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A note on continuous linear mappings between function spaces

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Abstract. Let θ: $C_p(X)\to C_p(Y)$ be a linear continuous function. If θ is an order preserving homeomorphism (an isometry from $C_u^*(X)$ to $C_u^*(Y)$), then the Tikhonov spaces X and Y are homeomorphic. This generalizes the well known theorem of Nagata that if $C_p(X)$ and $C_p(Y)$ are topologically isomorphic then X and Y are homeomorphic. If θ is 1-1 and Y has caliber (τ, λ) (resp. is pseudocompact) then X has caliber (τ, λ) (resp. is pseudocompact), proving in this way that if $L_p(X)$ has caliber (τ, λ) then so does X. Related results for L-embedded spaces are obtained.

Introduction. Everywhere below X, Y and Z stand for infinite Tikhonov topological spaces, $C_p(X)$ is the space of all continuous real-valued functions on X endowed with the topology of pointwise convergence, $C_p^*(X)=\{f\in C_p(X)\colon f \text{ bounded}\}$ and $C_u^*(X)=\{f\in C(X)\colon f \text{ bounded}\}$ endowed with the topology of uniform convergence. It is clear that the family of sets $V(x;G)=\{f\in C_p(X)\colon f(x)\in G\}$ where G is open in $\mathbf{R}$, is an open subbase of $C_p(X)$.

A space X has *caliber* (τ, λ), where τ, λ are infinite cardinals, if for every family γ of non-empty open subsets of X such that $|\gamma|=\tau$, there exists a subfamily $\gamma_1\subset\gamma$ with $\bigcap\gamma_1\neq\emptyset$ and $|\gamma_1|=\lambda$.

We denote by N_x the family of open basic neighbourhoods of x and by 1_X the unit function on $C_p(X)$. For $A\subset X$ and $f\in C_p(X)$, we write $f|A$ for the restriction of f on A, $\operatorname{supp} f=\{x\in X\colon f(x)\neq 0\}$ for the support of f, and $\overline{A}$ for the closure of A in X.

Let e: $X\to C_p(C_p(X))$ such that $e(x)=\hat{x}$, where $\hat{x}(f)=f(x)$ for f in $C_p(X)$. The set of all finite linear combinations $z=a_1\hat{x}_1+\ldots+a_n\hat{x}_n$ is denoted by $L_p(X)$. It is known ([1]) that $L_p(X)$ is the dual space of $C_p(X)$, while $C_p(X)$ is the dual space of $L_p(X)$. If θ is a continuous linear function from $C_p(X)$ to $C_p(Y)$, then the induced function θ^* from $L_p(Y)$ to $L_p(X)$ defined by $\theta^*(y)=y\circ\theta$ for $y\in L_p(Y)$ is also linear and continuous. For $y\in Y$, $\theta^*(y)(f)=\theta(f)(y)$ for every f in $C_p(X)$. Suppose that for $y\in Y$, $\theta^*(y)=a_1\hat{x}_1+\ldots+a_n\hat{x}_n$, where $a_1,\ldots,a_n\neq 0$. The *determining set* of y in X with

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respect to θ is defined to be the set $D(y, \theta) = \{x_1, \ldots, x_n\}$. When θ is clear from the context, we write $D(y)$ instead of $D(y, \theta)$. Obviously, for $f \in C_p(X)$, if $f(x_i) = 0$ for every $x_i \in D(y) = \{x_1, \ldots, x_n\}$, then $\theta(f)(y) = \theta^*(y)(f) = (a_1 \hat{x}_1 + \ldots + a_n \hat{x}_n)(f) = a_1 f(x_1) + \ldots + a_n f(x_n) = 0$. Thus, if $f|D(y) = g|D(y)$ then $\theta(f)(y) = \theta(g)(y)$ for $f, g \in C_p(X)$.

We claim that for $x \in X$, $x \in D(y)$ if and only if for every $G \in N_x$, there is an $f \in C_p(X)$ such that $\operatorname{supp} f \subset G$ and $\theta(f)(y) \neq 0$. Indeed, let $x \notin D(y)$. For $G \in N_x$ such that $G \cap D(y) = \emptyset$, if $\operatorname{supp} f \subset G$ then $\theta(f)(y) = 0$. Conversely, for $x \in D(y)$, any function $f \in C_p(X)$ such that $\operatorname{supp} f \subset G$, $\operatorname{supp} f \cap (D(y) \setminus \{x\}) = \emptyset$ and $f(x) \neq 0$, satisfies $\theta(f)(y) \neq 0$. But this implies that $D(y, \theta)$ is in this case the set $H(y, \theta)$ defined in [2]. ■

LEMMA 1. *Let θ: $C_p(X) \to C_p(Y)$ be a linear continuous function and $x_0 \in X$. Then the following assertions are true for $y_0 \in Y$.*

(a) *Let $x_0 \in D(y_0)$ and $f, g \in C_p(X)$ such that $f|D(y_0) \setminus \{x_0\} = g|D(y_0) \setminus \{x_0\}$. Then $f(x_0) = g(x_0)$ if and only if $\theta(f)(y_0) = \theta(g)(y_0)$.*

(b) *If $y_0 \in \bar{A}$ where $A \subset Y$, then $D(y_0) \subset \overline{\bigcup \{D(y)\colon y \in A\}}$.*

(c) *If θ is a homeomorphism, then $D(x_0) = \{y_0\}$ if and only if $D(y_0) = \{x_0\}$.*

Proof. (a) Immediate consequence of the definition of $D(y_0)$.

(b) Let $x \in D(y_0) \setminus \overline{\bigcup \{D(y)\colon y \in A\}}$. Consider an $f \in C_p(X)$ such that $f(x) = 1$ and $f|\bigcup \{D(y)\colon y \in A\} \cup (D(y_0) \setminus \{x\}) = 0$. Then $\theta(f)(y_0) \neq 0$ and $\theta(f)|A = 0$, contradicting the hypothesis.

(c) Let $D(x_0) = \{y_0\}$. If $x_0 \notin D(y_0)$, there exists an $f \in C_p(X)$ such that $f|D(y_0) = 0$ and $f(x_0) = 1$. Hence, $\theta(f)(y_0) = 0$, implying that $f(x_0) = 0$, a contradiction. Let $D(y_0) = \{x_0, x_1, \ldots, x_n\}$, $n \geq 1$. We now select disjoint open sets $V_0 \in N_{x_0}$ and $V_1 \in N_{x_1}$ such that $(V_0 \cup V_1) \cap \{x_2, \ldots, x_n\} = \emptyset$. Let $f_0 \in C_p(X)$ such that $\operatorname{supp} f_0 \subset V_0$ and $\theta(f_0)(y_0) = p \neq 0$. Then $f_0(x_0) \neq 0$. Let also $f_1 \in C_p(X)$ such that $\operatorname{supp} f_1 \subset V_1$ and $\theta(f_1)(y_0) \neq 0$. For a suitable $a \in \mathbf{R}$ we obtain $\theta(af_1)(y_0) = -p$. Thus $\theta(f_0 + af_1)(y_0) = 0$. Hence, $(f_0 + af_1)(x_0) = 0$. But $(f_0 + af_1)(x_0) = f_0(x_0) \neq 0$, a contradiction. Consequently, $D(y_0) = \{x_0\}$. ■

PROPOSITION 1. *Let θ: $C_p(X) \to C_p(Y)$ be a linear homeomorphism such that $\theta(f) \geq 0$ if and only if $f \geq 0$, for every $f \in C_p(X)$. Then the spaces X and Y are homeomorphic.*

Proof. Let $x_0 \in X$. Suppose that $D(x_0) = \{y_1, \ldots, y_n\}$, $n \geq 2$. According to Lemma 1(c), $D(y_i) \setminus \{x_0\} \neq \emptyset$ for every $i = 1, \ldots, n$. Let $F = \bigcup_{i=1}^{n} D(y_i) \setminus \{x_0\}$. For every $x \in F$, let $G_x \in N_x$ satisfying the conditions: $x_0 \notin G_x$ and $G_x \cap G_{x'} = \emptyset$ if $x \neq x'$. Let $f_x \in C_p(X)$ such that $f_x \geq 0$, $\operatorname{supp} f_x \subset G_x$ and $f_x(x) = 1$ for every $x \in F$. Then it is immediate from Lemma 1(a) and the hypothesis that $\theta(f_x)(y_i) > 0$ if $x \in D(y_i)$. Thus, the continuous function $f = \sum_{x \in F} f_x$ satisfies $\theta(f)(y_i) > 0$, for every $i = 1, \ldots, n$.

We now select, for every $i = 1, \ldots, n$, $V_i \in N_{y_i}$ such that $V_i \cap V_j = \emptyset$ if $i \neq j$, and $h_i \in C_p(Y)$ such that $h_i \geq 0$, $\operatorname{supp} h_i \subset V_i$ and $h_i(y_i) = \theta(f)(y_i) > 0$. Obviously, $\theta^{-1}(h_i)(x_0) > 0$ in view of Lemma 1(a). Hence, the function $h = \sum_{i=1}^{n} h_i$ satisfies $h|D(x_0) = \theta(f)|D(x_0)$. Thus, $\theta^{-1}(h)(x_0) = f(x_0) = 0$. However, we have $\theta^{-1}(h)(x_0) = \sum_{i=1}^{n} \theta^{-1}(h_i)(x_0) > 0$, a contradiction. Thus, $D(x_0) = \{y_0\}$ and according to Lemma 1(c), $D(y_0) = \{x_0\}$.

We may now define a function t: $Y \to X$ such that $t(y) = x$ if $D(y) = \{x\}$. Lemma 1 implies that t is a homeomorphism. ■

A linear function θ: $C_p(X) \to C_p(Y)$ is a lattice homomorphism, provided that $\theta(\max\{f, g\}) = \max\{\theta(f), \theta(g)\}$. Obviously, $g \geq 0$ then implies that $\theta(g) \geq 0$.

COROLLARY 1.1. *Let θ: $C_p(X) \to C_p(Y)$ be a lattice homomorphism and a homeomorphism. Then X, Y are homeomorphic.* ■

Remark. It is easily observed that all the previous statements are valid for θ: $C_p^*(X) \to C_p^*(Y)$. ■

It is known (the Banach–Stone theorem) that the compact spaces X and Y are homeomorphic if and only if there is an isometry T from $C_u(X)$ to $C_u(Y)$. One can easily observe, from the proof of this theorem (see [6]), that T implies the existence of a homeomorphism from $C_p(X)$ to $C_p(Y)$. The combination of these two properties gives an analogous result for the non-compact case.

COROLLARY 1.2. *Let θ: $C_u^*(X) \to C_u^*(Y)$ be an isometry. Suppose that θ is also a linear homeomorphism from $C_p^*(X)$ onto $C_p^*(Y)$ and $\theta(1_X) = 1_Y$. Then X, Y are homeomorphic.*

Proof. Let $\theta(f) \in C_p^*(Y)$ such that $\theta(f)(y) < 0$ for some $y \in Y$. Then

$$\left\|1_Y - \frac{\theta(f)}{\|\theta(f)\|}\right\| > 1 \Leftrightarrow \left\|1_Y - \frac{\theta(f)}{\|f\|}\right\| > 1 \Leftrightarrow \left\|\theta(1_X) - \frac{\theta(f)}{\|f\|}\right\| > 1 \Leftrightarrow \left\|1_X - \frac{f}{\|f\|}\right\| > 1.$$

Thus f also takes negative values. Therefore, since θ maps nonnegative functions to nonnegative functions in both directions — according to Proposition 1 — the spaces X, Y are homeomorphic. ■

COROLLARY 1.3 (Nagata [7]). *If θ: $C_p(X) \to C_p(Y)$ is a linear multiplicative homeomorphism, then X, Y are homeomorphic.*

Proof. Since every ring isomorphism from $C_p(X)$ to $C_p(Y)$ sends nonnegative functions to nonnegative functions ($f = h^2$ implies $\theta(f) = (\theta(h))^2$), this is immediate from Proposition 1. ■

The following proposition has also been proved in Corollaries 1.2.9 and 1.2.15 of [4].

PROPOSITION 2. *Let θ be a linear, continuous, 1-1 function from $C_p(X)$ to $C_p(Y)$. Then if Y is pseudocompact, so is X.*

Proof. Let $B = \bigcup \{D(y)\colon y \in Y\}$. Obviously $B \neq \emptyset$. If $\bar{B} \neq X$, there exists an f in $C_p(X)$ identically zero on $\bar{B}$ and $f(x) = 1$, for some $x \in X \setminus \bar{B}$. Hence, $\theta(f)$ is identically zero on Y, contradicting the assumption that θ is linear and 1-1. Thus $\bar{B} = X$. But, according to Proposition 2(a) of [2], B is bounded (the restriction to B of every $f \in C_p(X)$ is bounded), since θ is linear. Therefore, X contains a dense bounded subset and is pseudocompact (see [8]). ■

PROPOSITION 3. *Let θ: $C_p(X) \to C_p(Y)$ be continuous, linear and* 1-1, *and let τ, λ be infinite cardinals such that $\tau \geqslant \lambda \geqslant \omega$. Then if Y has caliber (τ, λ), so does X.*

Proof. Let G be an open subset of X. Since $\overline{\bigcup\{D(y): y \in Y\}} = X$ the set $V = \{y \in Y: D(y) \cap G \neq \emptyset\}$ is non-empty. We will prove that it is also open. Let $F = Y \setminus V$ and $y \in \bar{F} \setminus F$. Then $D(y) \cap G = \{x_1, \ldots, x_n\}$. For every x_k, $k = 1, \ldots, n$, we can find an $f_k \in C_p(X)$ such that $\operatorname{supp} f_k \subset G$ and $\theta(f_k)(y) > 0$. However, $\theta(\sum_{k=1}^n f_k)(y) = \sum_{k=1}^n \theta(f_k)(y) > 0$ and $\theta(\sum_{i=1}^n f_k)|F = 0$, since $f_k|D(z) = 0$ for every $z \in F$; a contradiction.

Now let $\{G_i: i < \tau\}$ be a family of open sets in X. Then there exists a family $\{V_i: i < \tau\}$ of open sets in Y such that $V_i = \{y \in Y: D(y) \cap G_i \neq \emptyset\}$ for every $i < \tau$. However, Y has caliber (τ, λ). Thus, there is an $A \subset \tau$, $|A| = \lambda$, such that $W = \bigcap\{V_j: j \in A\} \neq \emptyset$. Let $y_0 \in W$. Then $D(y_0) \cap G_j \neq \emptyset$, for every $j \in A$. Since $D(y_0)$ is finite, there is an $x_0 \in D(y_0)$ such that $x_0 \in \bigcap\{G_p: p \in A'\}$, where $A' \subset A$, $|A'| = \lambda$. Thus, X has caliber (τ, λ). ■

It has been proved in [9] that X has caliber τ (τ regular) if and only if τ is a caliber of $L_p(X)$. Proposition 3 gives a sufficient condition for X to have caliber (τ, λ).

COROLLARY 3.1. *If $L_p(X)$ has caliber (τ, λ), where $\tau \geqslant \lambda \geqslant \omega$, so does X.* ■

A space $Y \subset X$ is said to be *L-embedded* in X if there is a linear and continuous function θ from $C_p(Y)$ to $C_p(X)$ such that $\theta(f)|Y = f$ for every $f \in C_p(Y)$. The following proposition is known (see [3]) for regular cardinals.

PROPOSITION 4. *Let Y be L-embedded in X and let τ, λ be infinite cardinals. Then if X has caliber (τ, λ), so does Y.* ■

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