On composants of the bucket handle

by

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Abstract. The permutations of composants induced by homeomorphisms of the bucket handle are studied. It appears that the only possible permutations are induced by iterates of the shift homeomorphism. We also show that non-zero composants of the bucket handle can be mapped onto each other by continuous bijections.

1. Indroduction. The principal motivation for the results in this paper is an old question of Knaster and Kuratowski: "Are all composants in the bucket handle which do not contain the zero point homeomorphic?" (the bucket handle and its zero point are defined in Section 2 below). The only step towards an answer to this question is a result of Bellamy, which states in what way the shift map on the bucket handle permutes composants (see [Bel]). Subsequently Dębski proved that this is the best one can get by considering homeomorphisms of the bucket handle. In other words, if a homeomorphism of the bucket handle maps one composant S onto another composant T, then there exists an iterate of the shift which also maps S onto T. One of our main theorems states that Dębski's result also holds for homeomorphisms of locally compact subspaces of the bucket handle (see Theorem (3.7) below for the exact statement).

Another of our objectives is to show that there exist continuous bijections between any two composants which do not contain the zero point. This result can be proved by deleting the zero point from the bucket handle. We show that the complement of the zero point admits transformations which map one composant bijectively onto another composant. By deleting the composant of the zero point one gains even more freedom and we are led to the following conjecture: the bucket handle minus the composant of the zero point is homogeneous.

Section 2 contains the necessary definitions and a short review of the results of [A-M]. In Section 3 we study the homeomorphisms of locally compact subsets of the bucket handle. The approach of [A-M] is generalized to so-called minimal matchbox manifolds

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with the help of a theorem of Gutek. In Section 4 the homeomorphisms of the bucket handle are classified up to isotopy. This appears to be a new result. An interesting corollary of the results in this section is a different proof of Watkins's classification theorem for Knaster continua. Finally, in Section 5 continuous bijections between composants are constructed.

2. Notation and preliminaries. The set of positive integers is denoted by $\mathbf{N}$. The Cantor set $\{0, 1\}^{\mathbf{N}}$ is denoted by $\mathscr{C}$, the open interval $(-1, 1)$ by $\mathscr{J}$ and the interval $[-1, 1]$ by $\mathscr{I}$.

The bucket handle is defined as follows. Let $\mathscr{C}$ be the usual Cantor middle third set in $[0, 1]$, which we think of as a subset of the real axis in the plane. Connect elements of $\mathscr{C}$ by semicircles:

(1) the semicircles in the upper half plane with center $(1/2, 0)$ which connect elements of $\mathscr{C}$,

(2) for all $n \geqslant 1$, the semicircles in the lower half plane with center $(1/(2\cdot 3^n), 0)$ which connect elements of $\mathscr{C}$.

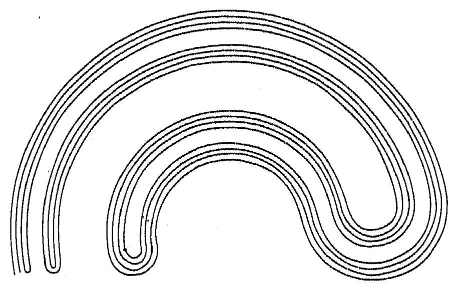

Fig. 1. The bucket handle

The continuum $\mathscr{K}$ which is the union of the semicircles in (1) and (2) is called the *standard Knaster continuum* or *bucket handle*. The element 0 of the Cantor middle third set $\mathscr{C} \subset \mathscr{K}$ is called the *zero point* of $\mathscr{K}$. It is the only element of $\mathscr{K}$ which does not lie in the interior of an open set homeomorphic to $\mathscr{C} \times \mathscr{J}$. Therefore all autohomeomorphisms of $\mathscr{K}$ fix the zero point.

There are various ways to define the bucket handle and we will encounter several throughout this paper. In Section 3 the bucket handle is represented following Gutek, in Section 4 it is defined as an inverse limit, and finally in Section 5 it is described as the attractor of the horseshoe map.

The 2-*solenoid* $\mathscr{S}_2$ is defined as the intersection of a descending sequence of solid tori $\mathscr{S}_2 = \bigcap\{T_n \mid n \in \mathbf{N}\}$. Each T_{n+1} is wrapped twice longitudinally in T_n without folding back (see [Dan]). It is well known that the topology of the 2-solenoid and the bucket handle are intimately connected. This will become apparent in Section 4.

A subset C of a continuum X is a *composant* if for some point p, C is the set of all points x such that x and p are elements of a proper subcontinuum of X. In the case of the bucket handle or the 2-solenoid the only proper subcontinua are arcs. Consequently, the composants are equal to the arc components.

We give a short review of the results and definitions of [A-M].

(2.1) Definition. A separable metric space X is called a *matchbox manifold* if for each $x \in X$ there is a zero-dimensional space S_x such that $S_x \times \mathscr{J}$ is homeomorphic to an open neighborhood of x. In this paper S_x is homeomorphic to the Cantor set for all $x \in X$.

The 2-solenoid and the bucket handle minus the zero point are examples of matchbox manifolds. Indeed, they are the only examples which we will encounter in this paper.

Let X be a matchbox manifold such that S_x is homeomorphic to $\mathscr{C}$ for all $x \in X$. Suppose that V is a closed neighborhood of $y \in X$ such that V is homeomorphic to $\mathscr{C} \times \mathscr{I}$ under a homeomorphism which induces a homeomorphism of the interior V° of V to $\mathscr{C} \times \mathscr{J}$. Then V is called a *matchbox neighborhood* of y or simply a *matchbox*. The arc components of V are called the *matches*. The topological boundary ∂V, corresponding to $\mathscr{C} \times \{-1, 1\}$, is called the set of *end points*. The set of end points is divided into a *top* which corresponds to $\mathscr{C} \times \{1\}$ and a *bottom* which corresponds to $\mathscr{C} \times \{-1\}$. These notions of top and bottom depend upon the homeomorphism between $\mathscr{C} \times \mathscr{I}$ and the matchbox. In general we will make no distinction between V and $\mathscr{C} \times \mathscr{I}$ (see for instance the notation in the lemma below). The following result is ubiquitous in the proofs in this paper, though hardly ever mentioned:

(2.2) Lemma (Lemma of the long box). *Let J be an arc in the matchbox manifold X with initial point x_1 and end point x_2. Suppose that V_1 and V_2 are matchbox neighborhoods of x_1 and x_2 respectively; V_i is homeomorphic to $S_i \times \mathscr{I}$ ($i = 1, 2$). There exists a matchbox V homeomorphic to $S \times \mathscr{I}$ such that:*

(1) *x_1 is an element of $S \times \{-1\} \subset S_1 \times \{-1\}$,*
(2) *x_2 is an element of $S \times \{1\} \subset S_2 \times \{1\}$,*
(3) *$S \times \{-1\}$ is a clopen subset of $S_1 \times \{-1\}$,*
(4) *$S \times \{1\}$ is a clopen subset of $S_2 \times \{1\}$.*

Stated less accurately, the lemma says that there is a long box V with bottom contained in the bottom of V_1 and top contained in the top of V_2.

The following result of Aarts and Martens generalizes a theorem of Keynes and Sears for compact spaces [K-S]. Recall that the suspension of a homeomorphism $h: X \to X$ of a topological space X is defined as the quotient space of $X \times \mathscr{I}$ obtained by the identification of the top and the bottom via h: $\Sigma(X, h) = X \times \mathscr{I}/\{(x, 1) \sim (h(x), -1)\}$.

(2.3) Theorem. *Let φ: $X \times \mathbf{R} \to X$ be a flow without rest points on a one-dimensional separable metric space X. Then there exists a zero-dimensional space S and a homeomorphism f: $S \to S$ such that X is topologically equivalent to the suspension $\Sigma(S, f)$.*

In the next section we shall see that there exists a representation of the bucket handle which is similar to a suspension. It is well known that the 2-solenoid admits a flow without rest points and can be represented as the suspension of a homeomorphism of the Cantor set (see Example (3.4)(a)).

3. Minimal matchbox manifolds. In this section we discuss an analogue of the theorem of Aarts and Martens for certain one-dimensional spaces which we call minimal matchbox manifolds.

(3.1) DEFINITION. Let X be a compact matchbox manifold. X is called a *minimal matchbox manifold* if:

(1) every arc component of X is dense,

(2) all matchboxes are homeomorphic to $\mathscr{C}\times\mathscr{I}$.

In particular, every point of X is two-sided recurrent, i.e. for each x in X the arc components of $X\setminus\{x\}$ are dense.

The important feature of a minimal matchbox manifold is that any one of its matchboxes determines the topological type of the entire space. That is the content of Theorem (3.3) below. Note that the 2-solenoid $\mathscr{S}_2$ is a minimal matchbox manifold and that $\mathscr{K}$ is not a minimal matchbox manifold because of its special point zero. However, $\mathscr{K}-\{0\}$ is a minimal matchbox manifold as all of its arc components are dense. We shall find that all results for minimal matchbox manifolds in this section hold for $\mathscr{K}$, if slightly modified as Example (3.4)(b) shows. This is a consequence of the fact that the results in this section also hold for locally compact spaces. For instance, a careful examination of the proof of Gutek's theorem shows that it remains valid in the case of locally compact spaces. Since we are interested only in the specific case of the bucket handle we do not go into these details any further.

Recall that a homeomorphism of a topological space is *minimal* if the only closed invariant subsets are the empty set and the space itself. In other words, all orbits are dense. In view of Theorem (2.3) a minimal matchbox manifold X admits a flow $\pi\colon X\times\mathbf{R}\to X$ without rest points if and only if it can be represented as the suspension over the Cantor set of a minimal homeomorphism h, i.e., $X\approx\Sigma(\mathscr{C}, h)$. Since X is a minimal matchbox, the elements $x\in X$ are two-sided recurrent. In other words, both $\pi(\{x\}\times[0,\infty))$ and $\pi(\{x\}\times(-\infty,0])$ are dense. This follows from the fact that the limit sets $\omega(x)=\bigcap\{\mathrm{cl}(\pi(\{x\}\times[n,\infty)))\mid n\in\mathbf{N}\}$ and $\alpha(x)=\bigcap\{\mathrm{cl}(\pi(\{x\}\times(-\infty,-n]))\mid n\in\mathbf{N}\}$ are invariant under π. We conclude from the minimality of the matchbox manifold that $\omega(x)$ and $\alpha(x)$ are equal to X.

There exists a correspondence between the autohomeomorphisms of X and the clopen subsets of $\mathscr{C}$. Since elements of X are two-sided recurrent, the *return map* h_A: $A\to A$ is well defined for any clopen subset A of $\mathscr{C}$:

$$h_A(x)=h^{n(x)}(x)\quad\text{with}\quad n(x)=\min\{n>0\mid h^n(x)\in A\}.$$

Note that A is homeomorphic to the Cantor set. The suspension $\Sigma(A, h_A)$ of the return map over A is also homeomorphic to X. Consequently, there are various choices for the homeomorphism h yielding a suspension $\Sigma(\mathscr{C}, h)$ homeomorphic to X. Suppose that the return maps h_A and h_B are conjugate: $\phi\circ h_A=h_B\circ\phi$ for some homeomorphism $\phi\colon A\to B$. In that case the suspensions $\Sigma(A, h_A)$ and $\Sigma(B, h_B)$ are homeomorphic. The homeomorphism $\Phi\colon \Sigma(A, h_A)\to\Sigma(B, h_B)$ is defined as $\Phi(a, t)=(\phi(a), t)$. As is shown in [Aar] the converse also holds, which leads to the following theorem:

(3.2) THEOREM. *Let X and Y be minimal matchbox manifolds represented by $\Sigma(\mathscr{C}, g)$ and $\Sigma(\mathscr{C}, h)$ respectively. Then X and Y are homeomorphic if and only if there exist clopen subsets $A, B\subset\mathscr{C}$ such that the return maps g_A and h_B are conjugate.*

This theorem was used in [A-F] to present an elementary proof of the classification theorem for solenoids (cf. [Bing] and [McC]). Since there are only countably many clopen subsets of $\mathscr{C}$, it is not so difficult to list all the possible return maps. This approach can also be used for minimal matchbox manifolds in general. Indeed, the following theorem of Gutek [Gut] can be used to formulate analogues of Theorems (2.3) and (3.2) for minimal matchbox manifolds.

(3.3) THEOREM. *Let X be a minimal matchbox manifold. There exists a continuous involution $\tau\colon \mathscr{C}\times\{-1,1\}\to\mathscr{C}\times\{-1,1\}$ without fixed points such that X is homeomorphic to $\mathscr{C}\times\mathscr{I}/\tau$, the quotient space of identified end points $x\sim\tau(x)$.*

The involution τ is an analogue of the return map for flows. In general it depends hon the choice of a matchbox in X, just as in the case of a return map, and can be described as follows.

Let $\mathscr{C}\times\mathscr{I}$ be a matchbox in X. Two end points of this matchbox are equivalent if they can be connected via an arc in the complement $X\setminus\mathscr{C}\times\mathscr{I}$. Every equivalence class contains precisely two elements and the involution τ simply switches the elements of each class. Consider the special case that X admits a flow without rest points. Things can be arranged in such a way that τ maps the top onto the bottom. In other words, in that case the quotient space $\mathscr{C}\times\mathscr{I}/\tau$ is equal to the suspension of a homeomorphism of the Cantor set.

The 2-solenoid and the bucket handle can be represented by means of an involution as in Theorem (3.3). This is shown in the following examples.

(3.4) EXAMPLES. (a) The 2-solenoid can be represented as a suspension over the Cantor set as follows: the homeomorphism $\alpha\colon\mathscr{C}\to\mathscr{C}$ is defined as

$$\begin{cases}\alpha(0, \delta_2, \delta_3, \ldots)=(1, \delta_2, \delta_3, \ldots),\\ \alpha(1, 1, 1, \ldots, 1, 0, \delta_i, \delta_{i+1}, \ldots)=(0, 0, 0, \ldots, 0, 1, \delta_i, \delta_{i+1}, \ldots),\\ \alpha(1, 1, 1, \ldots, 1, 1, \ldots)=(0, 0, 0, \ldots, 0, 0, \ldots).\end{cases}$$

The homeomorphism α is called the *adding machine* on the 2-adic integers. As a result the 2-solenoid $\mathscr{S}_2\approx\Sigma(\mathscr{C}, \alpha)$ admits a flow without rest points $\varphi\colon \Sigma(\mathscr{C}, \alpha)\times\mathbf{R}\to\Sigma(\mathscr{C}, \alpha)$ which can be described as follows. Let $[x]\in\mathbf{Z}$ denote the integer part of the real number x; the action of $\mathbf{R}$ on $\mathscr{S}_2$ is defined by $\varphi((\delta, s), t)=(\alpha^{[t+s]}(\delta), t+s-[t+s])$.

A homeomorphism h of $\mathscr{S}_2$ is *orientation preserving* if it preserves the sign of the action of φ, in other words, if for all γ and for all δ such that $\gamma = \varphi(\delta, t)$ for some $t > 0$, we have $h(\gamma) = \varphi(h(\delta), s)$ for some $s > 0$; otherwise h is *orientation reversing*.

It is to be observed that two orientation preserving homeomorphisms g, h on $\mathscr{S}_2$ are isotopic if and only if they induce the same permutation of the composants. The isotopy between g and h can be constructed by means of the flow φ: For every $x \in \mathscr{S}_2$ there exists a unique $t(x) \in \mathbf{R}$ such that $\varphi(g(x), t(x)) = h(x)$. Since g and h preserve the orientation of $\mathscr{S}_2$ the function $t\colon \mathscr{S}_2 \to \mathbf{R}$ is continuous. Therefore the isotopy $H\colon \mathscr{S}_2 \times [0, 1] \to \mathscr{S}_2$ given by $H(x, u) = \varphi(g(x), u \cdot t(x))$ is well defined.

(b) The bucket handle can be represented as follows: the involution $\tau\colon \mathscr{C} \times \{-1, 1\} \to \mathscr{C} \times \{-1, 1\}$ is defined as

$$\begin{cases} \tau((\delta_i), 1) = ((\bar{\delta}_i), 1) \quad \text{with } \bar{\delta}_i = 0 \text{ iff } \delta_i = 1, \\ \tau((\delta_i), -1) = \tau((0, 0, \ldots, 0, 1, \delta_n, \delta_{n+1}, \ldots), -1) = ((0, 0, \ldots, 0, 1, \bar{\delta}_n, \bar{\delta}_{n+1}, \ldots), -1), \\ \tau((0, 0, \ldots), -1) = ((0, 0, \ldots), -1). \end{cases}$$

The involution has a fixed point since $\mathscr{K}$ contains the special point zero. The quotient space $\mathscr{C} \times \mathscr{I}/\tau$ is homeomorphic to the bucket handle. Note that τ flips the tail of the elements (δ_i) of the Cantor set. We claim that two elements $((\delta_i), 0)$ and $((\varepsilon_i), 0)$ are in the same composant if and only if (δ_i) and (ε_i) have equal or opposite tail, i.e., for some $n \in \mathbf{N}$ either $\delta_i = \varepsilon_i$ for all $i \geqslant n$ or $\delta_i = \bar{\varepsilon}_i$ for all $i \geqslant n$. One of the inclusions is obvious: since τ flips the tails, two elements in the same composant must have the same or opposite tails. The reverse inclusion is not completely obvious, but it can be explained with the help of Figure 2 below. Suppose that x and y are elements of the Cantor set such that the tails of x and y are equal or opposite from the nth coordinate onwards. The Cantor set is divided into 2^{n-1} "blocks, in each of which the points have the same first $n-1$ coordinates." It is illustrated in Figure 2 in which way the involution τ ties

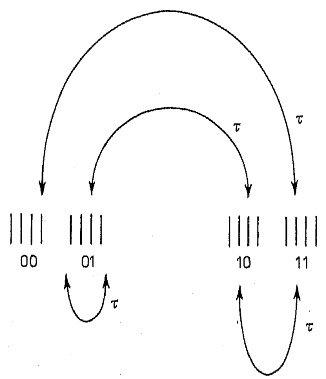

Fig. 2. The action of τ

these intervals together. Note that τ fixes the block of points whose first $n-1$ coordinates are equal to $00\ldots01$. On this interval it leaves the first n coordinates invariant and it flips the tail.

Apply τ as many times as needed to transport x into the interval which contains y. Now it may happen that the transport of x has a tail opposite to the tail of y. In that case one has to go on and transport x into the interval of y via the interval which corresponds to $00\ldots01$. Then the tail is reversed and x is transported onto y. This explains why x and y are elements of the same composant.

As explained above the involution depends on the choice of the matchbox. Let $\mathscr{C}_0$ be the subset of elements $(\delta_i) \in \mathscr{C}$ with first coordinate equal to 0. Consider the matchbox $\mathscr{C}_0 \times [-1/2, 1/2]$ in the bucket handle. The involution τ_0 on the end points of the matchbox $\mathscr{C}_0 \times [-1/2, 1/2]$ is identical to the involution τ itself. Only τ_0 leaves the first coordinate 0 invariant:

$$\begin{cases} \tau_0((\delta_i), \tfrac{1}{2}) = ((\bar{\delta}_i), \tfrac{1}{2}) \quad \text{with } \delta_1 = 0, \\ \tau_0((\delta_i), -\tfrac{1}{2}) = \tau((0, 0, \ldots, 0, 1, \delta_n, \delta_{n+1}, \ldots), -\tfrac{1}{2}) = ((0, 0, \ldots, 0, 1, \bar{\delta}_n, \bar{\delta}_{n+1}, \ldots), -\tfrac{1}{2}). \end{cases}$$

Therefore the map $\sigma\colon \mathscr{C}_0 \times [-1/2, 1/2] \to \mathscr{C} \times [-1/2, 1/2]$ defined by $\sigma((\delta_i)_{i=1}^{\infty}, t) = ((\delta_{i+1})_{i=1}^{\infty}, t)$ can be extended to a homeomorphism of the bucket handle onto itself. This homeomorphism σ is called the *shift homeomorphism* or simply the *shift* on the bucket handle.

As a result the composant of $((\delta_i), 0)$ is mapped onto the composant of $((\varepsilon_i), 0)$ by some iterate of the shift if and only if (δ_i) and (ε_i) have equal or opposite shifted tails. In other words, for some $n \in \mathbf{N}$ and $m \in \mathbf{Z}$ either $\delta_i = \varepsilon_{i+m}$ for all $i \geqslant n$ or $\delta_i = \bar{\varepsilon}_{i+m}$ for all $i \geqslant n$. The shift homeomorphism σ fixes two composants: the one which contains elements of the zero level $\mathscr{C} \times \{0\}$ with constant tail and the one which contains elements of $\mathscr{C} \times \{0\}$ with alternating tail. The other composants are not fixed by σ. This is essentially Bellamy's result [Bel].

Note that different iterates of the shift induce different permutations on composants.

It is a consequence of Gutek's theorem that this procedure to construct the shift homeomorphism can be applied more generally to minimal matchbox manifolds, as will be shown below.

(3.5) Definition. Let U, V be matchboxes in minimal matchbox manifolds X and Y respectively. The involutions $\tau_U\colon \partial U \to \partial U$ and $\tau_V\colon \partial V \to \partial V$ on the end points of U and V respectively are *conjugate with respect to matches* if there exists a homeomorphism $h\colon \partial U \to \partial V$ between the end points such that

(1) $h \circ \tau_U = \tau_V \circ h$,

(2) if u_1 and u_2 are elements of the same match in U, then $h(u_1)$ and $h(u_2)$ are elements of the same match in V.

The following theorem is an analogue of Theorem (2.3). It shows that the construction of the shift homeomorphism of $\mathscr{K}$ in Example (3.4)(b) is typical for minimal matchbox manifolds.

(3.6) THEOREM. *Two minimal matchbox manifolds X and Y are homeomorphic if and only if there exist matchboxes $U \subset X$ and $V \subset Y$ such that the involutions on the end points τ_U, τ_V are conjugate with respect to matches.*

Proof. *Necessity.* Suppose that $h\colon X \to Y$ is a homeomorphism; obviously h maps matchboxes onto matchboxes. Let U be an arbitrary matchbox in X and let V be its image in Y. The homeomorphism h restricted to the end points of U conjugates the involutions τ_U and τ_V with respect to matches.

Sufficiency. According to Gutek's theorem the quotient spaces U/τ_U and V/τ_V are homeomorphic to X and Y respectively. Since the conjugating homeomorphism between τ_U and τ_V respects matches, it can be extended to a homeomorphism between the quotient spaces.

In order to find the permutations induced by the autohomeomorphisms of $\mathscr{K}$ we could check the involutions on end points of matchboxes, following the approach in [A-F]. However, in the next section we will get these permutations by using results from algebraic topology. Instead Theorem (3.6) is employed to show that Bellamy's result cannot be improved by considering homeomorphisms of a locally compact subset of $\mathscr{K}$, as we will now point out.

(3.7) THEOREM. *Let X be a locally compact subset of the bucket handle $\mathscr{K}$ which contains at least one composant of $\mathscr{K}$. The permutation induced by an autohomeomorphism of X on the set $\{C \mid C \subset X$ is a composant of $\mathscr{K}\}$ corresponds to the permutation induced by an autohomeomorphism of $\mathscr{K}$.*

Proof. Note that the proof of the necessity in Theorem (3.6) holds for general matchbox manifolds (i.e., without the assumption that S_x as in Definition (2.1) equals the Cantor set). We apply this to X, a locally compact subset of $\mathscr{K}$ which contains a composant of $\mathscr{K}$.

Since X is a dense and, consequently, an open subset of $\mathscr{K}$, the matchboxes of X are matchboxes in $\mathscr{K}$. Suppose that $h\colon X \to X$ is an autohomeomorphism of X. Assume that it maps a matchbox U onto a matchbox V. The involutions τ_E and τ_F are defined on dense subsets E, F of the end points of U, V respectively. Two end points of E (or F) are equivalent if they can be connected by an arc in the complement of U (or V) in X. In other words, τ_E and τ_F are the restrictions of the involutions τ_U and τ_V to the end points of U and V respectively. The homeomorphism $h\colon \partial U \to \partial V$ restricted to $h\colon E \to F$ is a conjugating homeomorphism between τ_E and τ_F with respect to matches. Therefore h satisfies the conditions (1) and (2) of Definition (3.5) on a dense subset of the end points. By continuity h conjugates the involutions τ_U and τ_V with respect to end points. It follows from Theorem (3.6) that there exists a homeomorphism of $\mathscr{K}$ which permutes composants in the same way as h. In other words, the approach of Bellamy cannot be improved by considering homeomorphisms of a locally compact subspace of the bucket handle.

4. The class group of the bucket handle. We are interested in the permutations on composants that can be induced by a homeomorphism of the bucket handle. Obviously, if two such homeomorphisms are isotopic, they induce the same permutation on the composants. As we shall see below, the converse also holds. Therefore we can list all permutations if we can list all homeomorphisms up to isotopy. It is the objective of this section to calculate the class group $\mathscr{Cl}(K)$ which is defined as the group of homeomorphisms of $\mathscr{K}$ up to isotopy. We do this by lifting homeomorphisms of the bucket handle to the 2-solenoid $\mathscr{S}_2$: the class group of the 2-solenoid can be calculated by means of algebraic topology. This method to study homeomorphisms of the bucket handle is well known. It is employed by Bellamy [Bel] and in a slightly different setting by Dębski and Tymchatyn [D-T].

To this end we give yet another description of the bucket handle and the 2-solenoid: the bucket handle is topologically equivalent to the inverse limit of the unit interval $\mathscr{I} = [-1, 1]$ with bonding map $\tau\colon \mathscr{I} \to \mathscr{I}$ between successive terms given by $\tau(x) = 2x^2 - 1$ (τ is conjugate to the tent map). The zero point of the bucket handle corresponds to $(1, 1, 1, \ldots)$ and the shift on the bucket handle corresponds to the map $(x_i)_{i=1}^{\infty} \to (x_{i+1})_{i=1}^{\infty}$ of the inverse limit space.

Similarly, the 2-solenoid $\mathscr{S}_2$ can be defined as an inverse limit of the circle group $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$ with bonding map $\kappa\colon S^1 \to S^1$ given by $\kappa(z) = z^2$. It follows that the 2-solenoid is a compact abelian group with identity element $e = (1, 1, \ldots)$. The flow φ from Example (3.4)(a) is defined on the inverse limit space as $\varphi((z_j), t) = (e^{2^{2-j}\pi i t} \cdot z_j)$; at time 1 the first coordinate has rotated once, the second has rotated by the angle π, etc.

This description of the bucket handle and the 2-solenoid can be employed to define the map $\pi\colon \mathscr{S}_2 \to \mathscr{K}$ as $\pi((z_i)) = (\operatorname{Re} z_i)$. Note that π is well defined as $z \to \operatorname{Re} z$ commutes with the bonding maps. According to Lemma (4.1) below π has a lifting property. (Note: this map π was used by Bellamy in [Bel] in order to describe the action of the shift on the composants of the bucket handle. Bellamy attributed the definition of π to J. W. Rogers, Jr.)

Let $\gamma\colon \mathscr{S}_2 \to \mathscr{S}_2$ be the complex conjugation on coordinates which maps (z_i) onto $(\bar{z}_i)$. It has the identity element as its only fixed point. It is to be observed that γ is an orientation reversing homeomorphism of the 2-solenoid. Indeed, $\gamma(\varphi(x, t)) = \varphi(\gamma(x), -t)$ as follows from the definition of φ above.

It is a consequence of the definition of π that it is equal to the quotient map $q\colon \mathscr{S}_2 \to \mathscr{S}_2/\gamma$, which implies that π restricted to $\mathscr{S}_2 - \{e\}$ is a 2-1 covering map. The proof of the following homotopy lifting property of π is almost verbatim the same as the proof of the homotopy lifting property for covering maps on manifolds. A similar result is contained in [D-T].

(4.1) LEMMA. *Let $H\colon (\mathscr{K} - \{0\}) \times [0, 1] \to \mathscr{K} - \{0\}$ be a continuous map. Then there exists a lift $\tilde{H}\colon (\mathscr{S}_2 - \{e\}) \times [0, 1] \to \mathscr{S}_2 - \{e\}$ such that $H \circ \pi = \pi \circ \tilde{H}$.*

Proof. Let $f\colon \mathscr{K} - \{0\} \to \mathscr{K} - \{0\}$ be a continuous map. We show that f can be lifted to a map $\tilde{f}\colon \mathscr{S}_2 - \{e\} \to \mathscr{S}_2 - \{e\}$. It then follows in a standard way (e.g. [Spa]) that a homotopy $H\colon (\mathscr{K} - \{0\}) \times [0, 1] \to \mathscr{K} - \{0\}$ can be lifted to a homotopy $\tilde{H}\colon (\mathscr{S}_2 - \{e\}) \times [0, 1] \to \mathscr{S}_2 - \{e\}$. For every $x \in \mathscr{S}_2 - \{e\}$ there are two choices for $\tilde{f}$. A uniform choice has to be made.

We start with a point $x_0 \in \mathscr{S}_2-\{e\}$ and choose one of the preimages of $f(\pi(x_0))$ as $\tilde{f}(x_0)$. Since π is a local homeomorphism, there exists a map $\tilde{f}$: $U \to \mathscr{S}_2-\{e\}$ on a matchbox neighborhood U of x_0 such that it covers f on a neighborhood of $\pi(x_0)$. Now $\tilde{f}$ is extended to $\mathscr{S}_2-\{e\}$ in a way similar to the procedure in the case of a covering map on a manifold. Let x be an element of $\mathscr{S}_2-\{e\}$; there exists an arc A in $\mathscr{S}_2-\{e\}$ which connects x to an element $y \in U$. The arc $f\circ\pi(A)$ is covered by two arcs in $\mathscr{S}_2-\{e\}$. One of the arcs has end point $\tilde{f}(y)$. The image $\tilde{f}(x)$ is defined to be the other end point of this arc. As a consequence of the lemma of the long box, the map $\tilde{f}$ defined in this way is continuous.

Note that $\tilde{f}\circ\gamma$ also covers f, it corresponds to the other choice of $\tilde{f}(x_0)$. Indeed, for any continuous map g: $\mathscr{K}-\{0\} \to \mathscr{K}-\{0\}$ there exist exactly two lifted maps on $\mathscr{S}_2-\{e\}$. As a result the deck transformation γ commutes with lifts: $\gamma\circ\tilde{f} = \tilde{f}\circ\gamma$.

The *class group* $\mathscr{Cl}(\mathscr{S}_2)$ is the group of homeomorphisms which preserve the base point e, up to base point preserving isotopy. Note that homeomorphisms and isotopies of the bucket handle fix the zero point, hence so do the lifts. The zero point accidently corresponds to 1.

In the following lemma methods from algebraic topology and the theory of topological groups are employed to calculate $\mathscr{Cl}(\mathscr{S}_2)$. We want to emphasize, however, that the method from [A-F] combined with Theorem (3.6) can also be used to calculate $\mathscr{Cl}(\mathscr{K})$ more directly.

The group of topological isomorphisms on $\mathscr{S}_2$ is denoted by $\mathrm{Iso}(\mathscr{S}_2)$.

(4.2) LEMMA. *The class group $\mathscr{Cl}(\mathscr{S}_2)$ is naturally isomorphic to* $\mathrm{Iso}(\mathscr{S}_2)$.

Proof. First we indicate that every continuous map from $\mathscr{S}_2$ to the circle S^1 is homotopic to a continuous homomorphism. It follows from the continuity theorem for Čech cohomology that the first Čech cohomology group $H^1(\mathscr{S}_2)$ is isomorphic to the (discrete) group of 2-adic rationals $\mathbf{Q}_2 = \{n/2^m \mid n, m \in \mathbf{Z}\}$ (see [Spa], p. 358). The first Čech cohomology group is isomorphic to the group of continuous maps from $\mathscr{S}_2$ to S^1 up to homotopy. Also the group of 2-adic rationals $\mathbf{Q}_2$ is isomorphic to the Pontryagin dual group of $\mathscr{S}_2$. The Pontryagin dual is the group of homomorphisms from $\mathscr{S}_2$ to S^1 (see [H-R], p. 403). Consequently, $H^1(\mathscr{S}_2)$ is isomorphic to the Pontryagin dual of $\mathscr{S}_2$ (see also [McC], p. 198). Indeed, it is proved in [H-M], p. 211, that for a connected compact abelian group in general the first Čech cohomology group and the Pontryagin dual group are *naturally* isomorphic. Consequently, a map from $\mathscr{S}_2$ to the circle S^1 is homotopic to a continuous homomorphism and no two homomorphisms are homotopic.

Next we show that a continuous base point preserving map from $\mathscr{S}_2$ onto itself is homotopic to a continuous homomorphism. Let f: $\mathscr{S}_2 \to \mathscr{S}_2$ be a continuous base point preserving map. We denote the projection of f onto the ith coordinate of the inverse limit $\mathscr{S}_2$ by f_i. As pointed out above there exists a (base point preserving) homotopy F_1: $\mathscr{S}_2 \times [0, 1] \to S^1$ such that F_1 restricted to $\mathscr{S}_2 \times \{0\}$ equals f_1 and F_1 restricted to $\mathscr{S}_2 \times \{1\}$ is a homomorphism. The bonding map κ: $S^1 \to S^1$ is a covering map. It has the homotopy lifting property. The homotopy F_1 can be lifted to a homotopy F_2: $\mathscr{S}_2 \times [0, 1] \to S^1$ which is equal to f_2 on the bottom $\mathscr{S}_2 \times \{0\}$. By a connectedness argument the lift F_2 is a homomorphism on the top. Either $F_2(x\cdot y, 1)$ is equal to $F_2(x, 1)\cdot F_2(y, 1)$ or it is equal to $-F_2(x, 1)\cdot F_2(y, 1)$; these are the two elements in the fiber over $F_1(x\cdot y, 1)$. Since $\mathscr{S}_2$ is connected, there is one uniform choice of the sign for all $x, y \in \mathscr{S}_2$. If x, y are both chosen to equal the identity element, we see that the uniform choice has to be $F_2(x\cdot y, 1) = F_2(x, 1)\cdot F_2(y, 1)$.

The process of lifting homotopies can be continued by induction. There exist lifted homotopies F_i for all $i \in \mathbf{N}$ which are homomorphisms on the top. The infinite product of all these maps defines a homotopy between f and a continuous homomorphism.

Finally, we show that in the case that f is a homeomorphism it is isotopic to a topological isomorphism of $\mathscr{S}_2$. A homeomorphism permutes the composants of $\mathscr{S}_2$, hence so does the homomorphism homotopic to f. Therefore this homomorphism must be an isomorphism. It follows from Example (3.4)(a) that a homeomorphism on $\mathscr{S}_2$ is isotopic to an isomorphism.

(4.3) THEOREM. $\mathscr{Cl}(\mathscr{S}_2) \cong \mathbf{Z}\oplus\mathbf{Z}/2\mathbf{Z}$.

Proof. According to Lemma (4.2) the class group of the 2-solenoid is naturally isomorphic to the group of isomorphisms $\mathrm{Iso}(\mathscr{S}_2)$. By Pontryagin duality, this group is equivalent to the isomorphism group of the 2-adic integers $\mathrm{Iso}(\mathbf{Q}_2)$. The latter is generated by $x \to -x$ and $x \to 2x$. Under Pontryagin duality these correspond to the coordinatewise complex conjugation γ and the shift on the 2-solenoid, $\sigma((z_i)_{i=1}^{\infty}) = ((z_{i+1})_{i=1}^{\infty})$.

From Lemma (4.1) we obtain a map $\mathscr{Cl}(\mathscr{K}) \to \mathscr{Cl}(\mathscr{S}_2)$. For any homeomorphism h: $\mathscr{K} \to \mathscr{K}$ there exist two lifted homeomorphisms of $\mathscr{S}_2$ (recall that a homeomorphism of $\mathscr{K}$ fixes zero). One of these lifts is orientation preserving, the other reverses the orientation. For instance the orientation preserving lift of the shift on the bucket handle is the shift on the 2-solenoid. The orientation preserving lift of h is denoted by $\tilde{h}$. The homomorphism i: $\mathscr{Cl}(\mathscr{K}) \to \mathscr{Cl}(\mathscr{S}_2)$ which maps the isotopy class of h onto the isotopy class of $\tilde{h}$ is well defined. Its image equals the infinite cyclic subgroup of $\mathscr{Cl}(\mathscr{S}_2)$ generated by the shift on the 2-solenoid.

We claim that i is a monomorphism. Suppose that g and h are homeomorphisms of the bucket handle such that $\tilde{g}$ and $\tilde{h}$ are isotopic; it is to be shown that g and h are isotopic. According to Example (3.4)(a), $\tilde{g}$ and $\tilde{h}$ induce the same permutation on the composants of $\mathscr{S}_2$. It was also pointed out in Example (3.4)(a) that an isotopy $\tilde{H}$ between $\tilde{g}$ and $\tilde{h}$ can be defined by using the flow φ: $\tilde{H}(x, s) = \varphi(\tilde{g}(x), s\cdot t(x))$. We show that this isotopy covers an isotopy between g and h; this would certainly establish the claim that i is a monomorphism. In order to do that we excessively use the commutativity properties of the deck transformation γ.

As has been observed at the end of the proof of Lemma (4.1), $\tilde{g}(\gamma(x)) = \gamma(\tilde{g}(x))$ and $\tilde{h}(\gamma(x)) = \gamma(\tilde{h}(x))$. Also, as a consequence of the fact that γ reverses the flow φ, we have

$$\varphi(\tilde{g}(\gamma(x)), -t(x)) = \varphi(\gamma(\tilde{g}(x)), -t(x)) = \gamma(\varphi(\tilde{g}(x), t(x))) = \gamma(\tilde{h}(x)) = \tilde{h}(\gamma(x)).$$

Hence by definition $t(\gamma(x)) = -t(x)$. These two commutativity properties of γ add up to show that $\tilde{H}$ projects to an isotopy between g and h:

$$\tilde{H}(\gamma(x), s) = \varphi\big(\tilde{g}(\gamma(x)), s\cdot t(\gamma(x))\big) = \varphi\big(\gamma(\tilde{g}(x)), s\cdot -t(x)\big)$$
$$= \gamma\big(\varphi(\tilde{g}(x), s\cdot t(x))\big) = \gamma(\tilde{H}(x, t)).$$

This implies that the isotopy $\tilde{H}$ covers an isotopy in $\mathscr{K}$. It follows from Theorem (4.3) that the class group of $\mathscr{K}$ is generated by the shift on the bucket handle.

We have established the following result, announced in the introduction:

(4.4) THEOREM. *The class group of the bucket handle is an infinite cyclic group generated by the shift*: $\mathscr{C}l(\mathscr{K}) \cong \mathbf{Z}$.

As remarked in Example (3.4)(b), different iterates of the shift induce different permutations on the composants. Therefore two homeomorphisms of the bucket handle are isotopic if and only if they induce the same permutation on the composants.

The results of this section can be extended to arbitrary Knaster continua. These are defined by piecewise linear bonding maps on the interval (see [Wat]). If we do not consider the point zero, Knaster continua are minimal matchbox manifolds. As in the case of the bucket handle, Knaster continua are covered by solenoids. Lemma (4.1) on lifting continuous maps of the bucket handle can be generalized to the lifting of continuous maps between general Knaster continua. The proof remains the same. Consequently, two Knaster continua are homeomorphic if and only if the solenoids covering them are homeomorphic. The classification of Knaster continua as carried out in [Wat] thus corresponds to the classification of solenoids.

Also the class group of a general Knaster continuum can be calculated by the same methods as above. Some Knaster continua are very rigid. For instance, the class group of the $(2, 3, 5, 7, \ldots)$-solenoid (every prime number occurs exactly once) equals $\mathbf{Z}/2\mathbf{Z}$, and so does the class group of the underlying Knaster continuum.

5. Continuous surjections between composants. The following description of the bucket handle is well known in the theory of dynamical systems. The bucket handle $\mathscr{K}$ is described as the attractor of a Smale horseshoe map. The composants of $\mathscr{K}$ are the *unstable leaves* of the horseshoe. The construction is more geometrical in nature than the definitions of $\mathscr{K}$ we encountered above.

Let $R_0 = [0, 1]\times[0, 1]$ be the unit square in the plane. Now suppose $\lambda = [0, 1]\times\{1/2\}$ is the middle line of R_0, which we identify with $[0, 1]$ via the projection on the first coordinate. The polygonal region R_1 is the union of $A_1 = [0, 1/3]\times[0, 1]$, $M_1 = [1/3, 2/3]\times[2/3, 1]$ and $B_1 = [2/3, 1]\times[0, 1]$. We call A_1 the *left leg* of R_1 and B_1 its *right leg*. The horseshoe map Ω: $R_0 \to R_1$ maps R_0 homeomorphically onto R_1. It is a linear contraction on $[0, 1]\times[0, 1/3]$ and $[0, 1]\times[2/3, 1]$, in a way suggested by the figure below.

The descending sequence of regions $R_0 \supset R_1 \supset R_2 \supset \ldots$ is defined as $R_i = \Omega^i(R_0)$, the *left leg* of R_i is defined to be the set $\Omega^{i-1}(A_1)$ and similarly its *right leg* is $\Omega^{i-1}(B_1)$. The horseshoe is defined in such a way that the middle line λ intersects only the legs of

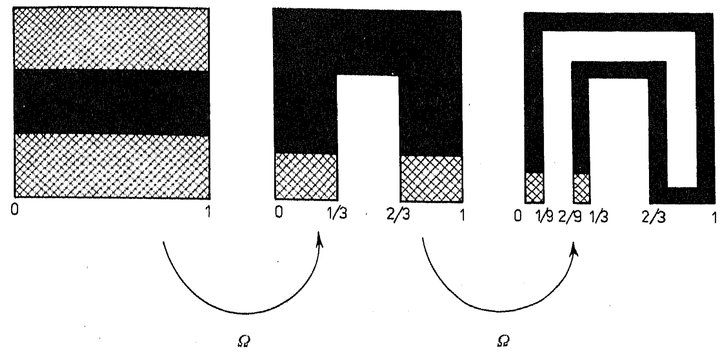

Fig. 3. The horseshoe map

each region R_i. Indeed, $\lambda \cap R_i$ is the union of 2^i intervals of which 2^{i-1} belong to A_i and 2^{i-1} belong to B_i. Without loss of generality we may assume that these 2^i intervals equal the standard intervals at the ith step of the construction of the Cantor middle third set. The bucket handle is defined as the intersection of the regions, $\mathscr{K} = \bigcap\{R_i \mid i\in\mathbf{N}\}$. The middle line λ intersects $\mathscr{K}$ in the Cantor middle third set. We picture the regions R_i to be polygonal, i.e., they are a finite union of rectangles in the plane. To every R_i an orientation is assigned: the bottom of R_i is the intersection of $A_i \cap [0, 1]\times\{0\}$ $= [0, 1/3^i]\times\{0\}$, the top is equal to $B_i \cap [0, 1]\times\{0\} = [2/3^i, 1/3^{i-1}]\times\{0\}$ for $i > 0$. The orientation of the legs is induced by R_i: the left leg A_i has bottom $[0, 1/3^i]\times\{0\}$, the right leg B_i has top $[2/3^i, 1/3^{i-1}]\times\{0\}$.

We exhibit a relation between the intervals in $R_i \cap \lambda$ and $B_{i+1} \cap \lambda$. The reason why we are interested in this relation becomes clear in the proof of Theorem (5.1) below. There the intervals in $R_i \cap \lambda$ are mapped onto intervals in $B_{i+1} \cap \lambda$.

The orientation of R_i induces an ordering of the 2^i intervals of $R_i \cap \lambda$: as the region runs from the bottom to the top, the intervals are arranged according to the order of intersection. For example the first interval is $[0, 1/3^i]$, the second is $[1-1/3^i, 1]$, etc. Similarly, the orientation of the right leg B_{i+1} in R_{i+1} induces an ordering of the 2^i intervals of $B_{i+1} \cap \lambda$. Notice that each interval in $R_i \cap \lambda$ contains exactly one interval of $B_{i+1} \cap \lambda$. We claim that the jth interval of $R_i \cap \lambda$ is a neighbor of the jth interval of $B_{i+1} \cap \lambda$. This follows from the observation that B_{i+1} runs in the opposite direction to R_i: its top is equal to $[2/3^{i+1}, 1/3^i]\times\{0\}$. This is a subset of $[0, 1/3^i]\times\{0\}$, the bottom of R_i. Hence, the last interval of $B_{i+1} \cap \lambda$ is contained in the first interval of $R_i \cap \lambda$. Similarly, the first interval of $B_{i+1} \cap \lambda$ is contained in the last interval of $R_i \cap \lambda$.

We conclude that the first interval in $B_{i+1} \cap \lambda$ is equal to $[2/3^i, 7/3^{i+1}] \times \{0\}$, the left third interval of the top of R_i. Consequently, the first interval in $B_{i+1} \cap \lambda$ is a neighbor of the first interval in $R_i \cap \lambda$. The right leg B_{i+1} continues to run up and down side by side along R_i. As a result the jth interval of $B_{i+1} \cap \lambda$ is a neighbor of the jth interval of $R_i \cap \lambda$. If R_i runs up, the interval in $B_{i+1} \cap \lambda$ is to the right, if R_i runs down, it is to the left.

The middle third set $\mathscr{C}$ and $\{0, 1\}^{\mathbf{N}}$ are identified in the standard way via the homeomorphism $(\delta_i) \to \sum_{i=1}^{\infty} 2\delta_i/3^i$. Elements of $\mathscr{C}$ are in the same interval of $R_i \cap \lambda$ if and only if they have equal first i coordinates. Expressed in terms of $\{0, 1\}^{\mathbf{N}}$, the neighbor relation between the intervals above becomes: (δ_i) is contained in the jth interval of $R_i \cap \lambda$ and (ε_i) is contained in the jth interval of $B_{i+1} \cap \lambda$ for some j if and only if

$$(*) \qquad (\delta_1, \delta_2, \ldots, \delta_{i-1}, \bar{\delta}_i, \delta_i) = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{i-1}, \varepsilon_i, \varepsilon_{i+1}).$$

This follows from the observation that $\delta_i = 0$ if R_i runs upwards through the interval containing (δ_i), and $\delta_i = 1$ if it runs downwards. The relation $(*)$ is needed in the proof of Theorem (5.1) below.

(5.1) THEOREM. *Let C and D be non-zero composants of $\mathscr{K}$. There exists an injective continuous map ϕ: $\mathscr{K}-\{0\} \to \mathscr{K}-\{0\}$ which maps C onto D.*

Proof. We show that for any pair of non-end points $\alpha, \beta \in \mathscr{C}$ there exists a continuous injection ϕ: $\mathscr{K}-\{0\} \to \mathscr{K}-\{0\}$ which maps α onto β. The map ϕ is an infinite composition of certain homeomorphisms ϕ_i: $\mathscr{K} \to \mathscr{K}$ which are defined in a similar fashion for all i. This is a well known method in topology (see [vM], Chapter 6). Every ϕ_i is a squeeze and stretch map from R_i onto R_{i+1}. The squeeze part is a contraction from R_i onto the left leg A_{i+1}. The stretch part expands a small neighborhood of 0 and contracts the complement (see Fig. 4). If these neighborhoods are chosen to be sufficiently small, then an infinite composition of ϕ_i's converges to a continuous map on $\mathscr{K}-\{0\}$. To be more precise: the small neighborhood of zero expanded by the stretch map is denoted by U_i. If the neighborhoods U_i satisfy the following two properties:

(1) $U_{i+1} \subset \phi_i(U_i)$,

(2) $\bigcap \{U_i \mid i \in \mathbf{N}\} = \{0\}$,

then every point of $\mathscr{K}-\{0\}$ is contained in a neighborhood which is expanded only finitely many times. Therefore an infinite composition of ϕ_i converges on $\mathscr{K}-\{0\}$. Here is the detailed description.

Let $S_i = [0, 1/3^i] \times [0, \varepsilon_i]$ be a small rectangle in R_i. Squeeze R_i into the left leg A_{i+1} in such a way that each of the 2^i intervals in $R_i \cap \lambda$ is mapped linearly into itself. Then the image of S_i is stretched onto the left leg A_{i+1}, the complement of the image of S_i is pushed into the right leg B_{i+1}. This is done in such a way that the composition ϕ_i of the squeezing and the stretching maps the intervals of $R_i \cap \lambda$ linearly onto the intervals of $B_{i+1} \cap \lambda$, preserving the order of the intervals as defined above.

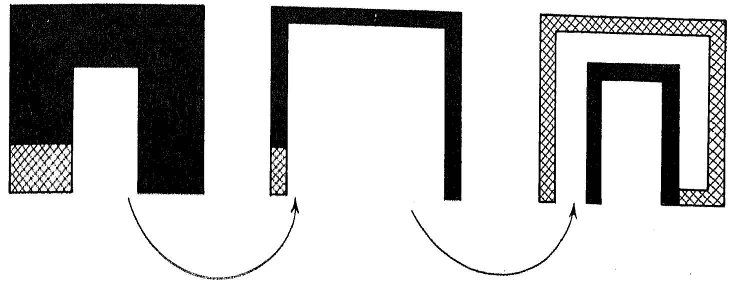

Fig. 4. Squeezing and stretching

According to the relation $(*)$ above in terms of (δ_i) the squeeze and stretch map ϕ_i acts as

$$(\delta_1, \delta_2, \ldots, \delta_{i-1}, \delta_i, \delta_{i+1}, \ldots) \to (\delta_1, \delta_2, \ldots, \delta_{i-1}, \bar{\delta}_i, \delta_i, \delta_{i+1}, \ldots),$$

i.e., it inserts $\bar{\delta}_i$ at the ith coordinate. Note that the composants of $\mathscr{K}$ are infinite unions of horizontal and vertical arcs since the R_n are polygonal. The squeezing map is defined in such a way that it maps horizontal arcs linearly onto horizontal arcs, and similarly verticals linearly onto verticals. Moreover, the conditions (1) and (2) above are satisfied.

Suppose that α and β are two non-end points of the middle third set. In other words, α and β have non-constant tail. Let $\alpha_1 = 0$ and $\beta_1 = 1$; there exists a first coordinate $n \in \mathbf{N}$ such that $\beta_n = 0$. The composition $\phi_{n-1} \circ \phi_{n-2} \circ \ldots \circ \phi_1$ adjusts α on the first n coordinates: $\phi_{n-1} \circ \phi_{n-2} \circ \ldots \circ \phi_1(\alpha)$ and β have the same first n coordinates. In general we can transform a coordinate α_i into a block $(\bar{\alpha}_i, \bar{\alpha}_i, \ldots, \bar{\alpha}_i, \alpha_i)$. Therefore if α and β are non-end points, an infinite composition of those transformations alters α into β. Hence, a suitable composition $\phi = \ldots \circ \phi_{i(n)} \circ \phi_{i(n-1)} \circ \ldots \circ \phi_{i(1)}$ of certain ϕ_i's maps α onto β. If the rectangles $S_{i(n)}$ are chosen sufficiently small, then ϕ is a well defined continuous map on $\mathscr{K}-\{0\}$.

Note that ϕ maps the middle third set injectively into itself. Let $x, y \in \mathscr{C}$ be two consecutive elements of a composant, hence x and y are connected by a polygon $P \subset \mathscr{K}$. For some $n \in \mathbf{N}$ the image $\phi_{i(n)} \circ \phi_{i(n-1)} \circ \ldots \circ \phi_{i(1)}(P)$ is disjoint from $S_{i(n+1)}$, hence P is stretched only a finite number of times. The subsequent squeezes are all linear on the edges of P. This implies that ϕ is injective on P, and since ϕ is also injective on $\mathscr{C}$, it is an injection on $\mathscr{K}-\{0\}$ which maps the composant of α onto the composant of β.

(5.2) Remark. *The image of ϕ is a subset of $\mathscr{K}-\{0\}$ of first category.*

The map in the proof of Theorem (5.1) is still far from a homeomorphism between composants. One would hope that the deletion of the zero composant would offer sufficient freedom to construct homeomorphisms between any pair of composants. Still, this is far from obvious and we do not even know the answer to the following question: is the 2-solenoid minus one composant homogeneous?

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A note on continuous linear mappings between function spaces

by

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Abstract. Let $\theta\colon C_p(X)\to C_p(Y)$ be a linear continuous function. If θ is an order preserving homeomorphism (an isometry from $C_u^*(X)$ to $C_u^*(Y)$), then the Tikhonov spaces X and Y are homeomorphic. This generalizes the well known theorem of Nagata that if $C_p(X)$ and $C_p(Y)$ are topologically isomorphic then X and Y are homeomorphic. If θ is 1-1 and Y has caliber (τ,λ) (resp. is pseudocompact) then X has caliber (τ,λ) (resp. is pseudocompact), proving in this way that if $L_p(X)$ has caliber (τ,λ) then so does X. Related results for L-embedded spaces are obtained.

Introduction. Everywhere below X, Y and Z stand for infinite Tikhonov topological spaces, $C_p(X)$ is the space of all continuous real-valued functions on X endowed with the topology of pointwise convergence, $C_p^*(X)=\{f\in C_p(X)\colon f \text{ bounded}\}$ and $C_u^*(X)=\{f\in C(X)\colon f \text{ bounded}\}$ endowed with the topology of uniform convergence. It is clear that the family of sets $V(x;G)=\{f\in C_p(X)\colon f(x)\in G\}$ where G is open in $\mathbf{R}$, is an open subbase of $C_p(X)$.

A space X has *caliber* (τ,λ), where τ, λ are infinite cardinals, if for every family γ of non-empty open subsets of X such that $|\gamma|=\tau$, there exists a subfamily $\gamma_1\subset\gamma$ with $\bigcap\gamma_1\neq\emptyset$ and $|\gamma_1|=\lambda$.

We denote by N_x the family of open basic neighbourhoods of x and by 1_X the unit function on $C_p(X)$. For $A\subset X$ and $f\in C_p(X)$, we write $f|A$ for the restriction of f on A, $\operatorname{supp} f=\{x\in X\colon f(x)\neq 0\}$ for the support of f, and $\overline{A}$ for the closure of A in X.

Let $e\colon X\to C_p(C_p(X))$ such that $e(x)=\hat{x}$, where $\hat{x}(f)=f(x)$ for f in $C_p(X)$. The set of all finite linear combinations $z=a_1\hat{x}_1+\ldots+a_n\hat{x}_n$ is denoted by $L_p(X)$. It is known ([1]) that $L_p(X)$ is the dual space of $C_p(X)$, while $C_p(X)$ is the dual space of $L_p(X)$. If θ is a continuous linear function from $C_p(X)$ to $C_p(Y)$, then the induced function θ^* from $L_p(Y)$ to $L_p(X)$ defined by $\theta^*(y)=y\circ\theta$ for $y\in L_p(Y)$ is also linear and continuous. For $y\in Y$, $\theta^*(y)(f)=\theta(f)(y)$ for every f in $C_p(X)$. Suppose that for $y\in Y$, $\theta^*(y)=a_1\hat{x}_1+\ldots+a_n\hat{x}_n$, where $a_1,\ldots,a_n\neq 0$. The *determining set* of y in X with

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