On nonparadoxical sets

by

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Abstract. Subsets A, B of $\mathbf{R}^n$ are *countably equidecomposable* if there is a partition $\{A_m: m \in \omega\}$ of A and isometries g_m of $\mathbf{R}^n$ such that $\{g_m A_m: m \in \omega\}$ is a partition of B. A set A is *paradoxical* if A contains two disjoint subsets each countably equidecomposable with A. We show the existence of nonparadoxical sets of full Lebesgue measure. We also prove that every set of positive measure contains an uncountable paradoxical subset of full measure. A subset A of $\mathbf{R}^n$ is *hereditarily nonparadoxical* if A has no uncountable paradoxical subsets. It is shown that the family of hereditarily nonparadoxical sets is a proper ideal and that, under $\neg$CH, the union of countably many hereditarily nonparadoxical sets has inner measure zero. This generalizes a result by Erdős and Kunen. We answer related questions concerning sets without repeated distances.

0. Introduction. We shall investigate the notion of countable equidecomposability in $\mathbf{R}^n$ (with the Euclidean metric denoted by d_n).

We use the standard set-theoretical notation. Ordinals are identified with sets of their predecessors and cardinals with initial ordinals. In particular, ω denotes the set of natural numbers and the first infinite cardinal, and ω_1 is the first ucountable cardinal.

If X is a set and $\varkappa$ is a cardinal, then $[X]^\varkappa$ is the family of subsets of X of cardinality $\varkappa$.

For a set X, $|X|$ denotes the cardinality of X. By 2^ω we denote $|\mathbf{R}|$. The set of rational numbers and the set of integers are denoted by $\mathbf{Q}$ and $\mathbf{Z}$, respectively.

By an *ideal* we mean a family of subsets of $\mathbf{R}^n$ closed under finite unions and taking subsets. An ideal is called a *σ-ideal* if it is closed under countable unions.

Definition 0.1. Let A and B be subsets of $\mathbf{R}^n$, let G be a subgroup of the group of isometries of $\mathbf{R}^n$ and let $\varkappa$ be a cardinal such that $\omega < \varkappa \leqslant 2^\omega$. We say that

(i) A is *$\varkappa$-G-equidecomposable* with B ($A \overset{G}{\underset{\varkappa}{\sim}} B$) if there exist $\lambda < \varkappa$, a partition $\{A_\xi: \xi \in \lambda\}$ of A, and a set of isometries $\{g_\xi: \xi \in \lambda\} \subseteq G$ such that $\{g_\xi A_\xi: \xi \in \lambda\}$ is a partition of B;

(ii) A is *countably G-equidecomposable* with B ($A \overset{G}{\underset{\infty}{\sim}} B$) if A is ω_1-G-equidecomposable with B;

(iii) A is *G-paradoxical* if there are two disjoint subsets A_1 and A_2 of A such that $A_1 \overset{G}{\underset{\infty}{\sim}} A$ and $A_2 \overset{G}{\underset{\infty}{\sim}} A$;

(iv) A is *G-nonparadoxical* if A is not G-paradoxical.

We write $A \prec^G_\infty B$ if there is a subset B_1 of B such that $A \overset{G}{\underset{\infty}{\sim}} B_1$. If no group is indicated, we consider the group of all isometries of $\mathbf{R}^n$.

The notion of countable equidecomposability was introduced by Banach and Tarski in their famous paper [1]. Our notation concerning countable equidecomposability is similar to the one used by Wagon in [10]; however, the word "countably" is omitted in Definition 0.1(iii),(iv) (compare with [10], p. 7; this will not cause a disagreement with Wagon's definition 1.1, p. 4).

Recall the following facts and theorems:

- $\overset{G}{\underset{\varkappa}{\sim}}$ is an equivalence relation.
- If $A \overset{G}{\underset{\infty}{\sim}} B$ and A is paradoxical, then so is B.
- $\overset{G}{\underset{\infty}{\sim}}$ preserves every G-invariant σ-ideal $\mathfrak{I}$ of subsets of $\mathbf{R}^n$ (i.e. $A \overset{G}{\underset{\omega}{\sim}} B$ and $A \in \mathfrak{I}$ implies that $B \in \mathfrak{I}$, see [1]).
- The Banach–Schröder–Bernstein Theorem: if $A \prec^G_\omega B$ and $B \prec^G_\omega A$, then $A \overset{G}{\underset{\omega}{\sim}} B$ (see [10], p. 136).
- For every n, $\mathbf{R}^n$ is paradoxical.
- Every set with nonempty interior contained in $\mathbf{R}^n$ is countably equidecomposable with $\mathbf{R}^n$ (see [1] or [10], p. 137).

The Banach–Schröder–Bernstein Theorem implies that $\prec_\omega$ is a partial order on $\{[A]_\sim : A \subseteq \mathbf{R}^n\}$. It is also known that being paradoxical is the property of $[A]_\sim$. In the first section we prove that above every set which is not countably equidecomposable with $\mathbf{R}^n$ there exist nonparadoxical sets and below every set which is "large" in the sense of measure or category there exist paradoxical sets which are still "large". The first property may be used to show the existence of nonparadoxical sets of full Lebesgue measure and nonparadoxical comeager sets. The second property is not true for arbitrary uncountable sets.

This gives rise to the notion of hereditarily nonparadoxical sets (i.e., sets without uncountable paradoxical subsets), which is introduced in the second section. An obvious example of such a set is a set without repeated distances, i.e. a set A contained in $\mathbf{R}^n$ such that d_n is 1-1 on $[A]^2$. We prove that the family of hereditarily nonparadoxical sets forms a proper ideal which is not contained in the ideal generated by sets without repeated distances.

Sets without repeated distances have been studied by many mathematicians. In particular, P. Erdős and S. Kakutani proved that the Continuum Hypothesis (CH) is equivalent to the following statement:

- The real line is the union of countably many sets each consisting of rationally independent numbers (i.e., numbers which are independent over the field $\mathbf{Q}$, see [4]).

R. O. Davis proved that under CH the plane may be partitioned into countably many sets each without repeated distances (see [2]). K. Kunen showed that the same is true for each $\mathbf{R}^n$ (see [6]). Thus

- (CH) For every n, $\mathbf{R}^n$ is the union of countably many hereditarily nonparadoxical sets.

On the other hand, P. Erdős showed that under $\neg$CH the real line (hence $\mathbf{R}^n$ for every n) is not the union of countably many sets each without repeated distances (see [3]). Moreover, P. Erdős and K. Kunen proved that under $\neg$CH the countable union of sets without repeated distances has inner measure zero.

In Section 3 we prove a generalization of the above result by Erdős and Kunen to hereditarily nonparadoxical sets, i.e. we show that, under $\neg$CH, the countable union of hereditarily nonparadoxical sets has inner measure zero and does not contain nonmeager subsets with the Baire property.

In Section 4 we answer related questions concerning sets without repeated distances. Our techniques enable us to prove:

- ($\neg$CH) If A is the union of countably many sets each without repeated distances, then for every linear space V over $\mathbf{Q}$ of size less than 2^ω there is a translation t such that $tV \cap A = \emptyset$.

On the other hand, for the existence of a linear space omitting A the translation t is superfluous:

- ($\neg$CH) If A is the union of countably many sets each without repeated distances and if $0 \notin A$, then there is a linear space V over $\mathbf{Q}$ of size 2^ω such that $V \cap A = \emptyset$.

Whenever we write "for every set" we mean "for every subset of (appropriate) $\mathbf{R}^n$". We denote by D^n the group of isometries of $\mathbf{R}^n$ and by T^n the group of translations. We consider only subgroups of D^n. Note that $(T^n, \cdot)$ is isomorphic to $(\mathbf{R}^n, +)$.

If G is a subgroup of D^n, then G acts on $\mathbf{R}^n$ in the obvious way. We use the following notation to describe the action of a group G on $\mathbf{R}^n$. If $x \in \mathbf{R}^n$, $A \subseteq \mathbf{R}^n$, G is a subgroup of D^n, and $g \in G$, then

$$Gx = \{gx:\ g \in G\} \text{ is the } G\text{-orbit of } x, \quad GA = \{gx:\ x \in A,\ g \in G\},$$

$$\operatorname{fix}(g) = \{x \in \mathbf{R}^n:\ gx = x\}, \quad \operatorname{Fix}(G) = \{x \in \mathbf{R}^n:\ \exists g \in G \setminus \{\mathrm{id}\}\ gx = x\}.$$

A *selector from G-orbits* is any set S such that $|Gx \cap S| = 1$ for every $x \in \mathbf{R}^n$.

If $C \subseteq D^n$, then $\langle C \rangle$ denotes the subgroup of D^n generated by isometries in C. If $A \subseteq \mathbf{R}^n$, then $\langle A \rangle = \langle \{t \in T^n:\ t(0) \in A\} \rangle$. Note that $\langle A \rangle$ is a subgroup of T^n.

If f is a function, then $f[X]$ is the image of X under f and $f^{-1}[X]$ is the inverse image of X.

If $A \subseteq \mathbf{R}^n$ then $\mathrm{Lin}_{\mathbf{Q}}(A)$ is the linear subspace of $\mathbf{R}^n$ spanned by A over $\mathbf{Q}$. If $A, B \subseteq \mathbf{R}^n$, then $A + B = \{x + y:\ x \in A,\ y \in B\}$.

The following lemmas are due to P. Zakrzewski (see [11], Lemma 2.2). We use them in almost every proof without indicating it.

LEMMA. *Let $\varkappa$ be a cardinal such that $\omega < \varkappa \leqslant 2^\omega$. Let A and B be subsets of $\mathbf{R}^n$ and let G be a subgroup of D^n. The following are equivalent:*

(i) $A \overset{G}{\underset{\varkappa}{\sim}} B$;

(ii) *there is a subgroup H of G such that $|H| < \varkappa$ and $|Hx \cap A| = |Hx \cap B|$ for each $x \in \mathbf{R}^n$.*

LEMMA. *Let A and B be subsets of $\mathbf{R}^n$ and let G be a subgroup of D^n. The following are equivalent:*

(i) $A \prec^G_\omega B$;

(ii) *there is a countable subgroup H of G such that $|Hx \cap A| \leqslant |Hx \cap B|$ for each $x \in \mathbf{R}^n$.*

Lemma. *Let A be a subset of $\mathbf{R}^n$ and let G be a subgroup of D^n. The following are equivalent*:

(i) *A is G-paradoxical*;

(ii) *there is a countable subgroup H of G such that $|Hx \cap A| = \omega$ for each $x \in A$.*

Since not all of Zakrzewski's lemmas can be found in [11], we give the proof of the first one as an example.

(i)$\Rightarrow$(ii). Let $\{A_\xi\colon \xi \in \lambda\}$ be a partition of A and let $\{g_\xi\colon \xi \in \lambda\} \subset G$ be such that $\lambda < \varkappa$ and $\{g_\xi A_\xi\colon \xi \in \lambda\}$ is a partition of B. Define a bijection $f\colon A \to B$ by $f(x) = g_\xi x$ if $x \in A_\xi$ and let $H = \langle\{g_\xi\colon \xi \in \lambda\}\rangle$. Then $|H| < \varkappa$ and $f(x) \in Hx \cap B$ for every $x \in A$, which shows that $|Hx \cap A| = |Hx \cap B|$ for every $x \in \mathbf{R}^n$.

(ii)$\Rightarrow$(i). Let H be a subgroup of G such that $|H| < \varkappa$ and $|Hx \cap A| = |Hx \cap B|$ for every $x \in \mathbf{R}^n$. Let $f\colon A \to B$ be any bijection such that $f(x) \in Hx$ for every $x \in A$. For each $h \in H$ define $A'_h = \{x \in A\colon f(x) = hx\}$. Then $A = \bigcup_{h \in H} A'_h$, so we can find $A_h \subseteq A'_h$ such that $\{A_h\colon h \in H\}$ is a partition of A. Obviously, $\{hA_h\colon h \in H\}$ is a partition of B. Thus, $A \overset{G}{\underset{\varkappa}{\sim}} B$. ∎

If H is a group from (ii) of one of the above lemmas then we say that H witnesses that the respective (i) holds.

Note that if H witnesses that $A \overset{G}{\underset{\varkappa}{\sim}} B$ (resp. $A \prec^G_\infty B$, A is paradoxical), then so does every group H_1 of size less than $\varkappa$ (resp. every countable group) containing H and contained in G. Indeed, any H_1-orbit is partitioned into H-orbits, so if $|Hx \cap A| = |Hx \cap B|$ for all x, then $|H_1 x \cap A| = |H_1 x \cap B|$ for all x.

1. The main purpose of this section is to show the existence of nonparadoxical sets of positive Lebesgue measure and the existence of nonparadoxical sets which are nonmeager and have the Baire property. Such sets have empty interior (see [1] or [10], Th. 9.14, p. 137, Th. 9.15, p. 139).

Theorem 1.1. *For every group G containing the group of translations of $\mathbf{R}^n$ and for every $A \subseteq \mathbf{R}^n$ the following are equivalent*:

(i) $A \overset{G}{\underset{\infty}{\sim}} \mathbf{R}^n$;

(ii) *there exists a countable subgroup H of G such that $HA \overset{G}{\underset{\infty}{\sim}} \mathbf{R}^n$*;

(iii) *there exists a countable subgroup H of G such that $|\mathbf{R}^n \setminus HA| < 2^\omega$*;

(iv) *A belongs to no proper G-invariant σ-ideal of subsets of $\mathbf{R}^n$*;

(v) *every set containing A is G-paradoxical.*

We shall need two lemmas.

Lemma 1.2. *Let A be the union of a countable family of at most k-dimensional affine subspaces of $\mathbf{R}^n$, where $k < n$. There exist $N \in \omega$, a partition $\{A_m\colon m < N\}$ of A and a set of translations $\{g_m\colon m < N\}$ such that*

$$\bigcup_{m<N} g_m A_m \subseteq \mathbf{R}^n \setminus A.$$

Proof. Proceed by induction on k. For $k = 0$ the set A is countable, so take any translation g such that $gA \cap A = \emptyset$. For $k > 0$ take a translation g such that no k-dimensional affine subspace which is contained in A is moved into A. Then what remains is $gA \cap A$, which is the union of a countable family of at most $(k-1)$-dimensional affine subspaces of $\mathbf{R}^n$. We cannot move any point from $gA \cap A$ into $g^{-1}(gA \cap A)$, so take $(gA \cap A) \cup (g^{-1}(gA \cap A))$ and use the induction hypothesis. ∎

Lemma 1.3. *For every countable subgroup G of D^n there is a countable subgroup H of T^n such that each H-orbit meets infinitely many G-orbits, i.e., for every x in $\mathbf{R}^n$ there is an infinite subset S of H such that $h_1 x \notin Ghx$ for any distinct $h, h_1 \in S$.*

Proof. Let G be a countable subgroup of D^n. Let $G_1 = G \cap T^n$. The group G_1 can be identified with the G_1-orbit of 0. Let V be the linear space generated by G_1 over $\mathbf{Q}$. Take $t \in \mathbf{R}^n \setminus V$ and let $H_1 = \langle\{t\}\rangle$. Then $H_1 \cap G = \{\mathrm{id}\}$.

Let $A = \operatorname{Fix}\langle H_1 \cup G\rangle$. For every $g \in \langle H_1 \cup G\rangle \setminus \{\mathrm{id}\}$, $\operatorname{fix}(g)$ is an affine subspace of $\mathbf{R}^n$ of dimension at most $n-1$, thus A is the union of a countable family of at most $(n-1)$-dimensional affine subspaces of $\mathbf{R}^n$. By 1.2 there exist $N \in \omega$, a partition $\{A_m\colon m < N\}$ of A and a set of translations $\{g_m\colon m < N\}$ such that

$$\bigcup_{m<N} g_m A_m \subseteq \mathbf{R}^n \setminus A.$$

Let $H = \langle H_1 \cup \{g_m\colon m < N\}\rangle$. We show that H has the required property. Take $x \in \mathbf{R}^n$. If $x \notin A$ then let $S = H_1$. If $hx = gh_1 x$ for distinct $h, h_1 \in S$ and $g \in G$ then $h = gh_1$, contrary to $H_1 \cap G = \{\mathrm{id}\}$. If $x \in A$ then there is m such that $x \in A_m$ and $g_m x \notin A$. Let $S = H_1 g_m$. ∎

Proof of 1.1 ((i)$\Rightarrow$(iv)$\Rightarrow$(iii)$\Rightarrow$(ii)$\Rightarrow$(i); (v)$\Rightarrow$(iii) and (i)$\Rightarrow$(v)).

(i)$\Rightarrow$(iv) is obvious.

(iv)$\Rightarrow$(iii). Suppose not (iii). Let $\mathfrak{I} = \{B \subseteq \mathbf{R}^n\colon$ there is a countable subgroup H of G such that $B \subseteq HA\}$. Then $\mathfrak{I}$ is a G-invariant σ-ideal and by the negation of (iii), $\mathfrak{I}$ is proper.

(iii)$\Rightarrow$(ii). Let H be a countable subgroup of G such that $|\mathbf{R}^n \setminus HA| < 2^\omega$. Take a translation $g \notin \langle \mathbf{R}^n \setminus HA\rangle$. Then $g[\mathbf{R}^n \setminus HA] \subseteq HA$. Thus $\langle\{g\} \cup H\rangle A = \mathbf{R}^n$ and $\langle\{g\} \cup H\rangle A \overset{G}{\underset{\infty}{\sim}} \mathbf{R}^n$.

(ii)$\Rightarrow$(i). Let H be a countable subgroup of G such that $HA \overset{G}{\underset{\infty}{\sim}} \mathbf{R}^n$, and let H_1 be a group which witnesses that. Each H_1-orbit meets HA, so each $\langle H_1 \cup H\rangle$-orbit meets A. Applying 1.3 to $\langle H_1 \cup H\rangle$ we get a countable subgroup H_2 of T^n such that each H_2-orbit meets infinitely many $\langle H_1 \cup H\rangle$-orbits. We shall show that for every $x \in \mathbf{R}^n$

$$|\langle H_2 \cup H_1 \cup H\rangle x \cap A| = \omega.$$

So take $x \in \mathbf{R}^n$ and an infinite set $S \subseteq H_2$ such that $h_1 x \notin \langle H_1 \cup H\rangle hx$ for any distinct $h, h_1 \in S$. For every $h \in S$, find $a_h \in A \cap \langle H_1 \cup H\rangle hx$. For distinct $h, h_1 \in S$, the elements $a_h, a_{h_1} \in A$ are different since they belong to different $\langle H_1 \cup H\rangle$-orbits. S being infinite yields that $|\langle H_2 \cup H_1 \cup H\rangle x \cap A| = \omega$. Thus $\langle H_2 \cup H_1 \cup H\rangle$ witnesses that $A \overset{G}{\underset{\infty}{\sim}} \mathbf{R}^n$.

(v)$\Rightarrow$(iii). Suppose not (iii). Let $\{G_\alpha\colon \alpha < 2^\omega\}$ be an enumeration of all countable subgroups of G. We define inductively a subset $\{x_\alpha\colon \alpha < 2^\omega\}$ of $\mathbf{R}^n \setminus A$ such that for every α the orbits $G_\alpha x_\alpha$ are pairwise disjoint and disjoint from A.

Suppose $\{x_\beta\colon \beta<\alpha\}$ is already defined. Then $|\mathbf{R}^n\backslash G_\alpha A| = 2^\omega$ and $|\bigcup\{G_\beta x_\beta\colon \beta<\alpha\}| < 2^\omega$. The set $\mathbf{R}^n\backslash G_\alpha A$ is the union of whole G_α-orbits and $\bigcup\{G_\beta x_\beta\colon \beta<\alpha\}$ is too small to meet all of them, so there is x_α such that the G_α-orbit of x_α omits $A\cup\bigcup\{G_\beta x_\beta\colon \beta<\alpha\}$.

Finally, let $B = A\cup\{x_\alpha\colon \alpha<2^\omega\}$. We claim that B is G-nonparadoxical. Indeed, if H is a countable subgroup of G, then there is α such that $H = G_\alpha$ and $Hx_\alpha\cap B = \{x_\alpha\}$. Thus H does not witness that B is paradoxical, contrary to (v).

(i)$\Rightarrow$(v). Every $B\supseteq A$ is countably G-equidecomposable with $\mathbf{R}^n$ by the Banach–Schröder–Bernstein Theorem and so is G-paradoxical. ■

Note that 1.1 ($\neg$(i)$\Rightarrow\neg$(ii)) implies that for every set A which is not countably equidecomposable with $\mathbf{R}^n$ there exists a paradoxical set $B\supseteq A$ which is not countably equidecomposable with $\mathbf{R}^n$. Indeed, take any countable group H such that each H-orbit is infinite and let $B = HA$.

Using the above arguments one can also prove that

- If $\omega<\varkappa\leqslant 2^\omega$ then for every set A and for every group $G\supseteq T^n$ the following are equivalent:
  (i) $A \overset{G}{\underset{\varkappa}{\sim}} \mathbf{R}^n$;
  (ii) A belongs to no proper G-invariant $\varkappa$-complete ideal of subsets of $\mathbf{R}^n$.

COROLLARY 1.4. *For every n there is a nonparadoxical set of full Lebesgue measure in* $\mathbf{R}^n$.

Proof. Let A be a meager set such that $\mathbf{R}^n\backslash A$ is null (see [8], Th. 1.6). Apply 1.1 ($\neg$(iii)$\Rightarrow\neg$(v)) to $G = D^n$ to find a nonparadoxical set $B\supseteq A$. ■

The dual argument gives

COROLLARY 1.5. *For every n there is a nonparadoxical comeager set in* $\mathbf{R}^n$. ■

In contrast with 1.4 and 1.5, every set of positive Lebesgue measure contains a paradoxical subset of full measure and every nonmeager set with the Baire property contains a paradoxical subset such that the difference is meager. Again, this will be a consequence of a more general

PROPOSITION 1.6. *Let G be a subgroup of* D^n *such that every nonempty open set is G-paradoxical and let* $\mathfrak{I}$ *be a G-invariant σ-ideal in* $\mathbf{R}^n$. *If A is such that*

$$A\backslash A_1 \overset{G}{\underset{\infty}{\sim}} U\backslash U_1$$

for some A_1, U_1 *in* $\mathfrak{I}$ *and a nonempty open set U, then there exists B in* $\mathfrak{I}$ *such that* $A\backslash B$ *is G-paradoxical.*

Proof. Let H be a countable subgroup of G which witnesses that $A\backslash A_1 \overset{G}{\underset{\infty}{\sim}} U\backslash U_1$ and that U is G-paradoxical. Let $B = H(A_1\cup U_1)$. Then $B\in\mathfrak{I}$ and $|Hx\cap(A\backslash B)| = |Hx\cap U| = \omega$ for every $x\in A\backslash B$. Thus H witnesses that $A\backslash B$ is G-paradoxical. ■

COROLLARY 1.7. *Every set of positive Lebesgue measure contains a paradoxical subset of full measure.*

Proof. Let A be a set of positive Lebesgue measure and let U be an open set with the same measure. By a theorem of Banach and Tarski, there are null sets A_1 and U_1 such that

$$A\backslash A_1 \underset{\infty}{\sim} U\backslash U_1$$

and all the pieces of the decomposition are Lebesgue measurable (see [10], Th. 9.17, p. 140, or [1]). By 1.6, A contains a paradoxical subset C such that $A\backslash C$ is null. ■

COROLLARY 1.8. *Every nonmeager set with the Baire property contains a paradoxical subset such that the difference is meager.*

Proof. Use the Baire property to verify the hypothesis of 1.6. ■

2. In the previous section we proved that above every set which is not countably equidecomposable with $\mathbf{R}^n$ there exist paradoxical sets which are not countably equidecomposable with $\mathbf{R}^n$. We also proved that below every set which is "large" in the sense of measure or category there exist paradoxical sets which are still "large". We shall see that the last property is no longer true if we consider arbitrary uncountable sets. This gives rise to the following definition.

DEFINITION 2.1. A set A is *hereditarily nonparadoxical* if A has no uncountable paradoxical subsets.

The restriction to uncountable subsets is reasonable, since every countably infinite set is paradoxical. For every uncountable set A, if A is hereditarily nonparadoxical then A is nonparadoxical and the implication fails only for countably infinite sets and the empty set. As we shall see (Prop. 2.4), the family of hereditarily nonparadoxical sets is an ideal and that is the reason why all countable sets are said to be hereditarily nonparadoxical. Note that all countable sets are contained in the ideal generated by uncountable hereditarily nonparadoxical sets.

Recall that d_n denotes the Euclidean metric in $\mathbf{R}^n$. An obvious example of an uncountable hereditarily nonparadoxical set is given by

PROPOSITION 2.2. *If X is a subset of* $\mathbf{R}^n$ *such that* d_n *is 1-1 on* $[X]^2$, *then X is hereditarily nonparadoxical.* ■

Every subset X of $\mathbf{R}^n$ such that d_n is 1-1 on $[X]^2$ will be called a *set without repeated distances.*

P. Erdős proved that every subset of $\mathbf{R}^n$ of cardinality $\varkappa\geqslant\omega$ contains a set without repeated distances of cardinality $\varkappa$ (see [3]). Thus

- every subset of $\mathbf{R}^n$ contains a hereditarily nonparadoxical set of the same cardinality.

Section 2 is mainly devoted to the study of the relationship between sets without repeated distances and sets without uncountable paradoxical subsets, but we shall begin with several comments.

Vitali's set S (i.e., a selector from $\mathbf{Q}$-orbits) is countably equidecomposable with $\mathbf{R}$, by 1.1, so S is not hereditarily nonparadoxical.

F. B. Jones showed that there is a Hamel basis of $\mathbf{R}$ which contains a perfect set ([5], see also [7]). Therefore

- there are perfect sets which are hereditarily nonparadoxical.

By 1.7,

- a hereditarily nonparadoxical set has inner measure zero

and by 1.8,

- a hereditarily nonparadoxical set contains no nonmeager subsets with the Baire property.

EXAMPLE 2.3. *For every n there is a set without repeated distances in* $\mathbf{R}^n$ *which is both nonmeasurable and without the Baire property.*

Proof. We shall construct a set without repeated distances which meets every nonmeager G_δ set and every G_δ set of positive Lebesgue measure. This is simple for $n = 1$, so assume that $n > 1$.

Let $\{F_\alpha: \alpha < 2^\omega\}$ be an enumeration of all G_δ subsets of $\mathbf{R}^n$ which either are of positive Lebesgue measure or are nonmeager. We define inductively an increasing sequence $\{A_\alpha: \alpha < 2^\omega\}$ of subsets of $\mathbf{R}^n$ such that for every α, d_n is 1-1 on $[A_\alpha]^2$, $|A_\alpha| = |\alpha|$, and $A_\alpha \cap F_\alpha \neq \emptyset$.

Suppose that $\{A_\beta: \beta < \alpha\}$ is already defined. For $z, t \in \bigcup_{\beta<\alpha} A_\beta$, $z \neq t$, let $Z_{z,t} = \{x: d_n(\{x, z\}) = d_n(\{x, t\})\}$; note that $Z_{z,t}$ is an $(n-1)$-dimensional affine subspace of $\mathbf{R}^n$. For $y, z, t \in \bigcup_{\beta<\alpha} A_\beta$, $z \neq t$, let $Y_{y,z,t} = \{x: d_n(\{x, z\}) = d_n(\{y, t\})\}$, an $(n-1)$-dimensional sphere in $\mathbf{R}^n$. Let

$$X_\alpha = \bigcup \{Z_{z,t}: z, t \in \bigcup_{\beta<\alpha} A_\beta, \ z \neq t\} \cup \bigcup \{Y_{y,z,t}: y, z, t \in \bigcup_{\beta<\alpha} A_\beta, \ z \neq t\}.$$

Choose an affine $(n-1)$-dimensional space V which is parallel to no $Z_{z,t}$. Again, choose an affine $(n-2)$-dimensional subspace of V which is parallel to no $Z_{z,t} \cap V$, and so on. After $n-1$ steps you will have chosen a line E such that the intersection of E with each $Z_{z,t}$ has cardinality 1.

Obviously, the intersection of E with any sphere contained in $\mathbf{R}^n$ has cardinality at most 2, hence $|E \cap X_\alpha| < 2^\omega$. Any line parallel to E has the same property.

Case 1: F_α is of positive Lebesgue measure. Choose a line E_1 parallel to E and such that $\lambda_1(E_1 \cap F_\alpha) > 0$. This can be done by Fubini's theorem (see [8], p. 53).

Case 2: F_α is nonmeager. Choose a line E_1 parallel to E and such that $E_1 \cap F_\alpha$ is nonmeager. This can be done by the Kuratowski–Ulam Theorem (see [8], Th. 15.4, p. 57).

In both cases take $a_\alpha \in E_1 \cap (F_\alpha \setminus X_\alpha)$ (see [8], Th. 5.1, p. 23) and let $A_\alpha = \bigcup_{\beta<\alpha} A_\beta \cup \{a_\alpha\}$. Clearly, d_n is 1-1 on $[A_\alpha]^2$.

Finally, let $A = \bigcup_{\alpha<2^\omega} A_\alpha$. Then A is a set without repeated distances and A meets every nonmeager G_δ set and every G_δ set of positive Lebesgue measure, hence A is not meager and is not null. By 1.7 it is nonmeasurable and by 1.8 it lacks the Baire property. ∎

The following proposition ensures the existence of uncountable hereditarily nonparadoxical sets which are not just sets without repeated distances.

PROPOSITION 2.4. *The family of hereditarily nonparadoxical sets is a proper invariant ideal.*

In the proof of 2.4 we shall need

LEMMA 2.5. *For every subset A of* $\mathbf{R}^n$, *the following are equivalent:*

(i) *A is hereditarily nonparadoxical;*

(ii) *for every countable group G*

$$|\{x \in \mathbf{R}^n: |Gx \cap A| = \omega\}| \leqslant \omega. \ ∎$$

Proof of 2.4. If both A and B satisfy 2.5(ii), then so does $A \cup B$. The ideal is obviously proper and invariant. ∎

Note that the family of hereditarily nonparadoxical sets is not a σ-ideal. Indeed, if A is uncountable and B is countably infinite then $A+B$ is $\langle B\rangle$-paradoxical.

The next proposition compared with 2.5 shows the combinatorial relationship between hereditarily nonparadoxical sets and sets without repeated distances.

PROPOSITION 2.6. *If X is such that* d_n *is 1-1 on* $[X]^2$, *then for every infinite subgroup G of* D^n

$$\big|\{x \in \mathbf{R}^n: |Gx \cap X| > 1\}\big| \leqslant |G|.$$

Proof. Suppose not. Let S be a selector from those G-orbits which intersect X in sets containing at least two elements. For every $s \in S$ choose distinct $x_s, y_s \in Gs \cap X$ and $g_s \in G \setminus \{\mathrm{id}\}$ such that $g_s x_s = y_s$. Since $|S| > |G|$, the function $s \to g_s$ cannot be 1-1. Take $s, t \in S$ such that $s \neq t$ and $g_s = g_t$. Then $g_s[\{x_s, x_t\}] = \{y_s, y_t\}$, thus $d_n(\{x_s, x_t\}) = d_n(\{y_s, y_t\})$, a contradiction. ∎

The ideal of hereditarily nonparadoxical sets is not generated by sets without repeated distances. Indeed, by an observation of A. Krawczyk, the set of all natural numbers is hereditarily nonparadoxical and is not the union of finitely many sets without repeated distances. The last property is a consequence of the following theorem due to van der Waerden:

- Suppose the set of natural numbers is the union of finitely many sets A_i, $i < k$. Then at least one of A_i contains arithmetic progressions of any finite length (see [9]).

However, all countable sets were added in the definition of hereditarily nonparadoxical sets in a somewhat unnatural way. Thus it seems interesting to ask whether the ideal of hereditarily nonparadoxical sets is generated by sets without repeated distances together with all countable sets. The following example answers negatively this question.

EXAMPLE 2.7. *For every n there exists a hereditarily nonparadoxical set A in* $\mathbf{R}^n$ *such that A is not the union of a countable set and finitely many sets each without repeated distances.*

Proof. Consider $\mathbf{R}$ canonically embedded in $\mathbf{R}^n$. Define inductively an uncountable set $B \subseteq [0, 1/2)$ such that the distance function d_1 is 1-1 on $[B]^2$. Partition B into a countably infinite number of uncountable sets A_k and let

$$A = \bigcup \{k + A_i: k < i; \ i, k \in \omega\}.$$

We shall show that A is not the union of a countable set and finitely many sets each without repeated distances. For suppose otherwise and let $A = C \cup \bigcup_{j<m} X_j$, where C is a countable set, $m \in \omega$, and d_1 is 1-1 on each $[X_j]^2$. By 2.6, for every j

$$\left|\{x \in \mathbf{R}\colon |\mathbf{Z}x \cap X_j| > 1\}\right| \leqslant \omega.$$

Take $i > m$. The set A_i is uncountable, hence there is $x \in A_i$ such that for every $j < m$, $|\mathbf{Z}x \cap X_j| \leqslant 1$ and $x \notin \mathbf{Z}C$. Then

$$|\mathbf{Z}x \cap (C \cup \bigcup_{j<m} X_j)| \leqslant m.$$

But $|\mathbf{Z}x \cap A| > m$ for every $x \in A_i$, contrary to the assumption that $A = C \cup \bigcup_{j<m} X_j$.

We claim that A is hereditarily nonparadoxical as a subset of $\mathbf{R}^n$.

Let G be a countable subgroup of D^n and let X be an uncountable subset of A. We shall show that G does not witness that X is paradoxical.

Fix t and s such that $|X \cap (s + A_t)| > \omega$. Let

$$Y = \{x \in X \cap (s+A_t)\colon |Gx \cap \bigcup \{k + A_t\colon k < t\}| < \omega\}.$$

By 2.4, $\bigcup \{k + A_t\colon k < t\}$ is hereditarily nonparadoxical. Thus, by 2.5, Y is uncountable.

CLAIM. $Y \not\prec^G_\infty \bigcup \{k + A_i\colon i \neq t;\ i, k \in \omega\}$.

Suppose that $Y \prec^G_\infty \bigcup \{k + A_i\colon i \neq t;\ i, k \in \omega\}$. Then there is a partition $\{Y_l\colon l \in \omega\}$ of Y and isometries $g_l \in G$ such that $\bigcup_{l \in \omega} g_l Y_l \subseteq \bigcup \{k + A_i\colon i \neq t;\ i, k \in \omega\}$.

At least one of the sets Y_l is uncountable and hence contains two different elements which are moved by the same isometry. Recall that $B \subseteq [0, 1/2)$ has no repeated distances, hence $\bigcup \{k + A_i\colon i \neq t;\ i, k \in \omega\}$ does not contain any two elements such that the distance between them is equal to the distance between some elements of Y, a contradiction.

Since $Y \not\prec^G_\infty \bigcup \{k + A_i\colon i \neq t;\ i, k \in \omega\}$, there is $x_0 \in Y$ such that

$$|Gx_0 \cap Y| > |Gx_0 \cap \bigcup \{k + A_i\colon i \neq t;\ i, k \in \omega\}|.$$

Since $x_0 \in Y$, $|Gx_0 \cap \bigcup \{k + A_t\colon k < t\}| < \omega$ and

$$|Gx_0 \cap \bigcup \{k + A_i\colon k < i;\ i, k \in \omega\}| < \omega.$$

Thus $|Gx_0 \cap X| < \omega$ and so G does not witness that X is paradoxical, since $x_0 \in X$. ∎

3. The main purpose of this section is to prove

THEOREM 3.1. ($\neg$CH) *For every n, the countable union of hereditarily nonparadoxical sets in $\mathbf{R}^n$ has inner measure zero and does not contain nonmeager subsets with the Baire property.*

Our theorem gives an apparent generalization of the following result by Erdős and Kunen:

- ($\neg$CH) The countable union of sets without repeated distances has inner measure zero (see [3], p. 136).

Unfortunately, the author does not know any example of a set which is hereditarily nonparadoxical and is not the union of countably many sets each without repeated distances.

QUESTION. Is every hereditarily nonparadoxical set the countable union of sets each without repeated distances?

Obviously, if we assume that $2^\omega = \omega_1$, then the answer is "yes" by a result of Kunen:

- (CH) For every n, $\mathbf{R}^n$ is the union of countably many sets each without repeated distances (see [6]).

LEMMA 3.2. *If $\varkappa > \omega_1$ and $f\colon \varkappa \to [\omega_1]^{\omega_1}$ then there exist an uncountable subset S of $\varkappa$ and a countable subset C of ω_1 such that $|f(\xi) \cap C| = \omega$ for each $\xi \in S$.*

Proof. For every ξ let $g_\xi\colon \omega_1 \to f(\xi)$ be the enumeration of elements of $f(\xi)$ such that if $\alpha < \beta$ then $g_\xi(\alpha) < g_\xi(\beta)$. Define $g\colon \varkappa \to \omega_1$ by $g(\xi) = g_\xi(\omega)$. There exists $\zeta \in \omega_1$ such that $|g^{-1}[\{\zeta\}]| > \omega_1$. Let $S = g^{-1}[\{\zeta\}]$ and let $C = [0, \zeta)$. Obviously, $|f(\xi) \cap C| = \omega$ for every $\xi \in S$. ∎

LEMMA 3.3. *If A is hereditarily nonparadoxical and G is a subgroup of D^n such that $|G| = \omega_1$, then*

$$\left|\{x \in \mathbf{R}^n \setminus \mathrm{Fix}(G)\colon |Gx \cap A| = \omega_1\}\right| \leqslant \omega_1.$$

Proof. Suppose, towards a contradiction, that

$$\left|\{x \in \mathbf{R}^n \setminus \mathrm{Fix}(G)\colon |Gx \cap A| = \omega_1\}\right| > \omega_1.$$

Let T be a selector from the G-orbits having uncountable intersection with A and omitting $\mathrm{Fix}(G)$. By our assumption $|T| > \omega_1$. Define $f\colon T \to [G]^{\omega_1}$ by $f(t) = \{g \in G\colon gt \in A\}$ for $t \in T$. By 3.2 there is an uncountable subset S of T and a countable subset C of G such that $|f(s) \cap C| = \omega$ for every $s \in S$, and because $Gs \cap \mathrm{Fix}(G) = \emptyset$ the last implies that $|Cs \cap A| = \omega$ for every $s \in S$.

Let $H = \langle C \rangle$ and let $B = \{x \in A\colon \exists g \in C\ \exists s \in S\ gs = x\}$. The group H is countable and B is an uncountable H-paradoxical subset of A, contrary to the assumption that A is hereditarily nonparadoxical. ∎

LEMMA 3.4. *If A is the union of countably many hereditarily nonparadoxical sets and G is a subgroup of D^n such that $|G| = \omega_1$, then*

$$\left|\{x \in \mathbf{R}^n \setminus \mathrm{Fix}(G)\colon |Gx \cap A| = \omega_1\}\right| \leqslant \omega_1.$$

Proof. Obvious from 3.3. ∎

We shall also use the following folklore lemma (see [3], p. 136).

LEMMA 3.5. *Let A be a subset of $\mathbf{R}^n$. If either*

(i) *A has positive Lebesgue measure, or*

(ii) *A is nonmeager and has the Baire property,*

then there are perfect sets P, Q such that $P + Q \subseteq A$.

Proof of 3.1. Let A be a set of positive Lebesgue measure and suppose that A is the union of countably many hereditarily nonparadoxical sets. By 3.5(i) there are perfect sets P, Q such that $P+Q \subseteq A$. Choose $P_1 \subseteq P$ of cardinality ω_1. Then

$$\left|\{x\in \mathbf{R}^n:\ |\langle P_1\rangle x \cap (P_1+Q)| = \omega_1\}\right| = |Q|$$

and, under $\neg$CH, $|Q| > \omega_1$. But P_1+Q, being a subset of A, is the union of countably many hereditarily nonparadoxical sets. Thus we get a contradiction with 3.4.

For category use 3.5(ii) instead of 3.5(i). ■

Note that for sets without repeated distances the proof may be much simpler. Indeed, use 2.6 instead of 3.3.

A simple consequence of 3.4 is the fact that, under $\neg$CH, no sphere in $\mathbf{R}^n$ $(n > 1)$ is the union of countably many hereditarily nonparadoxical sets.

We give another application of 3.4. Every set of rationally independent numbers contained in $\mathbf{R}$ is hereditarily nonparadoxical. We shall show that the situation may be completely different in $\mathbf{R}^n$, for $n > 1$.

Example 3.6. ($\neg$CH) *For every $n > 1$ there is a set of rationally independent vectors in $\mathbf{R}^n$ which is not the union of countably many hereditarily nonparadoxical sets.*

Proof. Assume $2^\omega > \omega_1$. Consider $\mathbf{R}^2$ canonically embedded in $\mathbf{R}^n$. Let S be the circle contained in $\mathbf{R}^2$ and let X be a subset of S, of cardinality ω_1, consisting of rationally independent vectors. We shall define inductively a sequence $\{A_\alpha:\ \alpha < 2^\omega\}$ such that:

(i) if $\alpha < \beta$ then $A_\alpha \subseteq A_\beta$,

(ii) for every α, A_α is a set of rationally independent vectors of cardinality $|\alpha|+\omega_1$,

(iii) for every α, $A_\alpha = \bigcup_{\beta<\alpha} A_\beta \cup (r_\alpha \cdot X)$, where $r_\alpha \in \mathbf{R}$ and $\cdot$ denotes multiplication of vectors from a linear space over $\mathbf{R}$ by elements of $\mathbf{R}$.

Let $A_\emptyset = X$ and suppose $\{A_\beta:\ \beta < \alpha\}$ is already defined. Let $V = \mathrm{Lin}_{\mathbf{Q}}(X)$. Note that for every $r \in \mathbf{R}$, $r\cdot V$ is the linear space generated by $r\cdot X$ over $\mathbf{Q}$.

Let W be the linear space generated by $\bigcup_{\beta<\alpha} A_\beta$ over $\mathbf{Q}$. Define the function $f\colon W\times V \to \mathbf{R}$ as follows: for $w\in W$ and $v \in V$

$$f(w, v) = \begin{cases} r & \text{if } r \text{ is such that } r\cdot v = w,\ v, w \neq 0, \\ 0 & \text{if there is no such } r. \end{cases}$$

Since $|W| < 2^\omega$ by the induction hypothesis and $|V| < 2^\omega$, there is $r_\alpha \notin f[W\times V] \cup \{0\}$.

We shall show that $A_\alpha = \bigcup_{\beta<\alpha} A_\beta \cup (r_\alpha \cdot X)$ satisfies (ii). The set $\bigcup_{\beta<\alpha} A_\beta$ is a basis of W and $r_\alpha \cdot X$ is a basis of $r_\alpha \cdot V$. By the choice of r_α, $W \cap (r_\alpha\cdot V) = \{0\}$, thus A_α is a basis of the linear space $W+(r_\alpha\cdot V)$.

Finally, let $A = \bigcup_{\alpha<2^\omega} A_\alpha$. Clearly, A consists of rationally independent vectors. We claim that A is not the union of countably many hereditarily nonparadoxical sets. Indeed, let G be the group of rotations about 0 generated by the rotations which move elements of X into elements of X. Then $|G| = \omega_1$, $\mathrm{Fix}(G) = \{0\}$ and

$$\left|\{x\in A\setminus \mathrm{Fix}(G):\ |Gx\cap A| = \omega_1\}\right| = 2^\omega.$$

By 3.4 the set A is not the union of countably many hereditarily nonparadoxical sets. ■

4. This section was inspired by the paper [3] of Erdős. We consider unions of sets without repeated distances.

Baumgartner proved that if A is a set without repeated distances, then $\mathbf{R}\setminus A$ contains an infinite arithmetic progression (see [3], p. 135). The following lemma ((i)$\Rightarrow$(iii)) shows that for every set A which is not countably T^n-equidecomposable with the real line and for every countable set X there exists a copy of X which is contained in $\mathbf{R}\setminus A$.

Lemma 4.1. *Let $\varkappa$ be a cardinal such that $\omega < \varkappa \leqslant 2^\omega$ and let A be a subset of $\mathbf{R}^n$. The following are equivalent:*

(i) $A \overset{T^n}{\underset{\varkappa}{\not\sim}} \mathbf{R}^n$;

(ii) *for every subgroup H of T^n, if $|H| < \varkappa$ then $HA \neq \mathbf{R}^n$;*

(iii) *for every set X of cardinality less than $\varkappa$ there is a translation t such that $tX \subseteq \mathbf{R}^n\setminus A$.*

Proof. (i)$\Rightarrow$(ii). (Compare with 1.1(ii)$\Rightarrow$(i).) Suppose that $HA = \mathbf{R}^n$. Let S be a selector from H-orbits and let S_1 be a subset of S of cardinality $|H|$. Obviously, $|\langle H\cup S_1\rangle x \cap A| = |H| = |\langle H\cup S_1\rangle x|$ for every $x\in\mathbf{R}^n$. But this means that $A \overset{T^n}{\underset{\varkappa}{\sim}} \mathbf{R}^n$, contrary to (i).

(ii)$\Rightarrow$(i) is obvious.

(iii) is a reformulation of (ii). Indeed, for (ii)$\Rightarrow$(iii) consider $H = \langle X\rangle$. Then take $t\in T^n$ such that $t(0)\notin HA$. For (iii)$\Rightarrow$(ii) consider $X = \{h(0):\ h\in H\}$. ■

Theorems 4.2 and 4.3 give a strengthening of the following result due to Erdős:

- ($\neg$CH) If A is the union of countably many sets each without repeated distances, then there are a linear space V over $\mathbf{Q}$ of size ω_1 and a translation t such that $tV\cap A = \emptyset$ (see [3]).

Theorem 4.2 states, under the same assumptions, that *for every* linear space V over $\mathbf{Q}$ of size *less than* 2^ω there is a translation t such that $tV\cap A = \emptyset$.

Theorem 4.2. *Let $\varkappa$ be a cardinal such that $\varkappa^+ < 2^\omega$ and let A be the union of $\varkappa$ many sets each without repeated distances. Then*

(i) $A \underset{2^\omega}{\not\sim} \mathbf{R}^n$ (*i.e. A generates a proper D^n-invariant 2^ω-complete ideal*),

(ii) *for every set X of cardinality less than 2^ω there is a translation t such that $tX \subseteq \mathbf{R}^n\setminus A$.*

Proof. (i) If $2^\omega = \omega_1$ then A is the union of finitely many sets each without repeated distances and thus A is hereditarily nonparadoxical by 2.4. Hence (i) follows from 1.1(i)$\Rightarrow$(v). So we may assume that $2^\omega > \omega_1$ and $\varkappa$ is an infinite cardinal.

Let $A = \bigcup_{\alpha<\varkappa} A_\alpha$ and suppose d_n is 1-1 on $[A_\alpha]^2$ for every α. Suppose not (i) and let G be a group of cardinality less than 2^ω which witnesses that $A \underset{2^\omega}{\sim} \mathbf{R}^n$. Without loss of generality we can assume that $|G| = \lambda \geqslant \varkappa^+$ and every G-orbit has cardinality λ, for some $\lambda < 2^\omega$ (we can enlarge G by adding $\varkappa^+$ translations).

By 2.6, $|\{x \in \mathbf{R}^n: |Gx \cap A_\alpha| > 1\}| \leqslant \lambda$ for every α. Hence $|\{x \in \mathbf{R}^n: |Gx \cap A| = \lambda\}| \leqslant \lambda$, since $\lambda > \varkappa$. But $\lambda < 2^\omega$, so there is $x_0 \in \mathbf{R}^n$ such that

$$|Gx_0 \cap A| < \lambda = |Gx_0|,$$

contrary to the assumption that G witnesses that $A \underset{2^\omega}{\sim} \mathbf{R}^n$.

(ii) Obviously, $A \underset{2^\omega}{\sim} \mathbf{R}^n$ implies that $A \overset{T^n}{\underset{2^\omega}{\sim}} \mathbf{R}^n$ and this gives (ii) by 4.1. ∎

Note that 4.3(i) implies that if $\varkappa^+ < 2^\omega$ then $\mathbf{R}^n$ is not the union of $\varkappa$ many sets each without repeated distances. Of course, using the same argument as in Section 3, it can be proved that such a union has inner measure zero and does not contain nonmeager subsets with the Baire property (if $2^\omega = \omega_1$ then the union is hereditarily nonparadoxical by 2.4).

The next theorem states that for existence of a linear space omitting the countable union of sets each without repeated distances the translation t is superfluous.

THEOREM 4.3. ($\neg$CH) *Let $\varkappa$ be a cardinal such that $\varkappa^+ < 2^\omega$ and let A be the union of $\varkappa$ many sets each without repeated distances. If $0 \notin A$, then there is a linear space V over $\mathbf{Q}$ of size 2^ω such that $A \cap V = \emptyset$.*

Proof (1). Assume that $0 \notin A$. We define inductively a sequence $\{X_\alpha: \alpha < 2^\omega\}$ such that:

(i) $X_\emptyset = \{0\}$,
(ii) if $\alpha < \beta$, then $X_\alpha \subseteq X_\beta$,
(iii) $|X_\alpha| = |\alpha| + 1$ for every α,
(iv) $\mathrm{Lin}_\mathbf{Q}(X_\alpha) \cap A = \emptyset$ for every α.

By the assumption that $0 \notin A$, (iv) is satisfied for $\alpha = \emptyset$.

Suppose that $\{X_\beta: \beta < \alpha\}$ is already defined for some $\alpha < 2^\omega$. If α is a limit ordinal, then let $X_\alpha = \bigcup_{\beta < \alpha} X_\beta$. If $\alpha = \beta + 1$, then let $B = \mathbf{Q} \cdot A$, where $\cdot$ denotes multiplication. The set B is the union of $\max(\varkappa, \omega)$ many sets each without repeated distances and $\max(\varkappa, \omega)^+ < 2^\omega$, since $\varkappa^+ < 2^\omega$ and $2^\omega > \omega_1$.

Consider $X = \mathrm{Lin}_\mathbf{Q}(X_\beta)$. Since X has cardinality less than 2^ω, by 4.2(ii), there is a translation t such that $tX \cap B = \emptyset$. Define $X_\alpha = X_\beta \cup \{t(0)\}$. We only need to check (iv).

Suppose $x \in \mathrm{Lin}_\mathbf{Q}(X_\alpha) \cap A$. Then $x = r \cdot t(0) + v$, where $r \in \mathbf{Q}$ and $v \in \mathrm{Lin}_\mathbf{Q}(X_\beta)$. By the induction hypothesis, we can assume that $r \neq 0$. Then $(1/r) \cdot x \in (t(0) + \mathrm{Lin}_\mathbf{Q}(X_\beta)) \cap B$, contradicting the choice of t.

Finally, let $V = \mathrm{Lin}_\mathbf{Q}(\bigcup_{\alpha < 2^\omega} X_\alpha)$. Then $|V| = 2^\omega$ by (iii) and $V \cap A = \emptyset$ by (iv). ∎

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(1) P. Zakrzewski, in a conversation with me, showed the existence of a subgroup of $\mathbf{R}^n$ of cardinality ω_1 which is disjoint from a given set of measure zero. We use a similar method in our proof.

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