On a theorem of Baumgartner and Weese

by

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Abstract. A model of ZFC+MA(σ-linked) is constructed in which the continuum c is large and each Boolean algebra of cardinality $\leqslant c$ is representable as the algebra of partitioners of some maximal antichain in $P(\omega)$/fin.

0. Let us consider the notion of partitioner-representability of Boolean algebras introduced by Baumgartner and Weese in [B-W]: if A is a m.a.d. (a maximal family of almost disjoint subsets of ω), then a set $x \subseteq \omega$ is called a *partitioner* of A if for each $a \in A$ either $a \subseteq_* x$ or $a \cap x =_* \emptyset$ (i.e. either $a \setminus x$ or $a \cap x$ is finite, respectively). Finite unions $a_1 \cup \ldots \cup a_n$ of elements of A as well as finite subsets of ω are called *trivial partitioners*. The family $F(A)$ of all the partitioners is a Boolean subfield of $P(\omega)$. We say that a given Boolean algebra B is *representable* on A if B is isomorphic to the factor algebra $F(A)/T$, where T denotes the ideal generated by trivial partitioners.

The fundamental theorem in [B-W] (see also [F-Z_1]) says that, assuming CH (the continuum hypothesis), each Boolean algebra of cardinality $\leqslant c$ is representable on some m.a.d. A in the sense described above.

In [F-Z_2] it is proved that—consistently with MA (Martin's Axiom)—the power set algebra $P(\omega_1)$ may not be representable. Hence, the assumption of CH in the theorem of Baumgartner and Weese cannot be replaced by MA.

On the other hand, a question arises (see Problem 7 in [B-W]) if the conclusion of the theorem is equivalent to CH. Here we answer this question negatively. Thus, we prove the following

THEOREM. *It is consistent with* ZFC+MA(σ-*linked*) *that the cardinality c is arbitrarily large and each Boolean algebra of cardinality $\leqslant c$ is representable on some m.a.d.*

In the proof we apply the technique of Laver [L] in a similar way to [B-F-Z].

1. Assume that the ground model V satisfies the generalized continuum hypothesis and choose a regular cardinal $\kappa > \omega_1$. Hence, κ satisfies $\kappa^{<\kappa} = \kappa$.

We define below a finite support iteration $\mathbf{P}_\kappa = \sum_{\alpha<\kappa} \mathbf{P}_\alpha$ having c.c.c. (countable chain condition) so that, in the corresponding generic extension $V[G]$, the conclusion of the Theorem will hold true. At some stages $\alpha < \kappa$ of the iteration we add generic subsets

$X_\alpha \subseteq \omega$ filling gaps in subalgebras of $P(\omega)/\mathrm{fin}$ generated by some previously added X_β, $\beta < \alpha$. Consequently, in $V[G]$ each algebra B, with $\operatorname{card} B \leqslant c$, will be embeddable in $P(\omega)/\mathrm{fin}$ (comp. [B-F-Z]).

More exactly, for Φ: $\kappa \to \{0, 1\}$, let $B(\Phi)$ be the subalgebra of $P(\omega)/\mathrm{fin}$ generated by $\{X_\alpha$: $\Phi(\alpha) = 1\}$. For each B with $\operatorname{card} B \leqslant c$, we find a Φ such that B is isomorphic to $B(\Phi)$. Simultaneously, at some other stages β, we add generic subsets $a_\beta \subseteq \omega$ so that, for each Φ as above, there is a m.a.d. $A(\Phi)$ consisting of some of the a's on which B is representable. In other words, for each B we choose some branch Φ of the binary tree $\bigcup_{\alpha<\kappa}\{0, 1\}^\alpha$ and force along Φ so that $B = B(\Phi)$ and B is representable on $A(\Phi)$. Since $\kappa^{<\kappa} = \kappa$ all this can be done in κ steps with the help of an ordinary (i.e. linear) iteration of length κ.

The definition of the iteration is inductive and uses a well known "booking" technique. At each step $\alpha < \kappa$ we fix an enumeration $e_\alpha = \{d(\alpha, \xi)$: $\xi < \kappa\}$ of some objects in V, so that each object $d(\alpha, \xi)$ occurs κ times in e_α.

Let J: $\kappa \times \kappa \to \kappa$ be a pairing function satisfying

$$\xi, \eta < J(\xi, \eta) \quad \text{for all } \xi, \eta < \kappa$$

and define $\operatorname{Nr}(d(\alpha, \xi)) = J(\alpha, \xi)$. Thus, each object d (enumerated at any stage) has arbitrarily large number $\operatorname{Nr}(d) < \kappa$ and each ordinal $\alpha < \kappa$ is the number $\alpha = \operatorname{Nr}(d)$ of some object d enumerated at some stage $< \alpha$.

Assume the following notation. For φ: $\alpha \to \{0, 1\}$ let $B(\varphi)$ be the subalgebra generated by $\{X_\beta$: $\varphi(\beta) = 1\}$. Thus, if s is a finite zero-one sequence with $\operatorname{dm}(s) \subseteq \{\beta$: $\varphi(\beta) = 1\}$ and

$$X(s) = \bigcap_{s(\xi)=0} X_\xi \cap \bigcap_{s(\eta)=1} (\omega \setminus X_\eta),$$

then $B(\varphi)$ consists of finite unions of sets of the form $X(s)$. A *gap* in $B(\varphi)$ is a system of the form

$$L = \langle \{X(s)\colon s \in S\};\ \{X(t)\colon t \in T\} \rangle$$

where $X(s) \cap X(t) =_* \emptyset$ for each $s \in S$ and $t \in T$. Moreover, we assume that the conditions $s_1, \ldots, s_n \in S$ and $X(s) \subseteq_* X(s_1) \cup \ldots \cup X(s_n)$ imply $s \in S$, and similarly for T.

Now, we can describe the inductive step. Assume that $\mathbf{P}_\alpha$ is already defined. We enumerate all the pairs $\langle X, \varphi\rangle$, $\langle Y, \varphi\rangle$, $\langle L, \varphi\rangle$ of $\mathbf{P}_\alpha$-names such that $\mathbf{P}_\alpha$ forces the following properties:

(1) $X, Y \subseteq \omega$, $X, Y \notin \mathrm{fin}$ and $\omega \setminus Y \notin \mathrm{fin}$.

(2) $X \in B(\varphi)$, $Y \notin B(\varphi)$ and L is a gap in $B(\varphi)$.

(3) $\varphi \in D_\alpha$, where D_α consists of all $\psi \in V^{(\mathbf{P}_\alpha)}$ with $\operatorname{dm}(\psi) \leqslant \alpha$ and such that if $\{\gamma_\xi$: $\xi < \delta\}$ is an increasing enumeration of $\{\gamma$: $\psi(\gamma) = 1\}$, then for each $\xi < \delta$ there is a gap L_ξ in $B(\psi|\gamma_\xi + 1)$ satisfying $\gamma_{\xi+1} = \operatorname{Nr}(L_\xi, \psi|\gamma_\xi + 1)$.

We extend the function Nr to the just enumerated objects as described above.

We also assume inductively that there are a.d. families $A_\alpha(\varphi) \subseteq V^{(\mathbf{P}_\alpha)}$ having the property

$$A_\alpha(\varphi) = \bigcup_{\xi<\delta} A_{\gamma_\xi+1}(\varphi|\gamma_\xi + 1)$$

where, as in (3) above, γ_ξ enumerates the set $\{\gamma$: $\varphi(\gamma) = 1\}$ and such that each infinite $X \in B(\varphi)$ is a partitioner of $A_\alpha(\varphi)$ and $X \notin [A_\alpha(\varphi)]$ (the ideal generated by $A_\alpha(\varphi)$).

Now, the ordinal α determines a pair $\langle X, \varphi\rangle$, $\langle Y, \varphi\rangle$ or $\langle L, \varphi\rangle$ enumerated at some stage $\beta < \alpha$. Accordingly, we distinguish three cases.

Case 1: $\alpha = \operatorname{Nr}(X, \varphi)$. We add a new element $a = a_\alpha(X, \varphi)$ under X and almost disjoint from $A_\alpha(\varphi)$. We may assume that we have a fixed ultrafilter $F \subseteq B(\varphi)$ containing X in $V^{(\mathbf{P}_\alpha)}$ (i.e. $\mathbf{P}_\alpha$ forces all this). Let $\mathbf{R}$ be the almost disjoint forcing over the family

$$H = \{\omega \setminus Z\colon Z \in F\} \cup A_\alpha(\varphi).$$

Thus, the conditions p of $\mathbf{R}$ have the form $p = \langle x_p, w_p\rangle$, where x_p is a zero-one sequence of some length $l(p)$ and w_p is a finite subset of H. The relation $p \leqslant q$ holds if and only if $x_q \subseteq x_p$, $w_q \subseteq w_p$ and for each i with $l(q) \leqslant i < l(p)$, the condition $i \in \bigcup w_q$ implies $x_p(i) = 0$.

It is well known that we have

$$\mathbf{P}_\alpha \Vdash \text{“}\mathbf{R} \text{ is } \sigma\text{-linked”}.$$

By the inductive assumption the set $\omega \setminus \bigcup w_p$ is infinite for each p. Hence, if $G \subseteq \mathbf{R}$ is a generic filter, then the set

$$a_\alpha = \{i \in \omega\colon \exists p \in G\,[x_p(i) = 1]\}$$

is infinite, almost disjoint from $A_\alpha(\varphi)$ and $a_\alpha \subseteq_* Z$ for each $Z \in F$. Define

$$\mathbf{P}_{\alpha+1} = \mathbf{P}_\alpha * \mathbf{R}, \qquad A_{\alpha+1}(\varphi) = A_\alpha(\varphi) \cup \{a_\alpha\}.$$

It follows easily that the inductive assumption is still valid for $B(\varphi)$ and $A_{\alpha+1}(\varphi)$.

Case 2: $\alpha = \operatorname{Nr}(Y, \varphi)$. Suppose that

$$\mathbf{P}_\alpha \Vdash \text{“}\forall x \in B(\varphi)\,[(Y \setminus x) \cup (x \setminus Y) \notin [A_\alpha(\varphi)]]\text{”},$$

i.e. Y is congruent mod $[A_\alpha(\varphi)]$ to no element of $B(\varphi)$. Then the complement $\omega \setminus Y$ has the same property and it follows that the set

$$J = \{x \in B(\varphi)\colon \text{either } x \cap Y \in [A_\alpha(\varphi)] \text{ or } x \setminus Y \in [A_\alpha(\varphi)]\}$$

is a proper ideal in $B(\varphi)$. Let F be any ultrafilter in $B(\varphi)$ extending the set $-J = \{\omega \setminus x\colon x \in J\}$. Thus, for each $x \in F$ we have $x \cap Y \notin [A_\alpha(\varphi)]$ and $x \setminus Y \notin [A_\alpha(\varphi)]$.

Hence, we may apply the almost disjoint forcing $\mathbf{R}(Y)$ which adds a set a under each $x \cap Y$ for $x \in F$, and $\mathbf{R}(\omega \setminus Y)$ adding a set b under the family $\{x \setminus Y\colon x \in F\}$ so that both a and b are almost disjoint from $A_\alpha(\varphi)$. Define

$$\mathbf{P}_{\alpha+1} = \mathbf{P}_\alpha * \mathbf{R}(Y) * \mathbf{R}(\omega \setminus Y), \qquad A_{\alpha+1}(\varphi) = A_\alpha(\varphi) \cup \{a_\alpha\}, \text{ where } a_\alpha = a \cup b.$$

It follows that the inductive assumption holds for $B(\varphi)$ and $A_{\alpha+1}(\varphi)$. Since Y intersects nontrivially the element $a \cup b \in A_{\alpha+1}(\varphi)$ we infer that Y is now not a partitioner of $A_{\alpha+1}(\varphi)$.

Case 3: $\alpha = \mathrm{Nr}(L, \varphi)$. Suppose that

$$L = \langle\{X(s)\colon s\in S\};\ \{X(t)\colon t\in T\}\rangle.$$

We enlarge L by adjoining all the elements of $A_\alpha(\varphi)$. Thus, let

$$S_A = \{\xi < \alpha\colon \forall t\in T[a_\xi \cap X(t) =_* \emptyset]\}, \quad T_A = \{\eta < \alpha\colon \exists t\in T[a_\eta \subseteq_* X(t)]\}.$$

Define

$$L^* = \langle\{X(s)\}_{s\in S} \cup \{a_\xi\}_{\xi\in S_A};\ \{X(t)\}_{t\in T} \cup \{a_\eta\}_{\eta\in T_A}\rangle.$$

Now, we apply Kunen's forcing filling the gap L^*:

$$\mathbf{P}_{\alpha+1} = \mathbf{P}_\alpha * \mathbf{E}(L^*).$$

The conditions $p\in\mathbf{E}(L^*)$ have the form $p = \langle u_p, x_p, w_p\rangle$, where u_p and w_p are finite subsets of $S\cup S_A$ and $T\cup T_A$, respectively, and x_p is a finite zero-one sequence of some length $l(p)$, so that $U\cap W \subseteq l(p)$ for each $U\in\bar{u}_p$ and $W\in\bar{w}_p$, where

$$\bar{u}_p = \{X(s)\colon s\in S\} \cup \{a_\xi\colon \xi\in S_A\}, \qquad \bar{w}_p = \{X(t)\colon t\in T\} \cup \{a_\eta\colon \eta\in T_A\}.$$

The ordering on $\mathbf{E}$ is defined thus:

$$p \leqslant q \quad \text{iff} \quad u_q \subseteq u_p,\ w_q \subseteq w_p,\ x_q \subseteq x_p$$

and for each i with $l(q) \leqslant i < l(p)$ the following implications hold:

$$\text{if } i\in \bigcup \bar{u}_q, \text{ then } x_p(i) = 1, \qquad \text{if } i\in\bigcup\bar{w}_q, \text{ then } x_p(i) = 0.$$

If $G\subseteq \mathbf{E}(L^*)$ is a generic filter, then it follows immediately that the set

$$X_\alpha = \{i\in\omega\colon \exists p\in G[x_p(i) = 1]\}$$

is a partitioner of $A_\alpha(\varphi)$ and fills the gap L:

$$X(s) \subseteq_* X_\alpha \quad \text{for each } s\in S, \qquad X_\alpha \cap X(t) =_* \emptyset \quad \text{for each } t\in T.$$

Let φ^α be an extension of φ such that $\varphi^\alpha(\xi) = 0$ for $\mathrm{dm}(\varphi) \leqslant \xi < \alpha$ and $\varphi^\alpha(\alpha) = 1$. Hence $\varphi^\alpha\in D_{\alpha+1}$ and each element of $B(\varphi^\alpha)$ is a partitioner of $A_{\alpha+1}(\varphi^\alpha) = A_\alpha(\varphi)$ (we add no elements to $A_\alpha(\varphi)$ in this case).

We have to show yet that no infinite element $Z\in B(\varphi^\alpha)$ is covered by a finite union of elements of $A_{\alpha+1}(\varphi^\alpha)$. By the inductive assumption this is true for $Z\in B(\varphi)$. Suppose that $Z\in B(\varphi^\alpha)$ has the form

$$Z = X(\sigma) \cap X_\alpha, \qquad \text{where } X(\sigma)\in B(\varphi),$$

and that $Z \subseteq_* a_{\xi_1} \cup \ldots \cup a_{\xi_n}$ for some $a_{\xi_1}, \ldots, a_{\xi_n} \in A_{\alpha+1}(\varphi^\alpha)$. Since $A_{\alpha+1}(\varphi^\alpha) = A_\alpha(\varphi)$ and Z is a partitioner we have $Z =_* a_{\xi_1} \cup \ldots \cup a_{\xi_n}$ and hence Z is in the ground model $V^{(\mathbf{P}_\alpha)}$. Note that $\mathbf{E}(L^*)$ is equivalent to $\mathbf{E}_0 \times \mathbf{E}_1$ where $\mathbf{E}_0$, $\mathbf{E}_1$ are gap-filling forcings for L^* restricted to $X(\sigma)$ and $\omega\backslash X(\sigma)$, respectively, in the obvious sense. Since $\mathbf{E}_0$ produces $X_\alpha \cap X(\sigma) = Z$ we see that $\mathbf{E}_0$ must have an atomic element $\langle u, x, w\rangle$ and hence (in this case of Z)

$$X_\alpha \cap X(\sigma) =_* X(\sigma) \cap \bigcup \bar{u} =_* \bigcup_{s\in u} X(\sigma)\cap X(s).$$

It follows that $Z = X_\alpha \cap X(\sigma)$ is in $B(\varphi)$ and hence (by the inductive assumption) must be finite. The case of $Z = X(\tau) \cap (\omega\backslash X_\alpha)$ is similar.

For limit ordinals $\alpha \leqslant \kappa$ we define $\mathbf{P}_\alpha$ as the direct limit of $\{\mathbf{P}_\beta\colon \beta < \alpha\}$. We also assume $\mathbf{P}_{\alpha+1} = \mathbf{P}_\alpha$ in all cases not mentioned above. This completes the definition of the iteration.

In the next section we prove that $\mathbf{P} = \mathbf{P}_\kappa$ has c.c.c. (in general, gap-filling forcings $\mathbf{E}$ do not have c.c.c.). Assuming this for the moment let us look how $\mathbf{P}$ works.

Let $G\subseteq \mathbf{P}_\kappa$ be a generic filter. It is clear that $V[G] \models$ "$c = \kappa$" and $V[G|\alpha] \models$ "$c < \kappa$" for each $\alpha < \kappa$. Let B be a Boolean algebra in $V[G]$, with $\mathrm{card}\, B = \kappa$. There are elements $b_\alpha\in B$ for $\alpha < \kappa$ such that $B = \bigcup_{\alpha<\kappa} B_\alpha$, where

$$B_0 = \{\mathbf{0}, \mathbf{1}\},$$

$$B_{\alpha+1} = [B_\alpha, b_\alpha] \quad \text{(the subalgebra generated by } B_\alpha \text{ and } b_\alpha),$$

$$B_\alpha = \bigcup_{\beta<\alpha} B_\beta \quad \text{for limit } \alpha < \kappa.$$

Assume inductively that we have an embedding $f\colon B_\alpha \to P(\omega)/\mathrm{fin}$ such that $f(b_\xi) = X_{\gamma_\xi}/\mathrm{fin}$ for each $\xi < \alpha$. Hence, $f[B_\alpha] = B(\varphi_\alpha)/\mathrm{fin}$, where φ_α is defined on $\sup\{\gamma_\xi\colon \xi < \alpha\}$, $\varphi_\alpha(\gamma_\xi) = 1$ for each $\xi < \alpha$ and φ_α is zero otherwise. Define

$$b(s) = \Big(\prod_{s(\xi)=0} b_\xi\Big)\cdot\Big(\prod_{s(\eta)=1} -b_\eta\Big)$$

where s is a finite zero-one function on α.

The next generator b_α determines a gap

$$K = \langle\{b(s)\colon b(s) \leqslant b_\alpha\};\ \{b(t)\colon b(t)\cdot b_\alpha = 0\}\rangle$$

in the algebra B_α. Let L be the image of K under f. Thus L is a gap in $B(\varphi_\alpha)$ and

$$L = \langle\{X(s_f)\}_S;\ \{X(t_f)\}_T\rangle,$$

where s_f is defined on $\{\gamma_\xi\colon \xi\in\mathrm{dm}(s)\}$ by the equality $s_f(\gamma_\xi) = s(\xi)$ and t_f is defined similarly.

Let $\beta < \kappa$ be so large that φ_α and L are in $V^{(\mathbf{P}_\beta)}$. Since the continuum at stage β is smaller than κ, we can find a number $\gamma_\alpha > \beta$ such that $\gamma_\alpha = \mathrm{Nr}(L, \varphi_\alpha)$ and

$$\mathrm{Nr}(X, \varphi_\alpha),\ \mathrm{Nr}(Y, \varphi_\alpha) < \gamma_\alpha, \quad \text{for all infinite } X\in B(\varphi_\alpha) \text{ and } Y\in V^{(\mathbf{P}_\beta)}\backslash B(\varphi_\alpha).$$

We define $f(b_\alpha) = X_{\gamma_\alpha}$ and $\varphi_{\alpha+1} = \varphi_\alpha^\alpha$. This extends f to an embedding from $B_{\alpha+1}$ onto $B(\varphi_{\alpha+1})/\mathrm{fin}$. Indeed, by the well known theorem on extension of homomorphisms, we have to show that the following equivalences hold:

$$b(s) \leqslant b_\alpha \equiv X(s_f) \subseteq_* X_{\gamma_\alpha}, \qquad b(t)\cdot b_\alpha = \mathbf{0} \equiv X(t_f)\cap X_{\gamma_\alpha} =_* \emptyset.$$

The implications from left to right are obvious, since X_{γ_α} fills the gap L. The converse implications follow immediately from the following

LEMMA 1. *Let $p = \langle u, x, w\rangle \in \mathbf{E}(L)$ and let X_γ stand for a generic subset added by* **E**. *Then we have*

$$\text{if}\quad p \Vdash \text{“}X(\sigma) \subseteq_* X_\gamma\text{”},\quad \text{then}\quad X(\sigma) \subseteq_* \bigcup_{s\in u} X(s),$$

$$\text{if}\quad p \Vdash \text{“}X(\tau) \cap X_\gamma =_* \emptyset\text{”},\quad \text{then}\quad X(\tau) \subseteq_* \bigcup_{t\in w} X(t).$$

Similarly, $p \Vdash$ “$a_\xi \subseteq_* X_\gamma$” *implies* $a_\xi \subseteq_* X(s)$ *for some* $s \in u$ *or* $\xi \in u$, *and also* $p \Vdash$ “$a_\eta \cap X_\gamma =_* \emptyset$” *implies* $a_\eta \subseteq_* X(t)$ *for some* $t \in w$ *or* $\eta \in w$.

Proof. We may assume $p \Vdash$ “$X(\sigma)\backslash k \subseteq X_\gamma$” for some k. It follows that

$$X(\sigma)\backslash \max\{k, l(p)\} \subseteq \bigcup \bar{u}.$$

If $u = \{s_1, \ldots, s_n, \xi_1, \ldots, \xi_m\}$, then we obtain

$$X(\sigma) \subseteq_* X(s_1) \cup \ldots \cup X(s_n) = \bigcup_{s\in u} X(s).$$

Indeed, $Z = X(\sigma)\backslash(X(s_1) \cup \ldots \cup X(s_n))$ is in $B(\varphi_\alpha)$ and hence $X(s) \subseteq_* Z$ for some s. Since $X(s) \subseteq_* a_{\xi_1} \cup \ldots \cup a_{\xi_n}$, $X(s)$ must be finite and consequently Z is finite. The remaining cases are proved in a similar way.

It is now clear that the algebra B is isomorphic to $B(\Phi)/\text{fin} = \bigcup_{\alpha<\kappa} B(\varphi_\alpha)/\text{fin}$. Let

$$A(\Phi) = \bigcup_{\alpha<\kappa} A_{\gamma_\alpha}(\varphi_\alpha).$$

By construction, each non-zero element b of $B(\Phi)$ is a nontrivial partitioner of $A(\Phi)$ (in fact there are κ many elements of $A(\Phi)$ under b), and each $Y \notin B(\Phi)$ is either a nonpartitioner or it is congruent to an element x of $B(\Phi) \bmod A(\Phi)$, i.e.

$$(Y\backslash x) \cup (x\backslash Y) =_* (a_{\xi_1} \cup \ldots \cup a_{\xi_n})\backslash(a_{\eta_1} \cup \ldots \cup a_{\eta_m})$$

for some $a_{\xi_1}, \ldots, a_{\xi_n}, a_{\eta_1}, \ldots, a_{\eta_m} \in A(\Phi)$. Note that such an x is uniquely determined. It follows that $A(\Phi)$ is an m.a.d. and the mapping $Y \to x$ is a homomorphism from the field of all the partitioners onto $B(\Phi)$, with kernel consisting of all trivial partitioners. Thus, $B(\Phi)$ and hence also B is representable on $A(\Phi)$.

For the representability of algebras B with $\operatorname{card} B < \kappa$ see the end of the next section.

2. The almost disjoint forcing **R** is a particular case of **E** (in which $u(p) = \emptyset$ for all p). Hence, all stages of the iteration can be treated uniformly, with an obvious modification in the case of $\alpha = \operatorname{Nr}(Y, \varphi)$.

Let $Q_\alpha \subseteq \mathbf{P}_\alpha$ consist of all $p \in \mathbf{P}_\alpha$ for which we have the following:

(1) For each $\gamma \in \operatorname{supp}(p)$ there are $u_\gamma(p)$, $x_\gamma(p)$, $w_\gamma(p)$ such that

$$p|\gamma \Vdash \text{“}p(\gamma) = \langle u_\gamma(p), x_\gamma(p), w_{\gamma(p)}\rangle\text{”}$$

and $\operatorname{dm}(s) \subseteq \operatorname{supp}(p)$, $\xi \in \operatorname{supp}(p)$ for each ξ, $s \in u_\gamma(p) \cup w_\gamma(p)$.

(2) For each $\gamma \in \operatorname{supp}(p)$, the number $l(x_\gamma(p))$ is constant. We write $l(p)$ for this value.

LEMMA 2. *For each $p \in \mathbf{P}_\alpha$ and $m \in \omega$, there is a $q \in Q_\alpha$ such that $q \leqslant p$ and $l(q) \geqslant m$.*

Proof. By induction on α. The only essential case is $\alpha = \beta+1$ and $\beta \in \operatorname{supp}(p)$. We find an $r \in \mathbf{P}_\beta$, $r \leqslant p|\beta$, such that

$$r \Vdash \text{“}p(\beta) = \langle u_\beta(p), x_\beta(p), w_\beta(p)\rangle\text{”}$$

for some $u_\beta(p)$, $x_\beta(p)$, $w_\beta(p)$. Extending r if necessary, we may assume that $\operatorname{supp}(r)$ contains each ξ, where $\xi \in u_\beta(p) \cup w_\beta(p)$ or $\xi \in \operatorname{dm}(s)$ for some $s \in u_\beta(p) \cup w_\beta(p)$. Now, by the inductive hypothesis, we may assume that $r \in Q_\beta$ and $l(r) \geqslant m$. If s, $\xi \in u_\beta(p) \cup w_\beta(p)$, then r determines $X(s)$ and a_β up to $l(r)$. Hence, if $l(r) > l(p)$, then there is an extension $\bar{x}_\beta \supseteq x_\beta(p)$ such that $l(\bar{x}_\beta) = l(r)$ and

$$r \Vdash \text{“}\langle u_\beta(p), \bar{x}_\beta, w_\beta(p)\rangle \leqslant p(\beta)\text{”}.$$

Thus, if $q|\beta = r$ and $q(\beta) = \langle u_\beta(p), \bar{x}_\beta, w_\beta(p)\rangle$, then $q \leqslant p$, $q \in Q_\alpha$ and $l(q) = l(r) \geqslant m$, which finishes the proof.

The next lemma shows that Q_α has c.c.c.

LEMMA 3. *If $p, q \in Q_\alpha$ are such that*

$$x_\gamma(p) = x_\gamma(q) = x \quad \textit{for each } \gamma \in \operatorname{supp}(p) \cap \operatorname{supp}(q),$$

then p, q are compatible. In fact, there is an $r \leqslant p$, q such that $r \in Q_\alpha$ and $l(r) = l(p) = l(q)$.

Proof. By induction on α. The only essential case is $\alpha = \beta+1$ and $\beta \in \operatorname{supp}(p) \cap \operatorname{supp}(q)$. We know that

$$p|\beta \Vdash p(\beta) = \langle u_\beta(p), x_\beta(p), w_\beta(p)\rangle, \qquad q|\beta \Vdash q(\beta) = \langle u_\beta(q), x_\beta(q), w_\beta(q)\rangle$$

and $x_\beta(p) = x_\beta(q) = x$. By the inductive hypothesis there is an $r_\beta \leqslant p|\beta$, $q|\beta$ such that $r_\beta \in Q_\beta$ and $l(r_\beta) = l(p) = l(q) = l$. It follows that

$$r_\beta \Vdash \text{“}U \cap W =_* \emptyset\text{”}$$

for each $U \in \bar{u}_\beta(p) \cup \bar{u}_\beta(q)$, $W \in \bar{w}_\beta(p) \cup \bar{w}_\beta(q)$. We have to prove that, for some $\bar{r} \leqslant r_\beta$, $\bar{r} \in Q_\beta$ and $l(\bar{r}) = l$, $\bar{r} \Vdash$ “$U \cap W \subseteq l$”, for then $r|\beta = \bar{r}$ and

$$r(\beta) = \langle u_\beta(p) \cup u_\beta(q), x, w_\beta(p) \cup w_\beta(q)\rangle$$

define r, as required.

We prove this by induction on β. Let e.g. $U = X(s)$ and $W = a_\eta$. If $\max\operatorname{dm}(s) < \eta$, then

$$r_\beta|\eta+1 \Vdash \text{“}X(s) \cap a_\eta =_* \emptyset\text{”},$$

and hence $r_\beta|\eta \Vdash$ “$X(s) \subseteq_* \{X(\tau)\colon \tau \in w_\eta(r_\beta)\}$”. Define $\bar{r}_\beta|\eta = r_\beta|\eta$, $\bar{r}_\beta(\eta) = \langle x_\eta(r_\beta), w_\eta(r_\beta) \cup \{s\}\rangle$ and $\bar{r}_\beta(\xi) = r_\beta(\xi)$ for $\eta < \xi < \beta$. Thus, $\bar{r}_\beta \in Q_\beta$, $\bar{r}_\beta \leqslant r_\beta$, $l(\bar{r}_\beta) = l$ and $\bar{r}_\beta \Vdash$ “$X(s) \cap a_\eta \subseteq l$”. If $\max\operatorname{dm}(s) > \eta$, then for some $\gamma \in \operatorname{dm}(s)$

$$r_\beta|\gamma+1 \Vdash \text{“}a_\eta \cap X_\gamma =_* \emptyset\text{”} \quad (\text{or } a_\eta \subseteq_* X_\gamma)$$

and hence

$$r_\beta|\gamma \Vdash \text{"}a_\eta \cap u =_* \emptyset\text{"} \quad \text{for each } u \in u_\gamma(r_\beta).$$

By the inductive hypothesis (applied for each $u \in u_\gamma(r_\beta)$), there is an $r_\gamma \leqslant r_\beta|\gamma$, $r_\gamma \in Q_\gamma$ and $l(r_\gamma) = l$, such that

$$r_\gamma \Vdash \text{"}a_\eta \cap u \subseteq l\text{"} \quad \text{for each } u \in u_\gamma(r_\beta).$$

Hence, we may define $\bar{r}_\beta|\gamma = r_\gamma$, $\bar{r}_\beta(\xi) = r_\beta(\xi)$ for $\gamma < \xi < \beta$ and $\bar{r}_\beta(\gamma) = \langle u_\gamma(r_\beta), x_\gamma(r_\beta), w_\gamma(r_\beta) \cup \{\eta\}\rangle$.

Consider yet the case $U = X(s)$ and $W = X(t)$. If $\sigma = s \cup t$ is not a function, then obviously $r_\beta \Vdash \text{"}X(s) \cap X(t) = \emptyset\text{"}$. Otherwise, let $\gamma = \max \operatorname{dm}(\sigma)$. We have $X(s) \cap X(t) = X(\sigma) = X(\sigma|\gamma) \cap \pm X_\gamma$ and hence

$$r_\beta|\gamma+1 \Vdash \text{"}X(\sigma|\gamma) \cap X_\gamma =_* \emptyset\text{"} \quad (\text{or } X(\sigma|\gamma) \subseteq_* X_\gamma).$$

Thus, $r_\beta|\gamma \Vdash \text{"}X(\sigma|\gamma) \cap u =_* \emptyset\text{"}$ for each $u \in u_\gamma(r_\beta)$ and, by the inductive hypothesis, there is an $r_\gamma \leqslant r_\beta|q$, $r_\gamma \in Q_\gamma$ and $l(r_\gamma) = l$, such that

$$r_\gamma \Vdash \text{"}X(\sigma|\gamma) \cap u \subseteq l\text{"} \quad \text{for each } u \in u_\gamma(r_\beta)$$

and hence $\bar{r}_\beta|\gamma = r_\gamma$, $\bar{r}_\beta(\gamma) = \langle u_\gamma(r_\beta), x_\gamma, w_\gamma(r_\beta) \cup \{\sigma|\gamma\}\rangle$ and $\bar{r}_\beta(\xi) = r_\beta(\xi)$ for $\gamma < \xi < \beta$ is as required. All the remaining cases are proved in a similar way.

Now, beginning with r_β, we apply the above to each pair U, W to obtain a decreasing sequence of corresponding $\bar{r}_\beta$'s. The last term $\bar{r}$ has the required properties, which finishes the proof.

Now, it is easy to see that $\mathbf{P}_\kappa$ has c.c.c. Indeed, suppose that $\{q_\alpha: \alpha < \omega_1\}$ is an uncountable antichain. By Lemma 2 we may assume that $\{q_\alpha: \alpha < \omega_1\} \subseteq Q_\kappa$. We apply the Δ-system lemma to $\{\operatorname{supp}(q_\alpha): \alpha < \omega_1\}$ and find a pair q_α, q_β satisfying the assumptions of Lemma 3.

To obtain MA(σ-linked) we force as described above at—say—all even stages, while at odd stages α we force with σ-linked forcings of cardinality $< \kappa$. The proof that this combined iteration has c.c.c. is very similar to that above. For details see [B-F-Z]. In particular, $P(c)$ holds in our model and hence each algebra of cardinality $< c$ is representable (see [B-W], Theorem 2.5).

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Received 10 April 1990;
in revised form 1 March 1991