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W R O C Ł A W S K A D R U K A R N I A N A U K O W A

Minimal paradoxical decomposition for Mycielski's square

by

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Abstract. We give a precise lower bound (six) for the number of pieces required in a bounded paradoxical subset of the hyperbolic plane, where the paradoxical subset is required to have positive Lebesgue measure. The lower bound is realized in particular by Mycielski's square. Analogous results are given for the sphere S^2. Let D be a closed disc of radius r in S^2. We give a precise lower bound for the number of pieces required in a paradoxical subset of D, where the paradoxical subset is required to have positive Lebesgue measure. The lower bound is six if $0 < r < \pi/2$, five if $\pi/2 \leqslant r < \pi$, and four if $r = \pi$.

DEFINITION 1. A subset X of the hyperbolic plane H^2 (resp. the sphere S^2) is said to be (m, n)-*paradoxical* if X is nonempty and there are subsets $A_1, \ldots, A_m, B_1, \ldots, B_n$ of X, and hyperbolic (resp. spherical) isometries $f_1, \ldots, f_m, g_1, \ldots, g_n$ such that $P_1 = \{A_i\}$, $P_2 = \{B_j\}$ and $P_3 = \{f_i(A_i)\} \cup \{g_j(B_j)\}$ are each partitions of X. For convenience we always consider $m \leqslant n$. We also consider the *pieces* A_i, B_j to be nonempty.

DEFINITION 2. We say that X is k-*paradoxical* if X is (m, n)-paradoxical for some m and n with $m+n = k$.

We are particularly interested in k-paradoxical sets which have positive (nonzero) Lebesgue measure. Such sets exist in S^2 according to the Banach–Tarski paradox.

The Banach–Tarski paradox is that any two subsets W, X of S^2 with nonempty interiors are equivalent by finite decomposition [BT]. This means that there is a finite partition $\{W_1, \ldots, W_N\}$ of W, and a finite partition $\{X_1, \ldots, X_N\}$ of X, such that W_i is isometric to X_i for $i = 1, \ldots, N$. It follows that any subset of S^2 with nonempty interior, in particular any Lebesgue measurable subset of S^2 with nonempty interior, is k-paradoxical for some sufficiently large k. For example, Robinson has shown that the whole of S^2 is 4-paradoxical [R].

Mycielski has proven the Banach–Tarski paradox for H^2: any two bounded sets with nonempty interiors in H^2 are equivalent by finite decomposition [M]. It follows that any bounded subset of H^2 with nonempty interior, in particular any Lebesgue measurable bounded subset of H^2 with nonempty interior, is k-paradoxical for some sufficiently large k. For example, and this is the key step of Mycielski's proof, a square of arbitrarily small diameter is k-paradoxical for some k.

We will show that Mycielski's square is in fact 6-paradoxical (Theorem 4). More importantly, we will show that $k=6$ is the absolute lower bound for a bounded k-paradoxical subset of H^2 with positive Lebesgue measure (Theorem 1).

We will show also that analogous results hold in S^2, where instead of requiring that the k-paradoxical set X be bounded, we require that X be contained within a (spherical) closed disc of radius r. The absolute lower bound (for a k-paradoxical subset of S^2 with positive Lebesgue measure contained within a disc of radius r) will depend on the radius r. If $r=\pi$ (so that there is no restriction on the size of X), the absolute lower bound is $k=4$. Certainly $k\leqslant 4$ from the work of Robinson, and $k\geqslant 4$ follows from Lemma 1 below. We will show that if $\pi/2\leqslant r<\pi$, the absolute lower bound is $k=5$ (Theorems 3, 6), and if $0<r<\pi/2$, the absolute lower bound is $k=6$ (Theorems 2, 5).

It is known that if we drop the requirement that the paradoxical set X has positive measure, the absolute lower bound is $k=4$ in the case of H^2, and in the case of S^2 there are absolute lower bounds of $k=2, 3, 4$ when $r=\pi$, $\pi/2\leqslant r<\pi$, $0<r<\pi/2$ respectively [S].

We will use the symbol μ for Lebesgue measure on both H^2 and S^2 and the symbol $\tilde{\mu}$ for a certain extension of μ which is now to be defined.

We work in the Poincaré disc model of H^2. (H^2 is identified with the open unit disc $x^2+y^2<1$ and the hyperbolic metric is given by $dh^2=4(1-x^2-y^2)^{-2}dx^2+4(1-x^2-y^2)^{-2}dy^2$.)

Let m_2 be Lebesgue measure in the plane. Then if E is any m_2-measurable subset of the disc, its hyperbolic Lebesgue measure is given by

$$\mu(E)=\int_E 4(1-x^2-y^2)^{-2}\,dm_2.$$

Hyperbolic Lebesgue measure μ is countably additive, complete, and invariant under hyperbolic isometries. Banach showed that m_2 may be extended to a finitely additive, total (defined on all subsets of the plane) measure $\tilde{m}_2$ which is invariant under Euclidean isometries [B]. For all subsets E of the disc, we define

$$\tilde{\mu}(E)=\int_E 4(1-x^2-y^2)^{-2}\,d\tilde{m}_2.$$

The measure $\tilde{\mu}$ is finitely additive, total, and invariant under those hyperbolic isometries that fix (0, 0) in the model, since these are in fact Euclidean isometries, and the quantity $1-x^2-y^2$ is invariant under these isometries.

We identify S^2 with the unit sphere $x^2+y^2+z^2=1$ and consider the parametrization of the sphere given by

$$F(\theta, z)=\big((1-z^2)^{1/2}\cos\theta,\ (1-z^2)^{1/2}\sin\theta,\ z\big),\qquad \theta\in(0, 2\pi),\ z\in(-1, 1).$$

If E is any Lebesgue measurable subset of the sphere, we have

$$\mu(E)=m_2\big(F^{-1}(E)\big),$$

since cylindrical projection preserves area. Spherical Lebesgue measure μ is countably additive, complete, and invariant under spherical isometries. For all subsets E of the sphere we define

$$\tilde{\mu}(E)=\tilde{m}_2\big(F^{-1}(E)\big).$$

The measure $\tilde{\mu}$ is finitely additive, total, and invariant under those spherical isometries that fix or interchange the points (0, 0, 1) and (0, 0, −1).

We will use the word "measurable" to mean μ-measurable.

LEMMA 1. *There is no $(1, n)$-paradoxical subset of H^2 or S^2 with positive finite Lebesgue measure.*

Proof. If X is $(1, n)$-paradoxical, there are subsets $A, B_1, \ldots, B_n$ of X, and isometries $f, g_1, \ldots, g_n$ such that $P_1=\{A\}$, $P_2=\{B_j\}$ and $P_3=\{f(A)\}\cup\{g_j(B_j)\}$ are each partitions of X. From P_1, $\mu(A)=\mu(X)$, so from P_3, we have $\mu\big(\bigcup g_j(B_j)\big)=\mu(X)-\mu(X)=0$. Each $g_j(B_j)$ is measurable with measure zero, by completeness of μ. From P_2, we have $\mu(X)=0$. ■

THEOREM 1. *There is no bounded 5-paradoxical subset of H^2 with positive Lebesgue measure.*

Proof. A 5-paradoxical subset is either (1, 4)-paradoxical or (2, 3)-paradoxical. In light of Lemma 1, it suffices to show that there is no bounded (2, 3)-paradoxical subset of H^2 with positive Lebesgue measure.

We prove a slightly stronger result: let X and Y be bounded subsets of H^2 with $\mu(X)=\mu(Y)$. Suppose there are subsets A_1, A_2, B_1, B_2, B_3 of X, and isometries f_1, f_2, g_1, g_2, g_3 such that $P_1=\{A_1, A_2\}$ and $P_2=\{B_1, B_2, B_3\}$ are partitions of X, and $P_3=\{f_1(A_1), f_2(A_2), g_1(B_1), g_2(B_2), g_3(B_3)\}$ is a partition of Y. Then $\mu(X)=0$.

Without loss of generality, f_1 is the identity. We write $f=f_2$. Also let $M=g_1(B_1)\cup g_2(B_2)\cup g_3(B_3)$ so that $P_4=\{A_1, f(A_2), M\}$ is a partition of Y. There are two cases. The proof is most difficult when f is a rotation of infinite order, and this is reserved for Case 2. In both cases we will appeal to the following simple identity.

For any measurable sets E, T and injective measure-preserving function g, we have

$$\begin{aligned}(1)\qquad \mu\big(E\cap g(T)\big)&=\mu\big(g(g^{-1}(E)\cap T)\big)\quad\text{(since } g \text{ is injective)}\\ &=\mu\big(g^{-1}(E)\cap T\big)\quad\text{(since } g \text{ is measure-preserving)}\\ &=\int_T \chi_{g^{-1}(E)}(s)\,d\mu(s)=\int_T |E\cap\{g(s)\}|\,d\mu(s).\end{aligned}$$

Case 1: f is not a rotation of infinite order. Write $\langle f\rangle$ for the group of isometries generated by f. We claim that there is a measurable transversal T of the orbits determined by the action of $\langle f\rangle$ on H^2. Furthermore, if E is measurable, then

$$(2)\qquad \mu(E)=\int_T |E\cap \mathrm{Orb}(s)|\,d\mu(s).$$

If f is a translation or a parallel displacement, there is a pair of non-intersecting lines L_1, L_2 such that f is equivalent to reflection in L_1 followed by reflection in L_2. If f is a glide reflection, there is a pair of ultraparallel lines L_1, L_2 with a common perpendicular L_3 such that f is equivalent to reflection first in L_1, then in L_2, and then in L_3. In each of these cases, let T be the infinite rectangle bounded by L_1 and $f(L_1)$, including the boundary L_1 but not $f(L_1)$. Then T is a measurable transversal of the f-orbits as required. Applying (1) with $g = f^k$, and summing over all k in $\mathbf{Z}$, we obtain

$$\sum_{k\in\mathbf{Z}} \mu(E \cap f^k(T)) = \int_T \sum_{k\in\mathbf{Z}} |E \cap \{f^k(s)\}|\, d\mu(s),$$

which is equivalent to (2) since $\mathrm{Orb}(s)$ consists of the countably many distinct points $\ldots, f^{-1}(s), s, f(s), \ldots$ for all s in T.

If f is reflection in the line L, let T be one of the two closed half-planes with boundary L. When s lies in L, the orbit $\mathrm{Orb}(s)$ consists of a single point. Otherwise, $\mathrm{Orb}(s)$ consists of two distinct points. We apply (1) with $g = f^k$ for $k = 0, 1$ to obtain

$$\mu(E \cap T) + \mu(E \cap f(T)) = \int_T \big(|E \cap \{s\}| + |E \cap \{f(s)\}|\big)\, d\mu(s),$$

which is equivalent to (2) since $\mathrm{Orb}(s)$ consists of the two points $s, f(s)$ for almost all s in T.

If f is a rotation of finite order n with fixed point O, let T be an infinite sector with vertex O and angle $2\pi/n$, including all the points of one of its two bounding rays but no point of the other except the point O. Apply (1) with $g = f^k$ for k in $\{0, 1, \ldots, n-1\}$ to obtain

$$\sum_{k=0}^{n-1} \mu(E \cap f^k(T)) = \int_T \sum_{k=0}^{n-1} |E \cap \{f^k(s)\}|\, d\mu(s),$$

which is equivalent to (2) since $\mathrm{Orb}(s)$ consists of the n distinct points $s, f(s), \ldots, f^{n-1}(s)$ for all s in $T \setminus \{O\}$.

We have established the claim (2) in all cases.

Now observe that $|Y \cap \mathrm{Orb}(s)|$ is finite for each orbit $\mathrm{Orb}(s)$; if f is a reflection or a rotation of finite order, then $|\mathrm{Orb}(s)|$ is finite, and in the other cases we use the fact that Y is bounded.

From the partition $P_4 = \{A_1, f(A_2), M\}$ of Y we see that

$$|Y \cap \mathrm{Orb}(s)| = |A_1 \cap \mathrm{Orb}(s)| + |f(A_2) \cap \mathrm{Orb}(s)| + |M \cap \mathrm{Orb}(s)|$$
$$= |A_1 \cap \mathrm{Orb}(s)| + |A_2 \cap \mathrm{Orb}(s)| + |M \cap \mathrm{Orb}(s)| = |X \cap \mathrm{Orb}(s)| + |M \cap \mathrm{Orb}(s)|.$$

By (2) we have

$$\int_T |M \cap \mathrm{Orb}(s)|\, d\mu(s) = \int_T |Y \cap \mathrm{Orb}(s)|\, d\mu(s) - \int_T |X \cap \mathrm{Orb}(s)|\, d\mu(s)$$
$$= \mu(Y) - \mu(X) = 0.$$

This means that there is a subset P of T of measure zero such that $M \cap \mathrm{Orb}(s)$ is empty except when s is in P. Hence, M is contained in the set $\bigcup\{f^k(P)\colon k \in \mathbf{Z}\}$ which has measure zero. Therefore, M is measurable, and $\mu(M) = 0$ by completeness of μ. Since $g_i(B_i) \subset M$, we have $\mu(g_i(B_i)) = 0$ (again by completeness of μ), so $\mu(B_i) = 0$. We see from the partition P_2 of X that $\mu(X) = 0$.

Case 2: f is a rotation of infinite order. (It is futile to seek a measurable transversal of the f-orbits as in Case 1; every transversal is necessarily nonmeasurable.) Let O be the fixed point of f. Without loss of generality, we identify O with the center of the Poincaré disc, so that $\tilde{\mu}$ is invariant under those hyperbolic isometries that fix O.

Let $\sim$ be the smallest equivalence relation on $H^2 \setminus \{O\}$ which satisfies $s \sim f(s)$ for all s in X. This guarantees that each resulting equivalence class $[s]$ satisfies

$$X \cap [s] = X \cap f^{-1}([s]). \tag{3}$$

We describe the equivalence classes. If $[s]$ is a class of finite cardinality n, then $[s] = \{t, f(t), f^2(t), \ldots, f^{n-1}(t)\}$ for some t where the elements $t, f(t), f^2(t), \ldots, f^{n-2}(t)$ all lie in X, but $f^{-1}(t)$ and $f^{n-1}(t)$ do not lie in X. Indeed, the set of all "first elements" t, of classes of cardinality n, is the measurable set

$$T_n = f(X^c) \cap X \cap f^{-1}(X) \cap \ldots \cap f^{-n+2}(X) \cap f^{-n+1}(X^c).$$

(T_n is a transversal of the classes of cardinality n.) The union of all the classes of cardinality n is

$$S_n = T_n \cup f(T_n) \cup f^2(T_n) \cup \ldots \cup f^{n-1}(T_n).$$

There are three types of infinite class:

(i) $[s] = \{t, f(t), f^2(t), \ldots\}$ where $t, f(t), f^2(t), \ldots \in X$ but $f^{-1}(t) \in X^c$.
(ii) $[s] = \{t, f^{-1}(t), f^{-2}(t), \ldots\}$ where $f^{-1}(t), f^{-2}(t), \ldots \in X$ but $t \in X^c$.
(iii) $[s] = \{\ldots, f^{-2}(t), f^{-1}(t), t, f(t), f^2(t), \ldots\}$ where $f^k(t) \in X$ for all k in $\mathbf{Z}$.

Accordingly, we define

$$T_\omega = f(X^c) \cap X \cap f^{-1}(X) \cap f^{-2}(X) \cap \ldots,$$
$$S_\omega = T_\omega \cup f(T_\omega) \cup f^2(T_\omega) \cup \ldots,$$
$$T_{-\omega} = X^c \cap f(X) \cap f^2(X) \cap \ldots,$$
$$S_{-\omega} = T_{-\omega} \cup f^{-1}(T_{-\omega}) \cup f^{-2}(T_{-\omega}) \cup \ldots,$$
$$S_z = \ldots \cap f^{-2}(X) \cap f^{-1}(X) \cap X \cap f(X) \cap f^2(X) \cap \ldots$$

(T_ω is a transversal of the infinite classes of type (i); S_ω is the union of all infinite classes of type (i); etc. Note that we do not exhibit a measurable transversal of the classes of type (iii).)

Let I be the set of symbols $\{1, 2, 3, \ldots; \omega, -\omega, z\}$. Then $\{S_i\colon i \in I\}$ is a partition of $H^2 \setminus \{O\}$. We establish some properties of the sets S_i.

Sublemma 1. *We have*

(a) $\mu(S_\omega) = \mu(S_{-\omega}) = 0$.
(b) $\tilde{\mu}(M \cap S_i) = 0$, *for all* i *in* I.
(c) $\mu(X \cap S_i) = \mu(Y \cap S_i)$, *for all* i *in* I.
(d) $\mu(M \cap S_n) = 0$, *for all* n *in* $\{1, 2, \ldots\}$.
(e) $\mu(M \cap S_z^c) = 0$.

Proof. For (a), if D is any disc with center O, the set $S_\omega \cap D$ has finite measure but is also the disjoint union of the countably many congruent measurable sets $T_\omega \cap D$, $f(T_\omega \cap D), f^2(T_\omega \cap D), \ldots$ Therefore, $\mu(S_\omega \cap D) = 0$. Since this is true for D of arbitrarily large radius, we have $\mu(S_\omega) = 0$. Similarly, $\mu(S_{-\omega}) = 0$.

For (b) and (c), since each S_i is a union of equivalence classes, equation (3) implies $X \cap S_i = X \cap f^{-1}(S_i)$. Then, since $A_2 \subset X$, we have $A_2 \cap S_i = A_2 \cap f^{-1}(S_i)$. Therefore,

$$\tilde{\mu}(A_2 \cap S_i) = \tilde{\mu}(A_2 \cap f^{-1}(S_i)) = \tilde{\mu}(f(A_2) \cap S_i).$$

From P_4 we have

$$\begin{aligned}\mu(Y \cap S_i) &= \tilde{\mu}(A_1 \cap S_i) + \tilde{\mu}(f(A_2) \cap S_i) + \tilde{\mu}(M \cap S_i)\\ &= \tilde{\mu}(A_1 \cap S_i) + \tilde{\mu}(A_2 \cap S_i) + \tilde{\mu}(M \cap S_i)\\ &= \mu(X \cap S_i) + \tilde{\mu}(M \cap S_i).\end{aligned}$$

Summing both sides over I yields

$$\mu(Y) = \mu(X) + \sum_{i \in I} \tilde{\mu}(M \cap S_i),$$

so $\tilde{\mu}(M \cap S_i) = 0$ for all i in I, and hence, $\mu(X \cap S_i) = \mu(Y \cap S_i)$ for all i in I.

To prove (d), we first establish

$$\mu(E \cap S_n) = \int_{T_n} |E \cap [s]| \, d\mu(s), \tag{4}$$

for any measurable subset E of H^2. Equation (1) with $T = T_n$ and $g = f^k$ is

$$\mu(E \cap f^k(T_n)) = \int_{T_n} |E \cap f^k(s)| \, d\mu(s).$$

Summing over k in $\{0, 1, \ldots, n-1\}$, we have

$$\sum_{k=0}^{n-1} \mu(E \cap f^k(T_n)) = \int_{T_n} \sum_{k=0}^{n-1} |E \cap f^k(s)| \, d\mu(s),$$

which is equivalent to (4) since $[s]$ consists of the n distinct points $s, f(s), \ldots, f^{n-1}(s)$ for all s in T_n.

Since $A_2 \subset X$, we have $A_2 \cap [s] = A_2 \cap f^{-1}([s])$, for each class $[s]$ from (3). Therefore,

$$\begin{aligned}|A_2 \cap [s]| &= |A_2 \cap f^{-1}([s])|\\ &= |f(A_2 \cap f^{-1}([s]))| \quad &&\text{(since } f \text{ is injective)}\\ &= |f(A_2) \cap [s]| &&\text{(again since } f \text{ is injective).}\end{aligned}$$

Therefore, from P_4 we have

$$\begin{aligned}|Y \cap [s]| &= |A_1 \cap [s]| + |f(A_2) \cap [s]| + |M \cap [s]|\\ &= |A_1 \cap [s]| + |A_2 \cap [s]| + |M \cap [s]| = |X \cap [s]| + |M \cap [s]|.\end{aligned}$$

Then by (4),

$$\begin{aligned}\int_{T_n} |M \cap [s]| \, d\mu(s) &= \int_{T_n} |Y \cap [s]| \, d\mu(s) - \int_{T_n} |X \cap [s]| \, d\mu(s)\\ &= \mu(Y \cap S_n) - \mu(X \cap S_n) = 0.\end{aligned}$$

(Here we have used (c).) It follows that $\mu(M \cap S_n) = 0$ for each n in $\{1, 2, \ldots\}$.

Property (e) follows from (a) and (d), since $\{S_i \colon i \in I\}$ is a partition of $H^2 \backslash \{O\}$.

SUBLEMMA 2. $\tilde{\mu}(M) = 0$.

Proof. Property (b) holds in particular when $i = z$. Combine this with (e).

Sublemma 2 does not follow from (b) alone, since $\tilde{\mu}$ is only finitely additive. Also, we do not claim that M is measurable. If indeed $\mu(M) = 0$, then $\mu(X) = 0$ immediately, by the same reasoning as in Case 1.

Write D_r for the closed disc of radius r with center O. Let r be least such that $\mu(D_r^c \cap S_z) = 0$; this is possible since S_z is contained in the bounded set X.

SUBLEMMA 3. $\mu(M \cap D_r^c) = 0$.

Proof. Write

$$M \cap D_r^c = (M \cap D_r^c \cap S_z^c) \cup (M \cap D_r^c \cap S_z) \subset (M \cap S_z^c) \cup (D_r^c \cap S_z).$$

The last expression is a set of measure zero, by property (e) and the choice of r.

If $r = 0$, then $\mu(M) = 0$, and $\mu(X) = 0$ immediately, as mentioned above. Therefore, assume $r > 0$, and define the annulus $N_\varepsilon = D_r \backslash D_{r-\varepsilon}$, where $0 < \varepsilon < r$. Now let Q be the intersection of N_ε with a "small" sector of D_r. Call such a set Q a *bite* of D_r.

SUBLEMMA 4. *If Q is a bite of D_r, then* $\mu(Q \cap X) > 0$.

Proof. Since f has infinite order, the sets $f^k(Q)$ cover N_ε. Thus,

$$\begin{aligned}N_\varepsilon \cap S_z &= \bigcup_{k \in \mathbf{Z}} (f^k(Q) \cap S_z)\\ &\subset \bigcup_{k \in \mathbf{Z}} (f^k(Q) \cap f^k(X)) \quad \text{(since } S_z = \textstyle\bigcap f^k(X))\\ &= \bigcup_{k \in \mathbf{Z}} f^k(Q \cap X).\end{aligned}$$

Therefore, if $\mu(Q \cap X) = 0$, then $\mu(N_\varepsilon \cap S_z) = 0$, but $\mu(S_z \cap N_\varepsilon) > 0$ by minimality of r.

The proof now breaks into three subcases depending upon how many of the g_i's fix O.

Subcase 1: $g_1(O) \neq O$ and $g_2(O) \neq O$. (g_3 may or may not fix O.) The union of the discs $g_1^{-1}(D_r)$ and $g_2^{-1}(D_r)$ does not cover ∂D_r, so there is some bite Q, as described above, such that $g_1(Q) \subset D_r^c$ and $g_2(Q) \subset D_r^c$. Thus, if $j \in \{1, 2\}$,

$$g_j(B_j \cap Q) = g_j(B_j) \cap g_j(Q) \subset M \cap D_r^c.$$

By Sublemma 3, $\mu(M \cap D_r^c) = 0$, so we have $\mu(g_j(B_j \cap Q)) = 0$. Therefore, $\mu(B_1 \cap Q) = 0$ and $\mu(B_2 \cap Q) = 0$, so $B_3 \cap Q$ is measurable, and $\mu(B_3 \cap Q) = \mu(X \cap Q)$. By Sublemma 4, $\mu(X \cap Q) > 0$, so $g_3(B_3 \cap Q)$ is a subset of M with positive measure, but this is impossible since $\tilde{\mu}(M) = 0$ (by Sublemma 2).

Subcase 2: $g_1(O) \neq O$, but g_2 and g_3 both fix O. Then $\tilde{\mu}$ is invariant under g_2 and g_3 as well as under f. The disc $g_1^{-1}(D_r)$ does not cover ∂D_r, so there is some bite Q such that $g_1(Q) \subset D_r^c$. Now,

$$g_1(B_1 \cap Q) = g_1(B_1) \cap g_1(Q) \subset M \cap D_r^c.$$

Therefore by Sublemma 3, $\mu(g_1(B_1 \cap Q)) = 0$, so $\mu(B_1 \cap Q) = 0$. Thus,

$$\mu(X \cap Q) = \tilde{\mu}(B_1 \cap Q)+\tilde{\mu}(B_2 \cap Q)+\tilde{\mu}(B_3 \cap Q) = \tilde{\mu}(B_2 \cap Q)+\tilde{\mu}(B_3 \cap Q).$$

Then

$$\mu(X) = \mu(Y) = \tilde{\mu}(A_1)+\tilde{\mu}(f(A_2))+\tilde{\mu}(g_1(B_1))+\tilde{\mu}(g_2(B_2))+\tilde{\mu}(g_3(B_3))$$
$$= \tilde{\mu}(A_1)+\tilde{\mu}(A_2)+\tilde{\mu}(g_1(B_1))+\tilde{\mu}(B_2)+\tilde{\mu}(B_3)$$
$$\geqslant \tilde{\mu}(A_1)+\tilde{\mu}(A_2)+\tilde{\mu}(B_2 \cap Q)+\tilde{\mu}(B_3 \cap Q) = \mu(X)+\mu(X \cap Q),$$

and we have a contradiction since $\mu(X \cap Q) > 0$ by Sublemma 4.

Subcase 3: g_1, g_2, g_3 all fix O. Then $\tilde{\mu}$ is invariant under all the g_i as well as under f. Thus,

$$\mu(X) = \mu(Y) = \tilde{\mu}(A_1)+\tilde{\mu}(f(A_2))+\tilde{\mu}(g_1(B_1))+\tilde{\mu}(g_2(B_2))+\tilde{\mu}(g_3(B_3))$$
$$= \tilde{\mu}(A_1)+\tilde{\mu}(A_2)+\tilde{\mu}(B_1)+\tilde{\mu}(B_2)+\tilde{\mu}(B_3) = \mu(X)+\mu(X).$$

Therefore $\mu(X) = 0$. ∎

THEOREM 2. *Let D be a closed disc in S^2 of radius less than $\pi/2$. There is no 5-paradoxical subset of D with positive Lebesgue measure.*

The terminology in the proof is the same as that in the proof of Theorem 1.

Proof. In light of Lemma 1, it suffices to show that there is no (2, 3)-paradoxical subset of D with positive Lebesgue measure. We prove a slightly stronger result: let X, Y be subsets of S^2 with $\mu(X) = \mu(Y)$ and $X \subset D$. Suppose there are subsets A_1, A_2, B_1, B_2, B_3 of X and isometries f_1, f_2, g_1, g_2, g_3 such that $P_1 = \{A_1, A_2\}$ and $P_2 = \{B_1, B_2, B_3\}$ are partitions of X, and $P_3 = \{f_1(A_1), f_2(A_2), g_1(B_1), g_2(B_2), g_3(B_3)\}$ is a partition of Y. Then $\mu(X) = 0$.

Let f, M be as in Theorem 1. There are three cases.

Case 1: f has finite order. We find a measurable transversal T of the f-orbits.

If f is a rotation of order n with axis OO', or a rotatory reflection of order n with axis OO', let T be the lune with vertices O and O', and angle $2\pi/n$, including all the points of one of its two bounding great semicircles but no point of the other except the points O and O'.

If f is reflection in the great circle C, let T be one of the two closed hemispheres with boundary C.

Equation (2) holds in all cases as in Theorem 1. Also, $|Y \cap \mathrm{Orb}(s)|$ is finite since $|\mathrm{Orb}(s)|$ is finite. It follows that $\mu(M) = 0$, so $\mu(X) = 0$.

Case 2: f is a rotation of infinite order. Let OO' be the axis of f. Without loss of generality, we identify O with the point (0, 0, 1), so that $\tilde{\mu}$ is invariant under the spherical isometries that fix or interchange the points O and O'.

Let S_z be as in Theorem 1. Write $D_r(p)$ for the closed disc with radius r and center p. Let r be least such that $\mu(D_r(O)^c \cap S_z) = 0$. (Possibly $r \geqslant \pi/2$.) If $r = 0$, we get $\mu(M) = 0$ (and hence $\mu(X) = 0$), so assume $r > 0$. Every bite Q of $D_r(O)$ contains points of X (in fact $\mu(Q \cap X) > 0$ as in Sublemma 4), so the boundary of $D_r(O)$ is contained in the closure of X, but X is contained in the given closed disc D. We have

$$\partial D_r(O) \subset \mathrm{cl}(X) \subset D.$$

Similarly, let s be least such that $\mu(D_s(O')^c \cap S_z) = 0$, and obtain

$$\partial D_s(O') \subset D.$$

Since D has radius less than $\pi/2$, one of the radii r, s must be less than $\pi/2$. Without loss of generality, $r < \pi/2$.

The remainder of the proof is the same as in Theorem 1, Case 2. Note that the inequality $r < \pi/2$ is necessary in Subcase 1; if $r \geqslant \pi/2$, the union of the discs $g_1^{-1}(D_r)$ and $g_2^{-1}(D_r)$ might contain ∂D_r.

Case 3: f is a rotatory reflection of infinite order. Let S_z be as in Theorem 1. If S_z is empty, $\mu(M) = 0$, so let $s \in S_z$. Then $\mathrm{Orb}(s) \subset S_z \subset X \subset D$, so $\mathrm{cl}(\mathrm{Orb}(s)) \subset D$, but $\mathrm{cl}(\mathrm{Orb}(s))$ is either a great circle or a pair of antipodal circles and so cannot be contained in D. ∎

THEOREM 3. *Let D be a closed disc in S^2 of radius less than π. There is no 4-paradoxical subset of D with positive Lebesgue measure.*

Proof. In light of Lemma 1, it suffices to show that there is no (2, 2)-paradoxical subset of D with positive Lebesgue measure. We prove a slightly stronger result: let X, Y be subsets of S^2 with $\mu(X) = \mu(Y)$ and $X \subset D$. Suppose there are subsets A_1, A_2, B_1, B_2 of X and isometries f_1, f_2, g_1, g_2 such that $P_1 = \{A_1, A_2\}$ and $P_2 = \{B_1, B_2\}$ are partitions of X, and $P_3 = \{f_1(A_1), f_2(A_2), g_1(B_1), g_2(B_2)\}$ is a partition of Y. Then $\mu(X) = 0$.

Without loss of generality, f_1 is the identity. Write $f = f_2$. Let $M = g_1(B_1) \cup g_2(B_2)$. There are two cases.

Case 1: f has finite order. The proof is the same as in Theorem 2, Case 1.

Case 2: f has infinite order. Then f is either a rotation of infinite order or a rotatory reflection of infinite order. In either case let OO' be the axis of f, and identify O with the point (0, 0, 1) so that $\tilde{\mu}$ is invariant under the spherical isometries that fix or interchange the points O and O'. Let S_z be as in Theorem 1. There are two subcases.

Subcase 1: g_1 neither fixes nor interchanges the points O and O'. Let S be the semigroup generated by f and g_1. We show that S contains two rotations F, G of infinite order which have different axes. Let $F = f^2$. If g_1 has infinite order, let $G = g_1^2$. If g_1 has finite order n, the element $g_1^{-1} = g_1^{n-1}$ lies in S, so let $G = g_1 F g_1^{-1}$.

Consider the partition

$$\{S_z \cap g_1(X),\ S_z^c \cap g_1(B_1),\ S_z^c \cap g_1(B_2)\}$$

of $g_1(X)$. Since $S_z^c \cap g_1(B_1) \subset S_z^c \cap M$, it has measure zero (as in Sublemma 1(e)). The sets $S_z \cap g_1(X)$ and $g_1(X)$ are clearly measurable, so it must be that the set $S_z^c \cap g_1(B_2)$

is measurable. It has the same measure as its image under $g_2 g_1^{-1}$, but the latter is contained in M. Since $\tilde{\mu}(M) = 0$ (as in Sublemma 2), we have $\mu(S_z^c \cap g_1(B_2)) = 0$. Hence,

(5) $$\mu(S_z \cap g_1(X)) = \mu(g_1(X)).$$

Now,

$$\mu(S_z \cap g_1(X)) \leqslant \mu(X \cap g_1(X)) \quad (\text{since } S_z \subset X)$$
$$\leqslant \mu(g_1(X)),$$

but by (5) the inequalities become equalities and we have $\mu(X) = \mu(g_1(X)) = \mu(X \cap g_1(X))$. It follows that for any measurable subset E of S^2, we have

(6) $$\mu(E \cap X) = \mu(g_1(E) \cap X).$$

Also,

$$\mu(S_z \cap g_1(X)) \leqslant \mu(S_z)$$
$$= \mu(S_z \cap f(S_z)) \quad (\text{since } S_z = f(S_z))$$
$$\leqslant \mu(X \cap f(X)) \quad (\text{since } S_z \subset X)$$
$$\leqslant \mu(X).$$

Again by (5), the inequalities become equalities. We have $\mu(X) = \mu(f(X)) = \mu(X \cap f(X))$. It follows that for any measurable subset E of S^2, we have

(7) $$\mu(E \cap X) = \mu(f(E) \cap X).$$

A simple induction using (6) and (7) shows that

(8) $$\mu(E \cap X) = \mu(w(E) \cap X),$$

for all w in S and all measurable sets E. We wish to show that $\mu(X) = 0$. Suppose not. Let E be a disc of diameter ε such that $\mu(E \cap X) > 0$. Let c be the center of E. Since S contains F and G, the set $\{w(c)\colon w \in S\}$ is dense in S^2. Therefore the discs $w(E)$ cover S^2, and by (8), each disc contains points of X. Since ε may be taken arbitrarily small, it follows that X is dense in S^2, which contradicts the inclusion $X \subset D$.

Subcase 2: g_1 fixes or interchanges the points O and O', and g_2 fixes or interchanges the points O and O'. Then $\tilde{\mu}$ is invariant under both g_1 and g_2 as well as under f. Thus,

$$\mu(X) = \mu(Y) = \tilde{\mu}(A_1) + \tilde{\mu}(f(A_2)) + \tilde{\mu}(g_1(B_1)) + \tilde{\mu}(g_2(B_2))$$
$$= \tilde{\mu}(A_1) + \tilde{\mu}(A_2) + \tilde{\mu}(B_1) + \tilde{\mu}(B_2) = \mu(X) + \mu(X).$$

Therefore, $\mu(X) = 0$. ■

DEFINITION 3. Fix $\varepsilon > 0$. Let K and L be perpendicular geodesics in H^2. Let K_1 and K_2 be the two equidistant lines to K at a distance $\varepsilon/2$, and let L_1 and L_2 be the two equidistant lines to L at a distance $\varepsilon/2$. *Mycielski's square* consists of the open subset of H^2 bounded by K_1, K_2, L_1 and L_2, together with those points of the boundary which lie in $K_1 \cup L_1$.

THEOREM 4. *Mycielski's square is 6-paradoxical.*

Proof. Let X be Mycielski's square. Mycielski proved the existence of four hyperbolic translations f_1, f_2, g_1, g_2 with the following properties.

(i) f_1 and f_2 fix the line K and generate a free abelian group.
(ii) g_1 and g_2 fix the line L and generate a free abelian group.
(iii) $X \subset f_1^{-1}(X) \cup f_2^{-1}(X)$.
(iv) $X \subset g_1^{-1}(X) \cup g_2^{-1}(X)$.
(v) The group $\langle f_1, f_2, g_1, g_2 \rangle$ is the free product of the groups $\langle f_1, f_2 \rangle$ and $\langle g_1, g_2 \rangle$.

We will define subsets $A_1, A_2, A_3, B_1, B_2, B_3$ of X such that

$$P_1 = \{A_1, A_2, A_3\},$$
$$P_2 = \{B_1, B_2, B_3\},$$
$$\text{and } P_3 = \{f_1(A_1), f_2(A_2), A_3, g_1(B_1), g_2(B_2), B_3\}$$

are each partitions of X, thus showing that X is (3, 3)-paradoxical. Let G be the infinite graph with vertex set $V(G) = X$ and edge set

$$E(G) = \{\{x, \varphi(x)\}\colon \varphi \in \{f_1, f_2, g_1, g_2\},\ x \in X,\ \varphi(x) \in X\}.$$

Let G_f be the subgraph of G with edge set

$$E(G_f) = \{\{x, \varphi(x)\}\colon \varphi \in \{f_1, f_2\},\ x \in X,\ \varphi(x) \in X\}.$$

Let G_g be the subgraph of G with edge set

$$E(G_g) = \{\{x, \varphi(x)\}\colon \varphi \in \{g_1, g_2\},\ x \in X,\ \varphi(x) \in X\}.$$

Let G_0 be a connected component of G. It suffices to show that each vertex of G_0 can be assigned to exactly one set in each of the three desired partitions, in a manner consistent with the action of the translations.

Each connected component of G_f which meets G_0 must lie entirely within G_0, since each connecting path in G_f is also a path in G. Similarly, each component of G_g which meets G_0 lies entirely within G_0. Let $\{C_i\}$ be the set of connected components of G_f which lie in G_0. Let $\{D_j\}$ be the set of connected components of G_g which lie in G_0.

Let H be the infinite bipartite graph with vertex set $V(H) = \{C_i\} \cup \{D_j\}$ and edge set $E(H) = V(G_0)$, where x is an edge of H connecting C_i to D_j exactly when $x \in V(C_i) \cap V(D_j)$.

We show that the graph H is connected. To find a path in H connecting a component C of G_f to a component D of G_g, let $x_0 e_1 x_1 \ldots e_n x_n$ be a path in G_0 connecting some element x_0 of $V(C)$ to some element x_n of $V(D)$. Each edge e_i is either an edge of G_f or an edge of G_g. It follows that some subsequence S of $x_0, x_1, \ldots, x_n$ is a sequence of edges in H connecting C to D. Precisely, $x_0 \in S$ if $e_1 \in E(G_g)$; $x_n \in S$ if $e_n \in E(G_f)$; if $i \in \{1, \ldots, n-1\}$, then $x_i \in S$ if either $e_i \in E(G_f)$ and $e_{i+1} \in E(G_g)$, or $e_i \in E(G_g)$ and $e_{i+1} \in E(G_f)$.

We show that the graph H has at most one cycle. Suppose that H has more than one cycle. Then there is an edge x_0 of H such that $H\backslash x_0$ still has a cycle and is still connected. Let D_0 be the component of G_g which contains x_0. Let

$$D_0 x_1 C_1 x_2 D_2 x_3 C_3 \ldots x_{n-1} C_{n-1} x_n D_n$$

be a walk in $H\backslash x_0$ which starts at D_0, follows a path to the nearest vertex of a cycle of $H\backslash x_0$, travels once around the cycle and then returns along the same path back to D_0, so $D_n = D_0$. If D_0 lies *on* a cycle of $H\backslash x_0$, then the walk is simply around that cycle. Following the walk, we never need to backtrack immediately at a particular edge, that is, $x_i \neq x_{i+1}$ for i in $\{1, \ldots, n-1\}$. Also $x_0 \neq x_1$ and $x_0 \neq x_n$, since $x_1, x_n \in E(H\backslash x_0)$. It follows that there are nonidentity elements $w_0, w_2, w_4, \ldots, w_n$ of $\langle g_1, g_2\rangle$ and nonidentity elements $v_1, v_3, v_5, \ldots, v_{n-1}$ of $\langle f_1, f_2\rangle$, such that $w_0(x_0) = x_1$, $v_1(x_1) = x_2$, $w_2(x_2) = x_3, \ldots,$ $v_{n-1}(x_{n-1}) = x_n$, and $w_n(x_n) = x_0$. Thus, the direct isometry $w_n v_{n-1} \ldots v_1 w_0$ fixes x_0.

Similarly, let C_0 be the component of G_f which contains x_0. There is a walk

$$C_0 x_1 D_1 x_2 C_2 x_3 D_3 \ldots x_{m-1} D_{m-1} x_m C_m$$

in $H\backslash x_0$ which, by the same method as above, yields a direct isometry $v_m w_{m-1} \ldots w_1 v_0$ that fixes x_0. Since the group $\langle f_1, f_2, g_1, g_2\rangle$ is the free product of the groups $\langle f_1, f_2\rangle$ and $\langle g_1, g_2\rangle$, the two direct isometries fixing x_0 do not commute, and this contradicts the fact that the action of the group of direct hyperbolic isometries on H^2 is locally commutative.

We now describe the assignment of elements of $V(G_0)$ to sets of the partitions. There are two cases.

Case 1: H has exactly one cycle. Let the cycle be

$$D_0 x_1 C_1 x_2 D_2 x_3 C_3 \ldots x_{n-1} C_{n-1} x_n D_n$$

with $D_n = D_0$. The assignment of points of $V(G_0)$ to sets of the partitions is accomplished in countably many steps.

Step 1. We assign elements of $V(C_1)$ to sets of P_1. The point x_1 lies in $V(C_1)$. Since $X \subset f_1^{-1}(X) \cup f_2^{-1}(X)$, there is a sequence

$$x_1, \ \varphi_1(x_1), \ \varphi_2\varphi_1(x_1), \ \varphi_3\varphi_2\varphi_1(x_1), \ \ldots$$

of elements which lie in X, and hence in $V(C_1)$, where each φ_i is either f_1 or f_2. For $i, j > 0$, we have

$$\varphi_i \ldots \varphi_2\varphi_1(x_1) \neq \varphi_{i+j} \ldots \varphi_{i+1}\varphi_i \ldots \varphi_2\varphi_1(x_1),$$

since the translation $\varphi_{i+j} \ldots \varphi_{i+1}$ is not the identity of the free abelian group $\langle f_1, f_2\rangle$. This means that the points of the sequence are all different, so there is no ambiguity in the following assignment.

Assign x_1 to A_k if $\varphi_1 = f_k$. ($k = 1$ or $k = 2$.)

Assign $\varphi_i \ldots \varphi_2\varphi_1(x_1)$ to A_k if $\varphi_{i+1} = f_k$, for each i in $\{1, 2, 3, \ldots\}$.

Assign all other elements of $V(C_1)$ to A_3.

Each element of $V(C_1)$ has now been assigned to exactly one of the sets in P_1, and the process has necessarily assigned each element of $V(C_1)\backslash x_1$ to exactly one of the sets of P_3.

Step 2. Assign the elements of $V(D_2)$ to sets of P_2. Beginning with the element x_2 of $V(D_2)$, we obtain a sequence

$$x_2, \ \varphi_1(x_2), \ \varphi_2\varphi_1(x_2), \ \varphi_3\varphi_2\varphi_1(x_2), \ \ldots$$

of distinct elements of $V(D_2)$ where each φ_i is either g_1 or g_2.

Assign x_2 to B_k if $\varphi_1 = g_k$.

Assign $\varphi_i \ldots \varphi_2\varphi_1(x_2)$ to B_k if $\varphi_{i+1} = g_k$, for each i in $\{1, 2, 3, \ldots\}$.

Assign all other elements of $V(D_2)$ to B_3.

Each element of $V(D_2)$ has now been assigned to exactly one of the sets in P_2, and the process has necessarily assigned each element of $V(D_2)\backslash x_2$ to exactly one of the sets of P_3. (This completes Step 2.)

Observe that since x_2 lies in both C_1 and D_2, its place in all three partitions has now been assigned.

In Step 3 we assign elements of $V(C_3)$ to sets of P_1, by the method of Step 1, using a sequence beginning with the element x_3. In Step 4 we assign elements of $V(D_4)$ to sets of P_2, by the method of Step 2, using a sequence beginning with the element x_4. We continue in this way, modelling the odd-numbered steps after Step 1, and the even-numbered steps after Step 2, until we have completed Step n. At this point, each element of

$$V(C_1) \cup V(C_3) \cup \ldots \cup V(C_{n-1})$$

has been assigned to a set in P_1, each element of

$$V(D_2) \cup V(D_4) \cup \ldots \cup V(D_n)$$

has been assigned to a set in P_2, and each element of

$$V(C_1) \cup V(D_2) \cup V(C_3) \cup V(D_4) \cup \ldots \cup V(C_{n-1}) \cup V(D_n)$$

has been assigned to a set of P_3. Notice that the placement of x_1 in P_3 is not determined until Step n.

The remainder of the assignment consists of alternately applying Steps A and B below *ad infinitum*, beginning with Step A.

Step A. For each element x in $V(G_0)$ which has been assigned to P_2, but has not yet been assigned to P_1, let C be the component of G_f containing x. Since C does not lie in a cycle of H, no point of C other than x has been assigned to a set in a partition. Assign all the elements of $V(C)$ to sets in P_1 by the method of Step 1, using a sequence beginning at x.

Step B. For each element x which has been assigned to P_1, but has not yet been assigned to P_2, let D be the component of G_g containing x. Assign all the elements of $V(D)$ to sets in P_2 by the method of Step 2, using a sequence beginning at x.

Case 2: H is acyclic. Let D_0 be any component of G_g which lies in G_0. Assign all the elements in $V(D_0)$ to the set B_3 of P_2. Then alternately apply Steps A and B above *ad infinitum*, beginning with Step A. ∎

DEFINITION 4. Let K and L be perpendicular geodesics (great circles) in S^2. Let p be one of the two points of intersection of K and L. Let K_1 and K_2 be the two circles equidistant to K at a distance $d \leqslant \pi/4$, and let L_1 and L_2 be the two circles equidistant to L also at a distance d. The condition $d \leqslant \pi/4$ ensures that the K_i meet the L_j. The open subset of S^2 which contains p and is bounded by K_1, K_2, L_1 and L_2 is called a *square* in S^2.

THEOREM 5. *Every square in S^2 is 6-paradoxical.*

Proof. It suffices to find rotations f_1, f_2, g_1, g_2 satisfying the conditions (i), …, (v) listed in the proof of Theorem 4. Clearly, there are rotations g_1, g_2 which satisfy conditions (ii) and (iv). Project the sphere stereographically onto the complex plane so that K is identified with the unit circle, and L with the real line. There are real numbers a, b, c, d satisfying $a^2+b^2 = 1$ and $c^2+d^2 = 1$ such that we may write the g_i as maps from $\mathbf{C}$ to $\mathbf{C}$ as follows:

$$g_1(z) = (az-b)/(bz+a) \quad \text{and} \quad g_2(z) = (cz-d)/(dz+c).$$

Let u and v be complex numbers of modulus one which are independent transcendental elements over $\mathbf{Q}(a, b, c, d)$. The numbers u and v may be chosen so that the rotations

$$f_1(z) = uz \quad \text{and} \quad f_2(z) = vz$$

satisfy conditions (i) and (iii). It can be shown that property (v) then follows from the transcendence of u and v. ∎

THEOREM 6. *The open hemisphere is 5-paradoxical.*

Proof. Let K, L, f_2, g_1, g_2 be as in Theorem 5. Write $f = f_2$. Let X be one of the two open hemispheres bounded by K. The rotations g_1 and g_2 must be chosen so as to satisfy condition (iv), but in place of condition (iii), we have simply $f(X) = X$.

We will define subsets A_1, A_2, B_1, B_2, B_3 of X such that $P_1 = \{A_1, A_2\}$, $P_2 = \{B_1, B_2, B_3\}$, and $P_3 = \{A_1, f(A_2), g_1(B_1), g_2(B_2), B_3\}$ are each partitions of X, thus showing that X is (2, 3)-paradoxical. As in Theorem 4, define infinite graphs G, G_f, G_g each with vertex set X and with edge sets

$$E(G) = \{\{x, \varphi(x)\}\colon \varphi \in \{f, g_1, g_2\},\ x \in X,\ \varphi(x) \in X\},$$
$$E(G_f) = \{\{x, f(x)\}\colon x \in X\},$$
$$E(G_g) = \{\{x, \varphi(x)\}\colon \varphi \in \{g_1, g_2\},\ x \in X,\ \varphi(x) \in X\}.$$

Let G_0 be a connected component of G, and define the graph H as in Theorem 4. The hemisphere is more complicated than the squares for two reasons.

First, there is the possibility of a 2-cycle in H. Indeed, the rotation fg_1 has a fixed point x in X. If C is the component of G_f containing x, and D is the component of G_g containing x, then x and $g_1(x)$ are two edges connecting C to D in H. This actually

presents no problem, since the proof that H has at most one cycle applies to 2-cycles in the same way that it applies to larger cycles.

The second complication arises from the fixed point O of f, so assume for the moment that G_0 is a component of G which does not contain the vertex O. The assignment of elements of $V(G_0)$ to sets of the partitions proceeds as in Theorem 4, but the odd-numbered steps are somewhat simplified. In Step 1, for example, the sequence of vertices in C_1 beginning at x_1 is simply

$$x_1,\ f(x_1),\ f^2(x_1),\ f^3(x_1),\ \ldots$$

Every element of the sequence is assigned to A_2, and all other elements of C_1 (exactly the elements $f^{-1}(x_1), f^{-2}(x_1), \ldots$) are assigned to A_1.

Now suppose that G_0 is the component of G which contains the vertex O. We show that, in this case, H is acyclic. Let D_0 be the component of G_g which contains O. Suppose there is a cycle in H. Let

$$D_0 x_1 C_1 x_2 D_2 x_3 C_3 \ldots x_{n-1} C_{n-1} x_n D_n$$

be a walk in H which starts at D_0, follows a path to the nearest vertex of a cycle of H, travels once around the cycle, and then returns along the same path back to D_0, so $D_n = D_0$. If D_0 lies *on* a cycle of H, then the walk is simply around that cycle. As in Theorem 4, we obtain nonidentity elements $w_0, w_2, w_4, \ldots, w_n$ of $\langle g_1, g_2\rangle$ and nonidentity elements $v_1, v_3, v_5, \ldots, v_{n-1}$ of $\langle f\rangle$ such that the direct isometry $w_n v_{n-1} \ldots v_1 w_0$ fixes x_0. Since the group $\langle f, g_1, g_2\rangle$ is the free product of the groups $\langle f\rangle$ and $\langle g_1, g_2\rangle$, the isometry $w_n v_{n-1} \ldots v_1 w_0$ does not commute with f, and this contradicts local commutativity. Thus, H is acyclic.

Assign O to A_1. Then, alternately apply Steps A and B *ad infinitum*, beginning with Step B. ∎

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