviously, $g \geq 0$ then implies that $\theta(g) \geq 0$.

**Corollary 1.** Let $\theta: C(X) \to C(Y)$ be a lattice homomorphism and a homomorphism. Then $X, Y$ are homomorphic.

**Remark.** It is easily observed that all the previous statements are valid for $\theta: C(X) \to C(Y)$.

It is known (the Banach-Stone theorem) that the compact spaces $X$ and $Y$ are homeomorphic if and only if there is an isometry $T$ from $C(X)$ to $C(Y)$. One can easily observe, from the proof of this theorem (see [6]), that $T$ implies the existence of a homeomorphism from $C(X)$ to $C(Y)$. The combination of these two properties gives an analogous result for the non-compact case.

**Corollary 1.2.** Let $\theta: C(X) \to C(Y)$ be an isometry. Suppose that $\theta$ is also a linear homomorphism from $C(X)$ onto $C(Y)$ and $\theta(1) = 1$. Then $X, Y$ are homeomorphic.

**Proof.** Let $\theta(f) \in C_\mathcal{P}(Y)$ such that $\theta(f)(y) < 0$ for some $y \in Y$. Then $1 + \frac{\theta(f)}{1 + \theta(f)} = 1 \Rightarrow 1 + \frac{\theta(f)}{1 + \theta(f)} = 1 \Rightarrow \theta(f)^2 = \theta(f)$ and $\theta(f) = 0$. Thus $f$ also takes negative values. Therefore, since $\theta$ maps nonnegative functions to nonnegative functions in both directions—according to Proposition 1—the spaces $X, Y$ are homeomorphic.

**Corollary 1.3 (Nagata [7]).** If $\theta: C(X) \to C(Y)$ is a linear multiplicative homomorphism, then $X, Y$ are homeomorphic.

**Proof.** Since every ring isomorphism from $C(X)$ to $C(Y)$ sends nonnegative functions to nonnegative functions ($f = h^2$ implies $\theta(f) = \theta(h^2)$), this is immediate from Proposition 1.

The following proposition has also been proved in Corollaries 1.29 and 1.21 of [4].

**Proposition 2.** Let $\theta$ be a linear, continuous, 1-1 function from $C(X)$ to $C(Y)$. Then if $Y$ is pseudocompact, so is $X$.

**Proof.** Let $B = \bigcup \{D(y): y \in Y\}$. Obviously $B \neq \emptyset$. If $B \neq X$, there exists an $f$ in $C(X)$ identically zero on $B$ and $f(x) = 1$, for some $x \in X \setminus B$. Hence, $\theta(f)$ is identically zero on $Y$, contradicting the assumption that $\theta$ is linear and 1-1. Thus $B = X$. But, according to Proposition 2(a) of [2], $B$ is bounded (the restriction to $B$ of every $f \in C(X)$ is bounded), since $\theta$ is linear. Therefore, $X$ contains a dense bounded subset and is pseudocompact (see [X]).
PROPOSITION 3. Let $\theta$: $C_\lambda(X) \rightarrow C_{\lambda'}(Y)$ be continuous, linear and 1-1, and let $\tau$, $\lambda$ be infinite cardinals such that $\tau \geq \lambda > \omega$. Then if $Y$ has caliber $(\tau, \lambda)$, so does $X$.

Proof. Let $G$ be an open subset of $X$. Since $\bigcup \{D(y): y \in Y\} = X$ the set $V = \{y \in Y: D(y) \cap G \neq \emptyset\}$ is non-empty. We will prove that it is also open. Let $F = Y \setminus V$ and $y \in F$. Then $D(y) \cap G = \emptyset$. For every $x_k$, $k = 1, \ldots, n$, we can find an $f_k \in C_\tau(X)$ such that $\supp f_k \subset G$ and $\theta(f_k(y)) > 0$. However, $\theta(\sum_{k=1}^{n} f_k(y)) = \sum_{k=1}^{n} \theta(f_k(y)) > 0$ and $\theta(\sum_{k=1}^{n} f_k(y)) > 0$, since $f_k \in D(z) = 0$ for every $z \in F$; a contradiction.

Now let $\{G_i: i < \tau\}$ be a family of open sets in $X$. Then there exists a family $\{V_i: i < \tau\}$ of open sets in $X$ such that $V_i = \{y \in Y: D(y) \cap G_i \neq \emptyset\}$ for every $i < \tau$. However, $Y$ has caliber $(\tau, \lambda)$. Thus, there is an $A \subset \tau$, $|A| = \lambda$, such that $W = \bigcap \{V_j: j \in A\} \neq \emptyset$. Let $x_0 \in W$. Then $D(y_0) \cap G_j \neq \emptyset$, for every $j \in A$. Since $D(y_0)$ is finite, there is an $x_0 \in D(y_0)$ such that $x_0 \in \bigcap \{G_j: p \in A\}$, where $A \subset \tau$, $|A| = \lambda$. Thus, $X$ has caliber $(\tau, \lambda)$. ■

It has been proved in [9] that $X$ has caliber $(\tau, \lambda)$ if and only if $\tau$ is a caliber of $L_\mu(X)$. Proposition 3 gives a sufficient condition for $X$ to have caliber $(\tau, \lambda)$.

COROLLARY 3.1. If $L_\mu(X)$ has caliber $(\tau, \lambda)$, where $\tau \geq \lambda > \omega$, so does $X$. ■

A space $Y \subset X$ is said to be $L$-embedded in $X$ if there is a linear and continuous function $\theta$ from $C_\tau(Y)$ to $C_\lambda(X)$ such that $\theta(f) \upharpoonright Y = f$ for every $f \in C_\tau(Y)$. The following proposition is known (see [3]) for regular cardinals.

PROPOSITION 4. Let $Y$ be $L$-embedded in $X$ and let $\tau$, $\lambda$ be infinite cardinals. Then if $X$ has caliber $(\tau, \lambda)$, so does $Y$. ■

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References


DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ATHENS
Pantepontos, 121 84 Athens, Greece

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