such that $q_2 > 0$ and $\alpha_i \neq 0$ for all $q_1+1 \leqslant i \leqslant q_1+q_2$. Putting together (1)–(9), we conclude

$$p_{\iota_i}(y_\lambda^t - u_1 - u_2) = 0$$

for all $1 \leqslant i \leqslant q_1+q_2$. Furthermore, $\iota_i \notin \{\iota_1, \ldots, \iota_{q_1}\}$ for every $i \in \{q_1+1, \ldots, q_1+q_2\}$; for otherwise we would have $p_{\iota_i}(y_\lambda^t - u_1) = p_{\iota_i}(y_\lambda^t - u_1 - u_2) = 0$, thus $p_{\iota_i}(u_2) = 0$. This contradicts (10). We conclude $p_{\iota_i}(y_\lambda^t) \neq 0$ for all $1 \leqslant i \leqslant q_1+q_2$. Hence $q_1+q_2 \leqslant n$.

After finitely many steps we must arrive at $\sum_i q_i > n$. A contradiction.

COROLLARY. *Let $\omega < \gamma \leqslant \varkappa$ be regular, X a ladder space of dimension $\varkappa$. If $\Gamma(X) > 0$, then (X, σ_γ) has no continuous basis.*

Proof. By Theorem 1, 2.1, a continuous basis $(x_\iota)_{\iota \in I}$ of (X, σ_γ) is an algebraic basis. Because σ_γ is coarser than $\sigma_\varkappa$, the coordinate functions p_ι are continuous on $(X, \sigma_\varkappa)$; and hence, $(x_\iota)_{\iota \in I}$ would be a continuous basis of $(X, \sigma_\varkappa)$. This contradicts the Theorem.

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Received 5 July 1990;
in revised form 21 February 1991

Relative consistency results via strong compactness

by

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Abstract. We show in this paper that certain relative consistency proofs which had originally been done using supercompactness can be recast, using Henle's notion of modified Prikry forcing, in terms of strong compactness.

The notion of strongly compact cardinal is perhaps the most peculiar in the entire litany of large cardinal axioms. The well known results of Magidor [M] and Kimchi and Magidor [KM] show that strongly compact cardinals suffer from a severe identity crisis: The least strongly compact cardinal can be either the least measurable cardinal or the least supercompact cardinal, and the class of strongly compact cardinals can coincide precisely with the class of measurable cardinals or with the class of supercompact cardinals (except at limit points). It is further the case that the consistency strength of strongly compact cardinals is still unknown. Guesses on their consistency strength range from equiconsistent with supercompacts to a consistency strength far below that of supercompactness.

One of the most frustrating aspects of working with strongly compact cardinals is their intractability in forcing constructions due to a lack of the normality and closure properties associated with supercompactness. Very few forcing proofs for this reason have been done using strongly compact cardinals. A notable exception is Gitik's construction of [G1] in which, starting from a class of strongly compact cardinals, a model in which all uncountable cardinals are singular is constructed.

In [H], a notion of modified Prikry forcing in which normal measures are not used was developed. We adapt this forcing construction to show that certain theorems originally proven using supercompactness can be reproven using strong compactness. Specifically, we establish the following results.

THEOREM 1. Con (ZFC + *There exist cardinals $\varkappa < \lambda$ so that $\varkappa$ is λ strongly compact and λ is measurable*) $\Rightarrow$ Con(ZF + *$\varkappa$ is a strong limit cardinal of cofinality ω carrying a Rowbottom filter + $\varkappa^+$ is a measurable cardinal which carries a normal measure*).

THEOREM 2. Con (ZFC + *There exist cardinals $\varkappa < \lambda$ so that $\varkappa$ is λ strongly compact and λ is measurable*) $\Rightarrow$ Con(ZF + *$\aleph_\omega$ is a strong limit cardinal carrying a Rowbottom filter + $\aleph_{\omega+1}$ is a measurable cardinal which carries a normal measure*).

* The research of the authors was partially supported by NSF Grants DMS-8616774 and INT-8513211.

THEOREM 3. Con (ZFC + *There exists a regular limit of strongly compact cardinals*) $\Rightarrow$ Con(ZF + *For every successor ordinal* α, $2^{\aleph_\alpha}$ *is a countable union of sets of cardinality* $\aleph_\alpha$).

Theorem 3 says that the Specker property holds at every successor $\aleph$ and was first established in [AG] using a regular limit of supercompact cardinals. Theorems 1 and 2 were established in [A1] and [A2] using cardinals $\varkappa < \lambda$ so that $\varkappa$ is 2^λ supercompact and λ is measurable and technical hypotheses which follow from the existence of cardinals $\varkappa < \lambda$ so that $\varkappa$ is λ supercompact and λ is measurable.

Section 1 of this paper is devoted to a discussion of the modified Prikry forcing of [H] in the context of strongly compact cardinals. Section 2 will contain the proofs of Theorems 1 and 2, except for Rowbottomness, which will be addressed in Section 3. Theorem 3 will be proved in the last section.

Basically, our notation is fairly standard. For ordinals $\alpha < \beta$, $[\alpha, \beta]$, $(\alpha, \beta]$, $[\alpha, \beta)$, and (α, β) are as in the usual interval notation, and $R(\alpha)$ denotes the universe through stage α. For $\varkappa < \lambda$ cardinals, $P_\varkappa(\lambda) = \{x \subseteq \lambda\colon |x| < \varkappa\}$, and for $\alpha < \varkappa$ an ordinal (usually finite or ω), $[P_\varkappa(\lambda)]^\alpha$ is the collection of α sequences from $P_\varkappa(\lambda)$ and $[P_\varkappa(\lambda)]^{<\alpha}$ is the collection of $<\alpha$ sequences from $P_\varkappa(\lambda)$. AC_ω is well ordered choice of length ω. $\mathrm{DC}_\varkappa$ is dependent choice of length $\varkappa$.

When forcing, if p and q are conditions, $q \Vdash p$ will mean that q contains more information than p, and for φ a formula in the appropriate forcing language, $p \parallel \varphi$ will mean that p decides the statement φ. For a set s in the generic extension, $\dot{s}$ will be a term denoting s.

An ultrafilter $\mathcal{U}$ on $P_\varkappa(\lambda)$ is said to be *fine* if for all $\alpha < \lambda$, $\{p \in P_\varkappa(\lambda)\colon \alpha \in p\} \in \mathcal{U}$. $\varkappa$ is λ *strongly compact* if $P_\varkappa(\lambda)$ carries a $\varkappa$-additive fine ultrafilter $\mathcal{U}$ (sometimes referred to as a *strongly compact measure*), and $\varkappa$ is *strongly compact* if $\varkappa$ is λ strongly compact for all cardinals $\lambda \geqslant \varkappa$. The reader desiring more information on the notions of measurability, strong compactness, or supercompactness should consult either [A1], [G1], or [M].

For $\varkappa < \lambda$ regular cardinals, the *Lévy collapse* of λ to $\varkappa^+$, $\mathrm{Col}(\varkappa, \lambda) = \{f\colon \varkappa \times \lambda \to \lambda\colon f$ is a function so that $|\mathrm{domain}(f)| < \varkappa$ and $f(\langle \alpha, \beta\rangle) < \beta\}$, ordered by $q \Vdash p$ iff $p \subseteq q$. The trivial condition is the empty set $\emptyset$. If $\beta_0 \in (\varkappa, \lambda)$ is a regular cardinal and $p \in \mathrm{Col}(\varkappa, \lambda)$, $p \restriction \beta_0 = \{\langle\langle \alpha, \beta\rangle, \gamma\rangle \in p\colon \beta < \beta_0\}$. $p \restriction \beta_0$ is then a condition in $\mathrm{Col}(\varkappa, \beta_0)$, and for G generic over $\mathrm{Col}(\varkappa, \lambda)$, $G \restriction \beta_0 = \{p \restriction \beta_0\colon p \in G\} = \{p \in G\colon p \in \mathrm{Col}(\varkappa, \beta_0)\}$ is generic over $\mathrm{Col}(\varkappa, \beta_0)$.

A cardinal $\varkappa$ is *Rowbottom* if for all $\gamma < \varkappa$ and all functions F from $[\varkappa]^{<\omega}$ to γ, there is a set $X \subseteq \varkappa$, of cardinality $\varkappa$, such that the range of F on $[X]^{<\omega}$ is countable. X is called *homogeneous* for F. $\varkappa$ carries a Rowbottom filter if there is a filter G on $\varkappa$ so that a homogeneous set may always be found in G.

§ 1. Modified Prikry forcing. This section will be devoted to a discussion of the modified Prikry forcing of [H] in the context of strongly compact cardinals. In this context, modified Prikry forcing has been used by others; in particular, Gitik employs this technique in [G1] and [G2]. If $V \models$ "$\varkappa$ is λ strongly compact and $\mathcal{U}$ is a fine ultrafilter on $P_\varkappa(\lambda)$", then let $\mathcal{F} = \{f\colon f$ is a function from $[P_\varkappa(\lambda)]^{<\omega}$ to $\mathcal{U}\}$. *Modified Prikry forcing* $\mathcal{P}_{\mathcal{U}}$ is defined as:

$$\{\langle p_0, \ldots, p_n, f\rangle\colon \langle p_0, \ldots, p_n\rangle \in [P_\varkappa(\lambda)]^{<\omega},\ p_i \cap \varkappa \neq p_j \cap \varkappa \text{ for } 0 \leqslant i < j \leqslant n, \text{ and } f \in \mathcal{F}\}.$$

Given two conditions $p = \langle p_0, \ldots, p_n, f\rangle$ and $q = \langle q_0, \ldots, q_m, g\rangle$, $q \Vdash p$ iff the following conditions hold:

1) $n \leqslant m$,
2) $\langle p_0, \ldots, p_n\rangle = \langle q_0, \ldots, q_n\rangle$,
3) $g \subseteq f$, i.e., for $\langle r_0, \ldots, r_k\rangle \in [P_\varkappa(\lambda)]^{<\omega}$, $g(\langle r_0, \ldots, r_k\rangle) \subseteq f(\langle r_0, \ldots, r_k\rangle)$,
4) for $n+1 \leqslant i \leqslant m$, $q_i \in f(\langle p_0, \ldots, p_n, q_{n+1}, \ldots, q_{i-1}\rangle)$.

Let G be V-generic over $\mathcal{P}_{\mathcal{U}}$. A routine density argument shows that for any regular $\delta \in [\varkappa, \lambda]$, $r_\delta = \{\langle p_0 \cap \delta, \ldots, p_n \cap \delta\rangle\colon \exists f \in \mathcal{F}\ [\langle p_0, \ldots, p_n, f\rangle \in G]\}$ (sometimes written as $r \restriction \delta$) codes a cofinal ω sequence through δ. The following analog of the basic lemma of Prikry forcing shows that V and $V[G]$ contain the same bounded subsets of $\varkappa$.

LEMMA 1.1 (Lemma 4.1 of [H]). *Given any formula φ in the forcing language with respect to $\mathcal{P}_{\mathcal{U}}$ and any condition $\langle p_0, \ldots, p_n, f\rangle$ there is some $g \subseteq f$, $g \in \mathcal{F}$ so that $\langle p_0, \ldots, p_n, g\rangle \parallel \varphi$.*

Proof. Let s be the sequence $\langle p_0, \ldots, p_n\rangle$, and for $t = \langle q_0, \ldots, q_m\rangle \in [P_\varkappa(\lambda)]^{<\omega}$, $0 \leqslant i \leqslant m$, let $t \restriction i = \langle q_0, \ldots, q_{i-1}\rangle$, with $t \restriction 0$ the empty sequence. For any such t, call t *sufficient* if for some $g \in \mathcal{F}$, $\langle s^\frown t, g\rangle \parallel \varphi$, where $s^\frown t$ is just the sequence composed of the elements of s followed by the elements of t. For t sufficient, let g_t be a witness. If the empty sequence is sufficient, then we are done. If not, then for any appropriate $t = \langle t_0, \ldots, t_m\rangle$, sufficient or not, either $X_t = \{q \in P_\varkappa(\lambda)\colon t^\frown q$ is sufficient$\}$ or $Y_t = \{q \in P_\varkappa(\lambda)\colon t^\frown q$ is not sufficient$\}$ is in $\mathcal{U}$, since $\{q\colon$ for $0 \leqslant i \leqslant m$, $q \cap \varkappa \neq q_i \cap \varkappa\} \in \mathcal{U}$ by the fineness and $\varkappa$-additivity of $\mathcal{U}$. For A_t that set, define f' by

$$f'(t) = f(t) \cap \bigcap_{i \leqslant \mathrm{length}(t)} g_{t \restriction i}(t) \cap A_t.$$

Now let $t \in [P_\varkappa(\lambda)]^{<\omega}$ be sufficient and of minimal length $m+1$, and let t' be the sequence t without its last element q_m. It follows that $A_{t'}$ must be $X_{t'}$, so for every $q \in f'(t')$, $t'^\frown q$ is sufficient. Thus, one of the sets $X = \{q\colon \langle s^\frown t'^\frown q, g_{t'^\frown q}\rangle \Vdash \varphi\}$ or $Y = \{q\colon \langle s^\frown t'^\frown q, g_{t'^\frown q}\rangle \Vdash \neg\varphi\}$ is in $\mathcal{U}$. For Z that set, form g from f' by letting $g(t') = f(t') \cap Z$ and $g(x) = f'(x)$ otherwise. It is now the case that $\langle s^\frown t', g\rangle \parallel \varphi$. To see this, suppose that Z is Y, and suppose further that some extension of $\langle s^\frown t', g\rangle$ forces φ. Such a condition must add to $s^\frown t'$ since t' is not sufficient. The next element added to $s^\frown t'$, q, must come from Y, giving a condition $\langle s^\frown t'^\frown q^\frown u, g'\rangle$ forcing φ. By construction, however, $\langle s^\frown t'^\frown q^\frown u, g'\rangle \Vdash \langle s^\frown t'^\frown q, f'\rangle \Vdash \langle s^\frown t'^\frown q, g_{t'^\frown q}\rangle$, and this last condition forces $\neg\varphi$, a contradiction.

We therefore have $\langle s^\frown t', g\rangle \parallel \varphi$. This, however, contradicts the minimality of the length of t for sufficiency. This contradiction proves Lemma 1.1. ∎

By the fact that $\mathcal{U}$ is $\varkappa$-additive, the same proof as in ordinary Prikry forcing [P] shows that V and $V[G]$ contain the same bounded subsets of $\varkappa$.

The Mathias criterion [Ma] for Prikry genericity states that an ω-sequence $\langle \alpha_n \colon n<\omega\rangle$ is Prikry generic with respect to Prikry forcing defined using the measure μ iff if $A\in\mu$, then there is some $m<\omega$ so that $\langle \alpha_n\colon m\leqslant n<\omega\rangle\subseteq A$, i.e., iff every μ measure-one set contains a "tail" of the ω-sequence. There is an analogous property for modified Prikry forcing. If $r=\langle p_n\colon n<\omega\rangle\in[P_\varkappa(\lambda)]^\omega$, then using the notation $r\restriction n=\langle p_0,\ldots,p_{n-1}\rangle$, we can define the set $H_r=\{\langle r\restriction n,f\rangle\colon f\in\mathscr{F}$ is so that if $n<k$ and $i\in(n,k]$ then $p_k\in f(r\restriction i)\}$. The Mathias-like property is then stated as follows.

LEMMA 1.2 (Lemma 4.3 of [H]). *For all $r\in[P_\varkappa(\lambda)]^\omega$, H_r is V-generic over $\mathscr{P}_{\mathscr{U}}$ iff for all $f\in\mathscr{F}$ there is an $n<\omega$ with $\langle r\restriction n,f\rangle\in H_r$.*

Proof. If H_r is V-generic over $\mathscr{P}_{\mathscr{U}}$, then given $f\in\mathscr{F}$, let $\mathscr{D}=\{\langle t,g\rangle\colon g\subseteq f\}$. Since $\mathscr{D}$ is dense open, $\mathscr{D}\cap H_r\neq\emptyset$. Thus, for some n, $\langle r\restriction n,g\rangle\in H_r$, so since $g\subseteq f$, $\langle r\restriction n,f\rangle\in H_r$.

Now suppose for all $f\in\mathscr{F}$ there is an $n<\omega$ with $\langle r\restriction n,f\rangle\in H_r$, and suppose $\mathscr{D}\subseteq\mathscr{P}_{\mathscr{U}}$ is dense open. For $t\in[P_\varkappa(\lambda)]^{<\omega}$, $n<\omega$, we will say that t is *n-capturable* iff for some f and for all $u\in[P_\varkappa(\lambda)]^n$, $\langle t^\frown u,f\rangle\Vdash\langle t,f\rangle$ implies $\langle t^\frown u,f\rangle\in\mathscr{D}$. We consider two cases:

Case 1: $r\restriction k$ is n-capturable for some k and some n. If this is true, then let f be the witness. By assumption, there is some m so that $\langle r\restriction m,f\rangle\in H_r$. Let $m'>\max(k+n,m)$. We will still have $\langle r\restriction m',f\rangle\in H_r$, and since $\langle r\restriction m',f\rangle\in\mathscr{D}$ (by the fact $\mathscr{D}$ is open), $H_r\cap\mathscr{D}\neq\emptyset$.

Case 2: For no k and n is $r\restriction k$ n-capturable. In this case, for each appropriate $t\in[P_\varkappa(\lambda)]^{<\omega}$, let $A_t\in\mathscr{D}$ be either $B^t_\omega=\{q\colon t^\frown q$ is not k-capturable for all $k\}$ or $B^t_k=\{q\colon t^\frown q$ is k-capturable$\}$, whichever has measure 1, and define $f\colon[P_\varkappa(\lambda)]^{<\omega}\to\mathscr{U}$ by $f(t)=\bigcup_{i\leqslant\mathrm{length}(t)}A_{t\restriction i}$ if $t=\langle t_0,\ldots,t_p\rangle$ has the property that for $0\leqslant i<j\leqslant p$, $t_i\cap\varkappa\neq t_j\cap\varkappa$ and $f(t)=P_\varkappa(\lambda)$ otherwise. By assumption, there is an m so that $\langle r\restriction m,f\rangle\in H_r$. As $\mathscr{D}$ is dense, there is an extension $\langle(r\restriction m)^\frown u,f'\rangle$ in $\mathscr{D}$. This u is then 0-capturable. Choose now a $t\in[P_\varkappa(\lambda)]^{<\omega}$ of minimal length so that $(r\restriction m)^\frown t$ is k-capturable for some k. Since we are not in case 1, $\mathrm{length}(t)=n+1>0$. Let $t'=t\restriction n$ be the first n elements of t. We must have $q\in f((r\restriction m)^\frown t')$, so $q\in A_{(r\restriction m)^\frown t'}$, and this means $A_{(r\restriction m)^\frown t'}=B_k^{(r\restriction m)^\frown t'}$. It follows that f witnesses that $(r\restriction m)^\frown t'$ is $(k+1)$-capturable. This contradicts our choice of t, and so case 2 cannot occur.

Since H_r is compatible and closed under weakening of conditions, H_r is V-generic over $\mathscr{P}_{\mathscr{U}}$. This proves Lemma 1.2. ■

We note that as with ordinary Prikry forcing, the generic sequence r completely determines the generic object G, and $G=H_r$. This is easily seen. By the genericity of G, if $\langle s,f\rangle=\langle r\restriction n,f\rangle\in G$, then $\langle s,f\rangle\in H_r$. Thus, $G\subseteq H_r$, so H_r intersects every dense open subset of $\mathscr{P}_{\mathscr{U}}$ in V. Hence, by the remarks concluding the proof of Lemma 1.2, H_r is V-generic over $\mathscr{P}_{\mathscr{U}}$, meaning that $G=H_r$. This observation will be key in the proofs of Theorems 1, 2 and 3.

For $\delta\in[\varkappa,\lambda)$, δ an inaccessible cardinal, we discuss a notion of restricted genericity through δ. If $x\subseteq P_\varkappa(\lambda)$, let $x\restriction\delta=\{Z\cap\delta\colon Z\in x\}$, and let $\mathscr{U}\restriction\delta=\{x\restriction\delta\colon x\in\mathscr{U}\}$. Since $\mathscr{U}$ is

a strongly compact measure on $P_\varkappa(\lambda)$, $\mathscr{U}\restriction\delta$ is a strongly compact measure on $P_\varkappa(\delta)$. We can thus define a modified Prikry forcing $\mathscr{P}_{\mathscr{U}\restriction\delta}$. Our goal is to show that r_δ is V-generic over $\mathscr{P}_{\mathscr{U}\restriction\delta}$ and to define the appropriate notion of restriction of condition.

Towards this end, given $p^*=\langle p_0,\ldots,p_n,f\rangle=\langle p,f\rangle\in\mathscr{P}_{\mathscr{U}}$, we wish to define $p^*\restriction\delta\in\mathscr{P}_{\mathscr{U}\restriction\delta}$. First, let $p\restriction\delta=\langle p_0\cap\delta,\ldots,p_n\cap\delta\rangle$. The second coordinate, which we will call $f\restriction^p\delta$, depends on p as well as f. We define it inductively. To begin, if there is no $n<\omega$ so that x is of the form $(p\restriction\delta)^\frown r^*$ for $r^*\in[P_\varkappa(\lambda)]^n$ we let $(f\restriction^p\delta)(x)=P_\varkappa(\delta)$. For all $r^*\in[P_\varkappa(\delta)]^n$, $n<\omega$, we define inductively and simultaneously $(f\restriction^p\delta)((p\restriction\delta)^\frown r^*)\in\mathscr{U}\restriction\delta$ and a set $S_{r^*}\subseteq[P_\varkappa(\lambda)]^n$ as follows:

If $r^*=\emptyset$, then $S_{r^*}=\emptyset$, and $(f\restriction^p\delta)(p\restriction\delta)=f(p)\restriction\delta$.

If $r^*=u^\frown\{a\}$, then $S_{r^*}=\{s^\frown\{t\}\colon s\in S_u$ and $t\in f(p^\frown s)$ is so that $t\cap\delta=a\}$, and $(f\restriction^p\delta)((p\restriction\delta)^\frown r^*)=[\bigcup_{s\in S_{r^*}}f(p^\frown s)]\restriction\delta$ if $S_{r^*}\neq\emptyset$. If $S_{r^*}=\emptyset$, then $(f\restriction^p\delta)((p\restriction\delta)^\frown r^*)=P_\varkappa(\delta)$.

Note that $s\in S_{r^*}$ implies $s\restriction\delta=r^*$. We define $p^*\restriction\delta$ as $\langle p\restriction\delta,f\restriction^p\delta\rangle$.

LEMMA 1.3. *For any $\langle p,f\rangle\in\mathscr{P}_{\mathscr{U}}$ and any $\langle(p\restriction\delta)^\frown r^*,g\rangle\in\mathscr{P}_{\mathscr{U}\restriction\delta}$ extending $\langle p,f\rangle\restriction\delta$ in $\mathscr{P}_{\mathscr{U}\restriction\delta}$ there is a function $h\in\mathscr{F}$ and an $s\in S_{r^*}$ so that $\langle p^\frown s,h\rangle\in\mathscr{P}_{\mathscr{U}}$ extends $\langle p,f\rangle$ in $\mathscr{P}_{\mathscr{U}}$ and $h\restriction\delta\subseteq g$. Further, $s\restriction\delta=r^*$.*

Proof. To construct h, we first define $k(u)=\{x\cup y\colon x\in P_\varkappa(\lambda\setminus\delta)$ and $y\in g(u)\}\in\mathscr{U}$ if $u\in P_\varkappa(\delta)$ and $k(u)=P_\varkappa(\lambda)$ otherwise. Next, we let $h=k\cap f$, i.e., for $x\in\mathrm{domain}(k)\cap\mathrm{domain}(f)$, $h(x)=k(x)\cap f(x)$. Clearly, $h\restriction\delta\subseteq g$. Since it is equally clear that $h\subseteq f$, it remains only to find an appropriate s. This is done by induction on the length of r^*.

For $r^*=\emptyset$, this is trivial. Now let $r^*=\langle r^*_0,\ldots,r^*_i\rangle$ and suppose we have established the lemma for all shorter sequences. Since $\langle(p\restriction\delta)^\frown r^*,g\rangle\Vdash\langle p\restriction\delta,f\restriction^p\delta\rangle$ we have

$$r^*_i\in(f\restriction^p\delta)((p\restriction\delta)^\frown(r^*\restriction i))=\Big[\bigcup_{v\in S_{r^*\restriction i}}f(p^\frown v)\Big]\restriction\delta;$$

this is true since by applying the induction hypothesis to $\langle(p\restriction\delta)^\frown(r^*\restriction i),g\rangle$, we have $S_{r^*\restriction i}\neq\emptyset$. Thus, there is a $v\in S_{r^*\restriction i}$ and a $t\in f(p^\frown v)$ so that $t\cap\delta=r^*_i$. We then have $v^\frown\{t\}\in S_{r^*}$, and we will show that $s=v^\frown\{t\}$ satisfies the conditions of the lemma.

It is immediately true that $s\restriction\delta=r^*$. Now suppose that $0\leqslant j\leqslant i$. As $s\restriction(j+1)\in S_{r^*\restriction(j+1)}$, $s_j\in f(p^\frown(s\restriction j))$. This is enough to show $\langle p^\frown s,h\rangle\Vdash\langle p,f\rangle$, proving Lemma 1.3. ■

LEMMA 1.4. *If $\langle p^\frown q,h\rangle\Vdash\langle p,f\rangle$, then $\langle p^\frown q,h\rangle\restriction\delta\Vdash\langle p,f\rangle\restriction\delta$.*

Proof. It is easy to see inductively that S_{r^*} as defined for the first condition is contained in the corresponding S_{r^*} for the second condition. It then follows that $h\restriction^{p^\frown q}\delta\subseteq f\restriction^p\delta$. It remains only to show that for all i in the domain of q, $q_i\cap\delta=(f\restriction^p\delta)((p\restriction\delta)^\frown((q\restriction\delta)\restriction i))$. For this, we need that $q_i\in\bigcup_{s\in S_{(q\restriction\delta)\restriction i}}f(p^\frown s)$, and for this, it suffices to show that $q\restriction i\in S_{(q\restriction\delta)\restriction i}$ since we know that $q_i\in f(p^\frown(q\restriction i))$. This last fact follows easily by induction. This proves Lemma 1.4. ■

Now, for G V-generic over $\mathscr{P}_{\mathscr{U}}$, let $G\restriction\delta=\{p^*\restriction\delta\colon p^*\in G\}$.

LEMMA 1.5. *$G\restriction\delta$ is V-generic over $\mathscr{P}_{\mathscr{U}\restriction\delta}$.*

Proof. We note first that Lemma 1.4 shows that $G\restriction\delta$ is compatible. In addition, the prior definitions ensure that $G\restriction\delta$ is closed under weakening of conditions.

Suppose now that $\mathcal{D}$ is a dense open subset of $\mathcal{P}_{\mathcal{U}\restriction\delta}$ which fails to meet $G\restriction\delta$. Let $\mathcal{D}^+ = \{\langle p,f\rangle\in\mathcal{P}_{\mathcal{U}}: \langle p,f\rangle\restriction\delta\in\mathcal{D}\}$. We will show that $\mathcal{D}^+$ is dense open in $\mathcal{P}_{\mathcal{U}}$, which will automatically suffice since there will then be a $\langle p,f\rangle\in\mathcal{D}^+\cap G$, which immediately yields that $\langle p,f\rangle\restriction\delta\in\mathcal{D}\cap G\restriction\delta$.

Let $\langle p,f\rangle$ be any condition. Since $\mathcal{D}$ is dense, there is a $\langle(p\restriction\delta)^\frown r^*, g\rangle\in\mathcal{D}$ extending $\langle p\restriction\delta, f\restriction^p\delta\rangle$. Let h and s be as in Lemma 1.3. Then $\langle p^\frown s,h\rangle\Vdash\langle p,f\rangle$ and $\langle p^\frown s,h\rangle\in\mathcal{D}^+$, since $\langle p^\frown s,h\rangle\restriction\delta\Vdash\langle(p\restriction\delta)^\frown r^*,g\rangle\in\mathcal{D}$ and $\mathcal{D}$ is open. This proves Lemma 1.5. ∎

By the definition of $G\restriction\delta$, it is now clear that r_δ is the generic sequence generated by $G\restriction\delta$. As noted earlier, r_δ completely determines $G\restriction\delta$.

§ 2. Making the successor of a singular cardinal measurable. The proofs of Theorems 1 and 2 are quite similar to the proofs given in [A1] and [A2]. For Theorem 1, let $\varkappa<\lambda$ be so that $V\models$ "$\varkappa$ is λ strongly compact and λ is measurable", and let $\mathcal{U}\in V$ and $\nu\in V$ be, respectively, a strongly compact measure on $P_\varkappa(\lambda)$ and a normal measure on λ. Let $\mathcal{P}_{\mathcal{U}}$ be as before, and for G V-generic over $\mathcal{P}_{\mathcal{U}}$, $\delta\in[\varkappa,\lambda)$ an inaccessible cardinal, let r_δ be as defined in Section 1. The model N_1 witnessing the conclusions of Theorem 1 will be the least model of ZF extending V and containing r_δ, for each inaccessible $\delta\in[\varkappa,\lambda)$ (but *not* the λ-sequence of the r_δ's). More formally, let $\mathcal{L}$ be the forcing language associated with $\mathcal{P}$, and let $\mathcal{L}_1\subseteq\mathcal{L}$ be the ramified sublanguage of $\mathcal{P}$ containing symbols $\mathring{v}$ for each $v\in V$, a unary predicate symbol $\mathring{V}$ (interpreted as $\mathring{V}(\mathring{v})\Leftrightarrow v\in V$), and symbols $\mathring{r}_\delta$ for each $\delta\in[\varkappa,\lambda)$ an inaccessible cardinal. N_1 is then defined inductively inside $V[G]$ as follows:

$$N_{1,0}=\emptyset,$$
$$N_{1,\delta}=\bigcup_{\alpha<\delta}N_{1,\alpha}\ \text{for}\ \delta\ \text{a limit ordinal},$$
$$N_{1,\alpha+1}=\{x\subseteq N_{1,\alpha}: x\in V[G]\ \text{and}\ x\ \text{can be defined over}\ \langle N_{1,\alpha},\in,c\rangle, c\in N_{1,\alpha},\ \text{using a forcing term}\ \tau\in\mathcal{L}_1\ \text{of rank}\ \leqslant\alpha\},$$
$$N_1=\bigcup_{\alpha\in\text{Ordinals}^V}N_{1,\alpha}.$$

Standard arguments shows that $N_1\models$ ZF. Also, as usual, each $\mathring{v}$ for $v\in V$ may be chosen so as to be invariant under any automorphism of $\mathcal{P}_{\mathcal{U}}$, and terms τ mentioning only $\mathring{r}_\delta$ may be chosen so as to be invariant under any automorphism of $\mathcal{P}_{\mathcal{U}}$ which preserves the meaning of r_δ.

Lemma 2.1. *Let $x\in N_1$ be a set of ordinals. Then for some $\delta\in[\varkappa,\lambda)$, δ inaccessible, $x\in V[r_\delta]$.*

Proof. Let τ be a term for x and let η be an ordinal so that for some $p=\langle p_0,\ldots,p_n,f\rangle$, $p\Vdash$ "$\tau\subseteq\eta$". Since $x\in N_1$, we can assume without loss of generality that τ mentions only finitely many terms of the form $\mathring{r}_\delta$. By the usual coding tricks, τ can be assumed to mention only one term of the form $\mathring{r}_\delta$. We show that $x\in V[r_\delta]$.

Assume that $p\in G$. Let σ be the term defined by $q\Vdash$ "$\check{\alpha}\in\sigma$" iff $q\Vdash p$, $q\restriction\delta\in G\restriction\delta$, and $q\Vdash$ "$\check{\alpha}\in\tau$". Clearly, σ denotes a set in $V[r_\delta]$. We show that $p\Vdash$ "$\sigma=\tau$". First, if $q\in G$, $q\Vdash p$, and $q\Vdash$ "$\check{\alpha}\in\tau$", then $q\restriction\delta\in G\restriction\delta$. Thus, $q\Vdash$ "$\check{\alpha}\in\sigma$", so $p\Vdash$ "$\tau\subseteq\sigma$".

Let now $q=\langle q_0,\ldots,q_l,g\rangle\Vdash p$ be so that $q\restriction\delta\in G\restriction\delta$ and $q\Vdash$ "$\check{\alpha}\in\tau$". By the genericity of G, there is some $q'=\langle q'_0,\ldots,q'_m,g'\rangle\in G$ so that $q'\parallel$ "$\check{\alpha}\in\tau$". Without loss of generality, we can assume that $l<m$. If $q'\Vdash$ "$\check{\alpha}\in\tau$", then we are done, so assume that $q'\Vdash$ "$\check{\alpha}\notin\tau$". Since $q'\restriction\delta\in G\restriction\delta$, $q\restriction\delta\in G\restriction\delta$, and $l<m$, we know that for $0\leqslant i\leqslant l$, $q_i\cap\delta=q'_i\cap\delta$, and also that there is some $\langle q_0\cap\delta,\ldots,q_l\cap\delta,r^*_{l+1},\ldots,r^*_m,g''\rangle\in G\restriction\delta$ extending $q\restriction\delta$ with $r^*_i=q_i\cap\delta$ for $l+1\leqslant i\leqslant m$. Thus, by Lemma 1.3, there is some $q''=\langle q_0,\ldots,q_l,s_{l+1},\ldots,s_m,h\rangle$ extending q so that for $l+1\leqslant i\leqslant m$, $s_i\cap\delta=r^*_i=q'_i\cap\delta$. Since $q''\Vdash q$, $q''\Vdash$ "$\check{\alpha}\in\tau$".

Consider now the permutation Ψ of $P_\varkappa(\lambda)$ given by $\Psi(q_i)=q'_i$ and $\Psi(q'_i)=q_i$ for $0\leqslant i\leqslant l$, $\Psi(s_i)=q'_i$ and $\Psi(q'_i)=s_i$ for $l+1\leqslant i\leqslant m$, and Ψ is the identity otherwise. For any condition $\langle t_0,\ldots,t_k,h'\rangle\in\mathcal{P}_{\mathcal{U}}$, consider the function $\pi\colon\mathcal{P}_{\mathcal{U}}\to\mathcal{P}_{\mathcal{U}}$ given by $\pi(\langle t_0,\ldots,t_k,h'\rangle)=\langle\Psi(t_0),\ldots,\Psi(t_k),\Psi(h')\rangle$, where the action of Ψ on h', $\Psi(h')$, is given as follows: If $h'(\langle u_0,\ldots,u_j\rangle)=A\in\mathcal{U}$, then $\Psi(h'(\langle u_0,\ldots,u_j\rangle))=h'(\langle\Psi(u_0),\ldots,\Psi(u_j)\rangle)$ has value $\{\Psi(u): u\in A\}$ which, since Ψ permutes only finitely many elements, is a $\mathcal{U}$ measure 1 set. It can be verified that π is an automorphism of $\mathcal{P}_{\mathcal{U}}$, and by construction, $\pi(q'')=\langle\Psi(q_0),\ldots,\Psi(q_l),\Psi(s_{l+1}),\ldots,\Psi(s_m),\Psi(h)\rangle=\langle q'_0,\ldots,q'_m,\Psi(h)\rangle$ is compatible with q'. Since $\langle q_0\cap\delta,\ldots,q_l\cap\delta,s_{l+1}\cap\delta,\ldots,s_m\cap\delta\rangle=\langle q'_0\cap\delta,\ldots,q'_m\cap\delta\rangle$, Ψ is the identity except possibly on $S=\{q_0,\ldots,q_l,s_{l+1},\ldots,s_m,q'_0,\ldots,q'_m\}$, and any $\langle q_0,\ldots,q_l,s_{l+1},\ldots,s_m,s_{m+1},\ldots,s_j,h'\rangle$ extending q'' must be so that $s_i\notin S$ for $m+1\leqslant i\leqslant j$ (by the fact $s_i\cap\varkappa$ for $m+1\leqslant i\leqslant j$ cannot equal $u\cap\varkappa$ for any $u\in S$), π does not affect the meaning of r_δ. Therefore, the properties of τ ensure that $\pi(q'')\Vdash$ "$\check{\alpha}\in\tau$". Since $q'\Vdash$ "$\check{\alpha}\notin\tau$", this is a contradiction. Hence, $p\Vdash$ "$\sigma\subseteq\tau$", so $p\Vdash$ "$\tau=\sigma$". This proves Lemma 2.1. ∎

Lemma 2.2. *V and N_1 contain the same bounded subsets of $\varkappa$.*

Proof. By Lemma 2.1, for any $x\in N_1$, x a bounded subset of $\varkappa$, $x\in V[r_\delta]$ for some $\delta\in[\varkappa,\lambda)$, δ inaccessible. Since $G\restriction\delta$ is V-generic over $\mathcal{P}_{\mathcal{U}\restriction\delta}$, Lemma 1.1 shows that $V[G\restriction\delta]$ (or $V[r_\delta]$) contains the same bounded subsets of $\varkappa$ that V does. Thus, $x\in V$. ∎

Lemma 2.3. *$N_1\models$ "$\varkappa$ is a strong limit cardinal of confinality ω".*

Proof. Since $V\models$ "$\varkappa$ is a strong limit cardinal", it immediately follows from Lemma 2.2 that $N_1\models$ "$\varkappa$ is a strong limit cardinal". Since $r_\varkappa\in N_1$, $N_1\models$ "cof$(\varkappa)=\omega$". ∎

Lemma 2.4. *$N_1\models$ "$\lambda\leqslant\varkappa^+$".*

Proof. Lemma 2.4 is proven by showing that no ordinal $\delta\in(\varkappa,\lambda)$ which is a cardinal in V remains a cardinal in N_1. To show this, we let $\beta\in(\delta,\lambda)$ be an inaccessible cardinal in V. We then show that in $V[r_\beta]$, δ is no longer a cardinal. As $V[r_\beta]\subseteq N_1$, the collapsing function for δ will be present in N_1, showing that $N_1\models$ "δ is not a cardinal".

Proceeding with the proof, we show that there are no cardinals in the interval $(\varkappa,\beta]$ in $V[r_\beta]$. To do this, let α be the least cardinal in V which remains a cardinal in $V[r_\beta]$, and consider two cases:

Case 1: α is a regular cardinal in V. Since $r_\alpha\in V[r_\beta]$, $V[r_\beta]\models$ "cof$(\alpha)=\omega$", so by the leastness of α, $V[r_\beta] \models$ "$\alpha = \varkappa^+$ and $\mathrm{cof}(\varkappa^+) = \omega$". As $V[r_\beta] \models$ ZFC, this is impossible.

Case 2: α is a singular cardinal in V. In this case, again $V[r_\beta] \models$ "$\alpha = \varkappa^+$", so $V[r_\beta] \models$ "ZFC + $\varkappa^+$ is singular", an impossibility.

Thus, no $\delta \in (\varkappa, \lambda)$ is a cardinal in N_1. This proves Lemma 2.4. ∎

LEMMA 2.5. *$N_1 \models$ "$\nu^* = \{x \subseteq \lambda$: x contains a ν measure 1 set$\}$ is a normal measure on λ".*

Proof. If $x \subseteq \lambda$ is a set in N_1, then by Lemma 2.1 let $\delta \in [\varkappa, \lambda)$ be so that $x \in V[r_\delta]$. Since $|\mathscr{P}_{\mathscr{U}_\delta}| < \lambda$, by the Lévy–Solovay results [LS], $V[r_\delta] \subseteq N_1$ satisfies "Either x or $\lambda \backslash x$ contains a ν measure 1 set". If $N_1 \models$ "$\langle x_\alpha$: $\alpha < \gamma < \lambda\rangle$ is a sequence of ν measure 1 sets", then since $\langle x_\alpha$: $\alpha < \gamma < \lambda\rangle$ can be coded by a set of ordinals, there is some $\delta \in [\varkappa, \lambda)$ so that $\langle x_\alpha$: $\alpha < \gamma < \lambda\rangle \in V[r_\delta]$. Since the results of [LS] imply that $V[r_\delta] \models$ "$\nu_\delta^* = \{x \subseteq \lambda$: x contains a ν measure 1 set$\}$ is a normal measure on λ", $V[r_\delta] \models$ "$\bigcap_{\alpha<\gamma} x_\alpha \in \nu_\delta^*$", i.e., $V[r_\delta] \models$ "$\bigcap_{\alpha<\gamma} x_\alpha$ contains a ν measure 1 set". Thus, $N_1 \models$ "$\bigcap_{\alpha<\gamma} x_\alpha \in \nu^*$". Finally, if $N_1 \models$ "f: $\lambda \to \lambda$ is a regressive function", then since f can be coded by a set of ordinals, let δ be so that $f \in V[r_\delta]$. By our earlier remarks, $V[r_\delta] \models$ "f is constant on a ν measure 1 set", i.e., both $V[r_\delta]$ and N_1 satisfy "f is constant on a ν measure 1 set", so $N_1 \models$ "f is constant on a ν^* measure 1 set". This proves Lemma 2.5. ∎

Lemmas 2.1–2.5 complete the proof of Theorem 1, except for Rowbottomness.

Since the proof of Theorem 2 from Theorem 1 is exactly the same as in [A1] and [A2], we only sketch it here and refer the reader to these papers for further details. As V and N_1 contain the same bounded subsets of $\varkappa$, any cardinal $\delta < \varkappa$ which is (strongly) inaccessible in V remains (strongly) inaccessible in N_1. Since in V the inaccessible cardinals below $\varkappa$ are unbounded in $\varkappa$ and $N_1 \models$ "$\mathrm{cof}(\varkappa) = \omega$", we can first choose in N_1 a sequence $\langle \alpha_n$: $n < \omega\rangle$ cofinal in $\varkappa$ and then use this sequence to define canonically in N_1 a sequence $\langle \varkappa_n$: $n < \omega\rangle$ of (strongly) inaccessible cardinals cofinal in $\varkappa$. Using $\langle \varkappa_n$: $n < \omega\rangle$, we construct our model N_2 for Theorem 2 as follows. Working in N_1, let $Q_0 = \mathrm{Col}(\omega_1, \varkappa_0)$, and for $i > 0$ let $Q_i = \mathrm{Col}(\varkappa_{i-1}, \varkappa_i)$. Define $\hat{Q} = \prod_{i<\omega} Q_i$, and take as the forcing conditions Q the set $\{p \in \hat{Q}$: the ith coordinate of p, p_i, is non-empty only finitely often$\}$, ordered by $q \Vdash p$ iff $\forall i [q_i \Vdash p_i]$.

For any $n < \omega$, view Q as $Q_n \times Q^n$ where $Q_n = \prod_{i \leqslant n} Q_i$ and $Q^n = \{p \in \prod_{i>n} Q_i$: the kth coordinate of p, p_k, is non-empty only finitely often$\}$, with both partial orders ordered componentwise. Let H be N_1-generic over Q. It follows by the above remarks that $H_n = H \upharpoonright Q_n$ (the projection of H onto Q_n) is N_1-generic over Q_n.

The model N_2 which witnesses Theorem 2 is the least model of ZF extending N_1 containing each H_n (but *not* the ω-sequence of the H_n's). As in Theorem 1, we can talk about N_2 using a ramified sublanguage $\mathscr{L}_1 \subseteq \mathscr{L}$ of the forcing language with respect to Q, where $\mathscr{L}_1$ contains symbols $\check{v}$ for each $v \in N_1$, a unary predicate symbol $\mathring{N}_1$ (interpreted as $\mathring{N}_1(\check{v})$ iff $v \in N_1$), and symbols $\mathring{H}_n$ for each $n < \omega$. N_2 is then defined inductively inside $N_1[H]$ as follows:

$$N_{2,0} = \emptyset,$$

$$N_{2,\delta} = \bigcup_{\alpha<\delta} N_{2,\alpha}, \quad \text{for } \delta \text{ a limit ordinal},$$

$$N_{2,\alpha+1} = \{x \subseteq N_{2,\alpha}\colon x \in N_1[H] \text{ and } x \text{ can be defined over } \langle N_{2,\alpha}, \in, c\rangle_{c \in N_{2,\alpha}} \text{ using a forcing term } \tau \in \mathscr{L}_1 \text{ of rank } \leqslant \alpha\},$$

$$N_2 = \bigcup_{\alpha \in \mathrm{Ordinals}^{N_1}} N_{2,\alpha}.$$

The exact same arguments as in [A1] and [A2] then show that N_2 witnesses the conclusions of Theorem 2. Further, as in [A1] and [A2], the definition of N_2 ensures that AC_ω fails in N_2.

§ 3. Rowbottomness. Some additional notation: if f maps A to B, and $\mathscr{G}$ is a filter on A, then $f_*(\mathscr{G})$ is the filter on B defined by: for $X \subseteq B$, $X \in f_*(\mathscr{G})$ iff $f^{-1}(X) \in \mathscr{G}$.

Our measure $\mathscr{U}$ on $P_\varkappa(\lambda)$ may not be normal, but we may assume it has some degree of normality. Let k map $P_\varkappa(\lambda)$ to $\varkappa$ by $k(p) = p \cap \varkappa$. Let $\mathscr{U}_\varkappa$ be $k_*(\mathscr{U})$. We may at least assume that $\mathscr{U}_\varkappa$ is normal, since we can change $\mathscr{U}_\varkappa$ to make it normal as follows: let r: $P_\varkappa(\lambda) \to \varkappa$ be the least incompressible function, that is, the least (in the ultrapower of $P_\varkappa(\lambda)$) such that r is not constant on a set in $\mathscr{U}$, but $r(p) < k(p)$ on a set in $\mathscr{U}$ (r is the least counterexample to the failure of $\mathscr{U}_\varkappa$'s normality). Next, let l map $P_\varkappa(\lambda)$ to $P_\varkappa(\lambda)$ by setting $l(p) = (p \backslash \varkappa) \cup (p \cap r(p))$. It is routine to show that $l_*(\mathscr{U})$ is a fine measure on $P_\varkappa(\lambda)$, and that $k_*(l_*(\mathscr{U}))$ is a normal measure on $\varkappa$.

We will thus assume that $\mathscr{U}_\varkappa$ is normal on $\varkappa$. It also follows that $\mathscr{U}$ has a limited normality property: if s: $P_\varkappa(\lambda) \to \varkappa$ is *regressive*, i.e., $s(p) \in p$ on a set in $\mathscr{U}$, then s is constant on a set in $\mathscr{U}$. Equivalently, $\mathscr{U}$ has a weak diagonalization property, namely that if $\{X_\alpha\}_{\alpha<\varkappa} \subseteq \mathscr{U}$, then

$$\underset{\varkappa}{\triangle} X_\alpha = \{p\colon \alpha \in p \cap \varkappa \Rightarrow p \in X_\alpha\}$$

is in $\mathscr{U}$. These properties are easy to verify.

We begin by proving the following:

LEMMA 3.1. *If G is V-generic over $\mathscr{P}_{\mathscr{U}}$, then $V[G] \models$ "$\varkappa$ is a Rowbottom cardinal".*

Proof. Let the sequence defined by G be $p_0, p_1, \ldots$ Suppose that $\langle p_0, \ldots, p_r, u\rangle \in G$ forces that F: $[P_\varkappa(\lambda)]^{<\omega} \to \gamma < \varkappa$ is a counterexample to the Rowbottomness of $\varkappa$.

Let $\mathscr{F}_\varkappa = \{f\colon f\colon [P_\varkappa(\lambda)]^{<\omega} \to \mathscr{U}_\varkappa\}$. For any $f \in \mathscr{F}_\varkappa$ and any H V-generic over $\mathscr{P}_{\mathscr{U}}$, we can define a subset $X_{H,f}$ of $\varkappa$ as follows:

$$X_{H,f} = [f(\emptyset) \cap (p_0 \cap \varkappa)] \cup [f(p_0) \cap [(p_1 \cap \varkappa) \backslash (p_0 \cap \varkappa)]] \cup [f(p_0, p_1) \cap [(p_2 \cap \varkappa) \backslash (p_1 \cap \varkappa)]] \cup \ldots$$

CLAIM. *$\mathscr{U}_H = \{X_{H,f}\}_{f \in \mathscr{F}_\varkappa}$ is a uniform filter on $\varkappa$.*

Proof. It is clearly a filter, since $X_{H,f} \cap X_{H,f'}$ contains $X_{H,f \cap f'}$. Uniformity follows from a density argument. Let $\mathscr{D} = \{\langle r, g\rangle\colon \forall q_0, \ldots, q_k,\ p \in g(q_0, \ldots, q_k) \Rightarrow |f(q_0, \ldots, q_k) \cap [(p \cap \varkappa) \backslash (q_k \cap \varkappa)]| \geqslant q_k \cap \varkappa$. It is not hard to show that $\mathscr{D}$ is dense, and since G intersects $\mathscr{D}$, $|X_{H,f}| = \varkappa$, because $\varkappa = \bigcup |p_k \cap \varkappa|$.

Let $\mathscr{T}$ be the collection of all finite sequences of 0's and 1's. For $\pi \in \mathscr{T}$, $g \in \mathscr{F}$, $h \in \mathscr{F}_\varkappa$, $\sigma = \langle s_0, \ldots, s_k\rangle \in [P_\varkappa(\lambda)]^{<\omega}$ with each $s_i \cap \varkappa$ a cardinal, $\tau = \langle t_0, \ldots$

$\ldots, t_n\rangle \in [\varkappa]^{<\omega}$, we will say that $\langle\sigma, \tau\rangle$ is *appropriate for* π, g, h iff

1) $s_0 \cap \varkappa < s_1 \cap \varkappa < \ldots$,
2) $t_0 < t_1 < \ldots$,
3) $t_i \neq s_j \cap \varkappa$, for all i, j, and
4) when the $\{t_i\}_{i<n}$, $\{s_j \cap \varkappa\}_{<k}$ are arranged in order, we have a sequence, ϱ, with
   (a) $\mathrm{len}(\varrho) = \mathrm{len}(\pi)$,
   (b) if $\pi(i) = 0$, then $\varrho(i) = t_j$ for some j, and $\varrho(i) \in h(t_0, \ldots, t_{j-1})$,
   (c) if $\pi(i) = 1$, then $\varrho(i) \notin \tau$, and $\varrho(i) \in g(t_0, \ldots, t_j)$, where t_j is the greatest member of τ below $\varrho(i)$.

CLAIM. *For all $\pi \in \mathscr{T}$, and for all $\sigma \in [P_\varkappa(\lambda)]^{<\omega}$ extending $\langle p_0, \ldots, p_r\rangle$, there are $g \in \mathscr{F}$, $h \in \mathscr{F}_\varkappa$, $\alpha < \varkappa$ such that for all $\langle\sigma', \tau\rangle$ appropriate for π, g, h,*

$$\langle\sigma^\frown\sigma', g\rangle \Vdash \text{“}F(\tau) = \alpha\text{”}.$$

This claim will enable us to complete the proof of Lemma 3.1 as follows: take σ to be $\langle p_0, \ldots, p_r\rangle$, and for each $\pi \in \mathscr{T}$, choose g_π, h_π, α_π. Take g to be the intersection, $(\bigcap g_\pi) \cap f$, h to be $\bigcap h_\pi$, and $Z = \{\alpha_\pi\}_{\pi \in \mathscr{T}}$. Let H be generic, $\langle\sigma, g\rangle \in H$. Now for any $\tau \in [X_{H,h}]^{<\omega}$, we can find σ', a section of the generic sequence of H above σ, and a π such that $\langle\sigma', \tau\rangle$ is appropriate for g_π, h_π, π, so then $\langle\sigma^\frown\sigma', g_\pi\rangle \Vdash$ “$F(\tau) = \alpha_\pi$”, so $\langle\sigma^\frown\sigma', g\rangle \Vdash$ “$F(\tau) \in Z$”, and $\langle\sigma, g\rangle \Vdash$ “$F''[X_{H,h}]^{<\omega} \subseteq Z$”, which contradicts the assumption that $\langle\sigma, f\rangle$ forces that F is a counterexample to Rowbottomness, as $|Z| \leqslant \omega$.

The proof of the claim is by induction on the length n of π. For $n = 1$, we divide into two cases:

Case 1: $\pi(0) = 1$, i.e., we have a sequence of length 1 from $P_\varkappa(\lambda)$. The claim is then vacuously true. If $\langle\sigma', \tau\rangle$ is appropriate, τ is empty.

Case 2: $\pi(0) = 0$. Suppose $\sigma = \langle s_0, \ldots, s_m\rangle$. For each $\beta > s_m \cap \varkappa$ there is an $\alpha_\beta < \gamma$ and a $g_\beta \in \mathscr{F}$ such that $\langle\sigma', g_\beta\rangle \Vdash$ “$F(\beta) = \alpha_\beta$” (by the Prikry Lemma, Lemma 1.1). For every $\varrho \in P_\varkappa(\lambda)$, let $g(\varrho) = (\triangle_\varkappa g_\beta(\varrho)) \cap f(\varrho)$ and take any h so that for $\beta, \beta' \in h(\sigma)$, $\alpha_\beta = \alpha_{\beta'}$. Call this common value α. Then g, h, α satisfy the claim, since if $\langle\sigma', \tau\rangle$ is appropriate for π, g, h, then π must be of the form $\langle\emptyset, \{\beta\}\rangle$ with $\alpha_\beta = \alpha$, and $\langle\sigma, g_\beta\rangle \Vdash$ “$F(\beta) = \alpha_\beta$”, so $\langle\sigma, g\rangle \Vdash$ “$F(\beta) = \alpha_\beta$”, and so $\langle\sigma^\frown\sigma', g\rangle \Vdash$ “$F(\beta) = \alpha_\beta$”.

Now suppose we have established the claim for all π of length n, and we are given π of length $n+1$. Let π' be the sequence obtained from π by dropping its first element, $\pi(0)$.

Case 1: $\pi(0) = 1$. For every $p \in f(\sigma)$, apply the claim inductively to $\sigma^\frown\{p\}$, π', to get g_p, h_p, and α_p. Define g by:

1) if $x = \sigma^\frown\{p\}^\frown\sigma'$, then $g(x) = g_p(\sigma^\frown\{p\}^\frown\sigma')$;
2) if $x = \sigma$, then let $g(x)$ be a member of $\mathscr{U}$ such that $p, p' \in g(\sigma) \Rightarrow \alpha_p = \alpha_{p'}$;
3) otherwise, $g(x) = P_\varkappa(\lambda)$.

Define h by:

1) if $x = \sigma^\frown\{p\}^\frown\sigma'$, then $h(x) = h_p(\sigma^\frown\{p\}^\frown\sigma')$;
2) otherwise, let $h(x) = \varkappa$.

Let α be the common value of α_p for $p \in g(\sigma)$. Once again, these definitions suffice, since if $\langle\sigma', \tau\rangle$ is appropriate for π, g, h, then $\sigma' = \{p\}^\frown\sigma''$ for some p, σ'', and $\langle\sigma'', \tau\rangle$ is appropriate for π', g_p, and h_p, so $\langle\sigma^\frown\{p\}^\frown\sigma'', g_p\rangle \Vdash$ “$F(\tau) = \alpha$”, and therefore $\langle\sigma^\frown\sigma', g\rangle \Vdash$ ”$F(\tau) = \alpha$”.

Case 2: $\pi(0) = 0$. For each $\beta > s_m \cap \varkappa$, apply the lemma inductively to π', σ, and F_β, where F_β is defined by $F_\beta(\tau) = F(\{\beta\}^\frown\tau)$. We obtain g_β, h_β, α_β as before. Let $g(\varrho) = \triangle_\varkappa g_\beta(\varrho)$ and $h(\varrho) = \triangle_\varkappa h_\beta(\varrho)$, with the added restriction that $\beta, \beta' \in h(\sigma) \Rightarrow \alpha_\beta = \alpha_{\beta'}$. Let α be this common value. As before, this is sufficient. If $\langle\sigma', \tau\rangle$ is appropriate for π, g, and h, then $\tau = \{\beta\}^\frown\tau'$, for some β, τ', and so $\langle\sigma', \tau'\rangle$ is appropriate for g_β, h_β and π', hence $\langle\sigma^\frown\sigma', g_\beta\rangle \Vdash$ “$F_\beta(\tau') = \alpha_\beta$”, and thus $\langle\sigma^\frown\sigma', g\rangle \Vdash$ “$F(\tau) = \alpha$”.

This completes the proof of Lemma 3.1. ■

We have actually proved:

LEMMA 3.2. $V[G] \models$ “*$\varkappa$ carries a Rowbottom filter*”.

The filter $\mathscr{U}_G$ is the required filter. ■

LEMMA 3.3. $N_1 \models$ “*$\varkappa$ is a Rowbottom cardinal*”.

Proof. If F: $[P_\varkappa(\lambda)]^{<\omega} \to \gamma < \varkappa$ is a partition in N_1, then F is in $V[r_\delta]$ for some $\delta < \lambda$ by Lemma 2.1. By the proof above, there is a homogeneous set in $(\mathscr{U}\restriction\delta)_{r_\delta}$. ■

To prove that $\varkappa$ carries a Rowbottom filter in N_1, one merely notes that the definition of $X_{H,f}$ depends only on $H\restriction\varkappa$, and so $\mathscr{U}_G$ is in $V[r_\varkappa] \subseteq N_1$. This completes the proof of Theorem 1. ■

As indicated earlier, the fact that $\aleph_\omega$ carries a Rowbottom filter in N_2 follows in the same manner as in [A1] and [A2]. The proof of Theorem 2 is complete. ■

We remark here that the conclusions of Theorem 2 are slightly beyond the known consequences of the Axiom of Determinacy (AD). It is a theorem of Kleinberg [K] that $\aleph_\omega$ is Rowbottom in any model of AD, but it is still open under these circumstances whether or not a Rowbottom filter exists on $\aleph_\omega$. In addition, N_1 satisfies $DC_\varkappa$. This is essentially shown, in a somewhat different context, by Kofkoulos in his thesis [Ko].

§ 4. Specker's Problem. Before beginning the proof of Theorem 3, we need to introduce some new notions. For a condition $\pi = \langle p_0, \ldots, p_n, f\rangle$ in a modified Prikry partial order we will call $\langle p_0, \ldots, p_n\rangle$ the *p-part* of π. Also, for any (well-ordered) cardinal $\aleph$, the *Specker property* SP($\aleph$) will mean that $2^\aleph$ is a countable union of sets of cardinality $\aleph$.

We turn now to the proof of Theorem 3.

Proof of Theorem 3. The proof is quite similar to the one given in [AG]. Let $V \models$ “ZFC + There exists a regular limit of strongly compact cardinals”, and let α_0 be the least such limit, with $\langle\varkappa_\alpha\colon \alpha < \alpha_0\rangle$ the sequence of strongly compact cardinals whose limit is α_0. Let $\langle\mathscr{U}_\alpha\colon \alpha < \alpha_0\rangle$ be a sequence of strongly compact ultrafilters with $\mathscr{U}_\alpha$ defined over $P_{\varkappa_\alpha}(\varkappa_{\alpha+1})$. As before, $\mathscr{P}_{\mathscr{U}_\alpha}$ will be modified Prikry forcing defined using $\mathscr{U}_\alpha$.

Define now a sequence of partial orders $\langle \mathscr{P}_\alpha \colon \alpha < \alpha_0 \rangle$ as follows:

$$\mathscr{P}_0 = \mathrm{Col}(\omega, \varkappa_0),$$
$$\mathscr{P}_{\alpha+1} = \mathscr{P}_{\mathscr{U}_\alpha},$$
$$\mathscr{P}_\lambda = \mathrm{Col}\big(\big(\bigcup_{\alpha<\lambda} \varkappa_\alpha\big)^+, \varkappa_\lambda\big) \quad \text{for } \lambda \text{ a limit ordinal.}$$

Note that since α_0 is the least regular limit of strongly compact cardinals, the definition of $\mathscr{P}_\lambda$ makes sense.

We are now in a position to define the partial order $\mathscr{P}$ used in the proof of Theorem 3. $\mathscr{P}$ consists of all elements $p = \langle p_\alpha \colon \alpha < \alpha_0 \rangle$ of $\prod_{\alpha<\alpha_0} \mathscr{P}_\alpha$ so that the support of p is some ordinal $\beta < \alpha_0$, i.e., so that $\exists \beta < \alpha_0 \, \forall \gamma \geqslant \beta \, [p_\gamma$ is the trivial condition$]$. The ordering is the componentwise one.

Let G be V-generic over $\mathscr{P}$. The model $N_3 \subseteq V[G]$ which witnesses the conclusions of Theorem 3 can intuitively be thought of in the following manner. We wish to define N_3 in a fashion so that the $\varkappa_\alpha$'s and the $(\bigcup_{\alpha<\lambda} \varkappa_\alpha)^+$'s are the successor cardinals and so that each of these cardinals satisfies the Specker property. Thus, we will place in N_3 just enough information to be able to collapse each of the above cardinals, preserve the fact that they indeed remain cardinals in N_3, and define the sequence which witnesses the fact that they satisfy the Specker property. For any cardinal δ which becomes a successor cardinal in N_3, we will place in N_3 for each $n < \omega$, roughly speaking, the partial collapse map to δ^+ restricted to the nth element of a generic, cofinal ω-sequence through the least $\varkappa_\alpha > \delta$, together with the partial collapse map to γ^+ restricted to the nth element of a generic, cofinal ω-sequence through the least $\varkappa_\alpha > \gamma$ for every γ in a certain set of cardinals below δ.

Getting specific, let, for each $\alpha < \alpha_0$, G_α be the projection of G onto $\mathscr{P}_\alpha$. Let r_0 be the collapse map of $\varkappa_0$ to ω_1 generated by G_0. For λ, a limit ordinal, let r_λ be the collapse map of $\varkappa_\lambda$ to $(\bigcup_{\alpha<\lambda} \varkappa_\alpha)^{++}$ generated by G_λ. For $\beta = \alpha+1$, a successor ordinal, let $r_\beta^* = \langle r_{\alpha,n}^* \colon n < \omega \rangle$ be the generic ω-sequence generated by G_β which codes a cofinal ω-sequence through each regular cardinal in the interval $[\varkappa_\alpha, \varkappa_{\alpha+1}]$. By a routine density argument which uses the fact that $\mathscr{U}_\alpha$ is a $\varkappa_\alpha$-complete fine measure over $P_{\varkappa_\alpha}(\varkappa_{\alpha+1})$, we can let $r_\beta = \langle r_\alpha^n \colon n < \omega \rangle$ be a generic subsequence of r_β^* so that for each $n < \omega$, $r_\alpha^n \cap \varkappa_\alpha$ is an inaccessible cardinal in V, for $n < m < \omega$, $r_\alpha^n \cap \varkappa_\alpha < r_\alpha^m \cap \varkappa_\alpha$, and the sequence r_β codes a cofinal ω-sequence through each regular cardinal in the interval $[\varkappa_\alpha, \varkappa_{\alpha+1}]$. We can now define, for each $n < \omega$ and each $\beta < \alpha_0$, $s_n^\beta = \langle r_\alpha \restriction (r_\alpha^n \cap \varkappa_\alpha) \colon \alpha \leqslant \beta \rangle$. N_3 will then be defined as $R(\alpha_0)$ of the least model M of ZF extending V which contains, for every $n < \omega$ and every $\beta < \alpha_0$, the set s_n^β. More precisely, let $\mathscr{L}_1$ be a ramified sublanguage of the forcing language $\mathscr{L}$ with respect to $\mathscr{P}$ which contains symbols $\mathring{v}$ for every $v \in V$, a unary predicate symbol $\mathring{V}$ (interpreted as $\mathring{V}(\mathring{v}) \Leftrightarrow v \in V$), and all symbols of the form $\mathring{s}_n^\beta$ for $n < \omega$ and $\beta < \alpha_0$. As before, we can assume that each $\mathring{v}$ is invariant under any automorphism of $\mathscr{P}$. We can also assume that each $\tau \in \mathscr{L}_1$ which mentions only $\mathring{s}_n^\beta$ is invariant under any automorphism $\pi = \langle \pi_\alpha \colon \alpha < \alpha_0 \rangle$ of $\mathscr{P}$ so that π_α does not change the meaning of $r_\alpha \restriction (r_\alpha^n \cap \varkappa_\alpha)$ for $\alpha \leqslant \beta$ if there is enough information to determine all such ordinals.

Working in $V[G]$, we define an inner model M as follows:

$$M_0 = \varnothing,$$
$$M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha \quad \text{if } \lambda \text{ is a limit ordinal,}$$
$$M_{\alpha+1} = \{x \subseteq M_\alpha \colon x \in V[G] \text{ and } x \text{ is definable over } \langle M_\alpha, \in, c \rangle_{c \in M_\alpha} \text{ by a term } \tau \in \mathscr{L}_1 \text{ of rank } \leqslant \alpha\},$$
$$M = \bigcup_{\alpha \in \mathrm{Ordinals}^V} M_\alpha.$$

As in [AG], it will be the case that for $N_3 = R(\alpha_0)^M$, $N_3 \models \mathrm{ZF}$ (including the Axioms of Power Set and Replacement). We refer the reader to Lemma 1.5 of [AG] for further details.

LEMMA 4.1. *Assume that $x \in M$ is a set of ordinals. Then*:

(a) *$x \in V[s_n^\delta]$ for some $n < \omega$ and $\delta < \alpha_0$.*

(b) *If $x \subseteq \omega$, $x \in V[s_n^0]$ for some $n < \omega$.*

(c) *If $\alpha < \alpha_0$ and $x \subseteq \varkappa_\alpha$, $x \in V[s_n^\delta]$ for some $n < \omega$ and $\delta = \alpha+1$.*

(d) *If $\lambda < \alpha_0$ is a limit ordinal and $x \subseteq (\bigcup_{\alpha<\lambda} \varkappa_\alpha)^+$, $x \in V[s_n^\delta]$ for some $n < \omega$ and $\delta = \lambda$.*

Proof. The proof is a modification of the proof of Lemma 1.1 of [AG] in the context of modified Prikry forcing. Specifically, we first prove (a) and show how (b), (c) and (d) all follow from (a). Let $\tau \in \mathscr{L}_1$ and $p \in \mathscr{P}$ be so that τ denotes x and $p \Vdash$ "$\tau \subseteq \mathring{\gamma}_0$" for some ordinal γ_0. As before, using the standard coding tricks, we can assume that τ mentions only one term of the form $\mathring{s}_n^\delta$. We show that $p \Vdash$ "$x \in V[\mathring{s}_n^\delta]$".

Let $p = \langle p_\alpha \colon \alpha < \alpha_0 \rangle$ where $\gamma < \alpha_0$ is so that p_α is trivial for $\alpha \geqslant \gamma$. First, since $\delta < \alpha_0$, we can assume without loss of generality that $\gamma \geqslant \delta$ and for every $\alpha \leqslant \delta$, r_α^n is determined. (Simply extend p_α for $\alpha \leqslant \delta+1$ so that the p-part of $p_{\alpha+1}$ determines r_α^n). Next, define a function $f \colon \alpha_0 \to \alpha_0$ by $f(\beta) = r_\beta^n \cap \varkappa_\beta$ for $\beta \leqslant \delta$, $f(\lambda) = (\bigcup_{\alpha<\lambda} \varkappa_\alpha)^+$ for $\lambda > \delta$ a limit ordinal, and for $\beta = \alpha+1 > \delta$, a successor ordinal, $f(\beta) = \varkappa_\alpha$. Our first claim is that if $q = \langle q_\alpha \colon \alpha < \alpha_0 \rangle$, $s = \langle s_\alpha \colon \alpha < \alpha_0 \rangle$, $q \Vdash p$, $s \Vdash p$ are so that $\forall \alpha < \alpha_0 \, [q_\alpha \restriction f(\alpha) = s_\alpha \restriction f(\alpha)]$, then for any $\beta_0 < \gamma_0$, if $q \Vdash$ "$\beta_0 \in \tau$", then $s \Vdash$ "$\beta_0 \in \tau$".

Assume the claim is false, and let $u^0 = \langle u_\alpha^0 \colon \alpha < \alpha_0 \rangle$ be so that $u^0 \Vdash s$ and $u^0 \Vdash$ "$\beta_0 \notin \tau$". For each successor $\alpha = \beta+1$, $\alpha < \alpha_0$, let $u_\alpha^1 \in \mathscr{P}_\alpha$ be so that $u_\alpha^1 \Vdash q_\alpha$ and so that for $\langle t_0^\alpha, \ldots, t_k^\alpha \rangle$, the p-part of u_α^1, and $\langle \hat{t}_0^\alpha, \ldots, \hat{t}_k^\alpha \rangle$, the p-part of u_α^0, $t_j^\alpha \cap f(\alpha) = \hat{t}_j^\alpha \cap f(\alpha)$ for $0 \leqslant j \leqslant k$; that this is possible follows from $q_\alpha \restriction f(\alpha) = s_\alpha \restriction f(\alpha)$, Lemma 1.3, and the proof of Lemma 2.1. Form a condition $u^2 = \langle u_\alpha^2 \colon \alpha < \alpha_0 \rangle$ by $u_\alpha^2 = u_\alpha^1$ if α is a successor ordinal and $u_\alpha^2 = q_\alpha$ if α is a limit ordinal or $\alpha = 0$. Clearly, $u^2 \Vdash q$ and $u^2 \Vdash$ "$\beta_0 \in \tau$".

We define now an automorphism $\pi = \langle \pi_\alpha \colon \alpha < \alpha_0 \rangle$ of $\mathscr{P}$ so that $\pi(u^2)$ is compatible with u^0 and $\pi(u^2) \Vdash$ "$\beta_0 \in \tau$". If $\lambda = 0$ or λ is a limit ordinal, then by the homogeneity of the Lévy collapse, we can let π_λ be any automorphism of $\mathscr{P}_\lambda$ so that $\pi_\lambda(u_\lambda^2)$ is compatible with u_λ^0 and π_λ is generated by a function which is the identity on $f(\lambda)$. If $\alpha = \beta+1$ is a successor ordinal, then as in Lemma 2.1, let π_α be an automorphism of $\mathscr{P}_\alpha = \mathscr{P}_{\mathscr{U}_\beta}$ so that π_λ preserves the meaning of $r_\alpha \restriction (r_\alpha^n \cap \varkappa_\alpha)$ and $\pi_\alpha(u_\alpha^2)$ is compatible with u^0_α. $\pi = \langle \pi_\alpha\colon \alpha < \alpha_0 \rangle$ is thus an automorphism of $\mathscr{P}$ so that $\pi(u^2)$ is compatible with u^0, and by the construction of π and the invariance properties of τ, $\pi(u^2) \Vdash$ "$\beta_0 \in \tau$". Since $\pi(u^2)$ is compatible with u^0 and $u^0 \Vdash$ "$\beta_0 \notin \tau$", this is a contradiction. Thus, the claim is established. As in Lemma 2.1, we can therefore define $y = \{\varrho < \gamma_0\colon \exists q \Vdash p\ [q = \langle q_\alpha\colon \alpha < \alpha_0 \rangle,\ q_\alpha \restriction f(\alpha) \in G_\alpha \restriction f(\alpha)$, and $q \Vdash$ "$\varrho \in \tau$"$]\}$, a set definable in $V[\prod_{\alpha<\alpha_0} G_\alpha \restriction f(\alpha)]$, and show as in Lemma 2.1 that $x = y$. Hence, $x \in V[\prod_{\alpha<\alpha_0} G_\alpha \restriction f(\alpha)]$.

We next show that $x \in V[\prod_{\alpha\leqslant\delta} G_\alpha \restriction f(\alpha)]$. Since each $G_\alpha \restriction f(\alpha)$ is recoverable from $r_\alpha \restriction (r^n_\alpha \cap \varkappa_\alpha)$, this will show that $x \in V[s^\delta_n]$. To this end, let now σ be a canonical term for x in the forcing language associated with

$$Q = \prod_{\alpha\leqslant\delta} \mathscr{P}_\alpha \restriction f(\alpha) \times \prod_{\alpha+1\in[\delta+1,\alpha_0)} \mathscr{P}_{\alpha+1} \restriction f(\alpha+1) \times \prod_{\lambda\in[\delta+1,\alpha_0),\ \lambda \text{ a limit}} \mathscr{P}_\lambda \restriction f(\lambda).$$

Define a term η in the forcing language with respect to $\prod_{\alpha\leqslant\delta} \mathscr{P}_\alpha \restriction f(\alpha)$ by $p = \langle p_\alpha\colon \alpha \leqslant \delta \rangle \Vdash$ "$\varrho \in \eta$" iff $\langle p_\alpha\colon \alpha < \alpha_0 \rangle \Vdash$ "$\varrho \in \sigma$", where for $\alpha \geqslant \delta+1$, p_α is the trivial condition. Clearly, η will denote a subset of x which is an element of $V[\prod_{\alpha\leqslant\delta} G_\alpha \restriction f(\alpha)]$. The proof will be complete if we can show that $\Vdash_Q$ "$\sigma \subseteq \eta$".

To this end, let $q = \langle q_\alpha\colon \alpha < \alpha_0 \rangle$ force "$\varrho \in \sigma$". It suffices to show that $u = \langle q_\alpha\colon \alpha \leqslant \delta \rangle \times \langle u_\alpha\colon \alpha > \delta \rangle \Vdash$ "$\varrho \in \sigma$", where for $\alpha > \delta$, u_α is the trivial condition. If this is not the case, then let $s = \langle s_\alpha\colon \alpha < \alpha_0 \rangle \Vdash u$ be so that $s \Vdash$ "$\varrho \notin \sigma$", and let $\beta \in (\delta, \alpha_0)$ be so that for all $\gamma \leqslant \beta$, s_γ and q_γ are the trivial condition. Without loss of generality, assume that for all successor ordinals $\gamma \in (\delta, \beta)$, the p-parts of s_γ and q_γ have the same length.

We construct now an automorphism $\Psi = \langle \Psi_\alpha\colon \alpha < \alpha_0 \rangle$ of Q as follows. For ordinals $\alpha \geqslant \beta$, ordinals $\alpha \leqslant \delta$, and limit ordinals $\alpha \in (\delta, \beta)$, let Ψ_α be the identity. (Note that for $\alpha \in (\delta, \beta)$ a limit ordinal, $\mathscr{P}_\alpha \restriction f(\alpha)$ is the trivial partial order.) For $\alpha \in (\delta, \beta)$ a successor ordinal, as in Lemma 2.1, let Ψ_α be an automorphism of $\mathscr{P}_\alpha \restriction f(\alpha)$ so that $\Psi_\alpha(s_\alpha)$ is compatible with q_α. By the construction of Ψ_α, $\Psi = \langle \Psi_\alpha\colon \alpha < \alpha_0 \rangle$ is an automorphism of Q which preserves the meaning of each $r_\alpha \restriction (r^n_\alpha \cap \varkappa_\alpha)$ for $\alpha \leqslant \delta$ and hence the meaning of σ. Since $\Psi(s)$ is compatible with q, we have $\Psi(s) \Vdash$ "$\varrho \notin \sigma$" and $q \Vdash$ "$\varrho \in \sigma$", a contradiction. This shows that $x \in V[s^\delta_n]$.

To show (b), (c), and (d), let σ be either ω, $\varkappa_\alpha$, or $(\bigcup_{\alpha<\lambda} \varkappa_\alpha)^+$, and let γ be so that $x \subseteq \sigma$ and $x \in V[s^\gamma_n]$. If $\gamma \leqslant \delta$ for δ as defined in (b), (c), and (d), then the proof is complete since $V[s^\gamma_n] \subseteq V[s^\delta_n]$. Thus, assume that $\gamma > \delta$. As in part (a), we know that $x \in V[\prod_{\alpha\leqslant\gamma} G_\alpha \restriction f(\alpha)]$ for f as defined previously. We will show that $x \in V[s^\delta_n]$ by showing that $V[\prod_{\alpha\leqslant\gamma} G_\alpha \restriction f(\alpha)]$ and $V[\prod_{\alpha\leqslant\delta} G_\alpha \restriction f(\alpha)]$ contain the same subsets of σ and then using the canonical identification of $V[\prod_{\alpha\leqslant\delta} G_\alpha \restriction f(\alpha)]$ with $V[s^\delta_n]$.

To do this, we need to show that forcing over $V[\prod_{\alpha\leqslant\delta} G_\alpha \restriction f(\alpha)]$ with $V[\prod_{\alpha\in[\delta+1,\gamma]} \mathscr{P}_\alpha \restriction f(\alpha)]$ adds no new subsets of σ. Write $\prod_{\alpha\in[\delta+1,\gamma]} \mathscr{P}_\alpha \restriction f(\alpha)$ as $Q' \times Q''$, where

$$Q' = \prod_{\alpha+1\in[\delta+1,\gamma]} \mathscr{P}_{\alpha+1} \restriction f(\alpha+1), \quad Q'' = \prod_{\lambda\in[\delta+1,\gamma),\ \lambda \text{ a limit}} \mathscr{P}_\lambda \restriction f(\lambda).$$

This factorization generates a factorization of $\prod_{\alpha\in[\delta+1,\gamma]} G_\alpha \restriction f(\alpha)$ into $G' \times G''$. Since each component $\mathscr{P}_\lambda \restriction f(\lambda)$ of Q'' is associated with a $\lambda > \delta$, the closure properties of the Lévy collapse and the definition of Q'' ensure that the subsets of σ in $V[G'']$ and V are the same. Further, by the definition of f and each $\mathscr{P}_\eta$, for $\alpha = \beta+1$, a fixed but arbitrary successor ordinal in $[\delta+1, \gamma]$, $|\prod_{\eta\leqslant\beta} \mathscr{P}_\eta \restriction f(\eta)| < \varkappa_\beta$. Also, if $\lambda > \alpha$ is a limit ordinal, then the closure properties of the Lévy collapse and the definition of $\mathscr{P}$ ensure that each $\mathscr{P}_\lambda \restriction f(\lambda)$ is (at least) $2^{2^{f(\alpha)}}$-closed. Thus, since $\mathscr{P}_\alpha \restriction f(\alpha)$ is a modified Prikry ordering defined using a strongly compact measure on $P_{\varkappa_\beta}(f(\alpha))$ with $|\mathscr{P}_\alpha \restriction f(\alpha)| < 2^{2^{f(\alpha)}}$, an application of the closure properties of

$$\prod_{\lambda\in[\alpha,\gamma],\ \lambda \text{ a limit}} \mathscr{P}_\lambda \restriction f(\lambda) = Q'''$$

followed by an application of the results of [LS] shows that $V^{Q''' \times \Pi_{\eta<\beta} \mathscr{P}_\eta \restriction f(\eta)} \models$ "$\mathscr{P}_\alpha \restriction f(\alpha)$ is a partial order which satisfies the Prikry property and adds no new bounded subsets to $\varkappa_\beta$". Thus, Q' can be regarded in $V[G'']$ as a full support iteration of partial orders each of which satisfies the Prikry property and adds no new bounded subsets to $\varkappa_\delta$, so since α_0 is the least regular limit of strongly compact cardinals, the result of [G2] shows that forcing over $V[G'']$ with Q' adds no new bounded subsets to $\varkappa_\delta$, i.e., since $\sigma < \varkappa_\delta$, $V[G''][G'] = V[G'][G''] = V[\prod_{\alpha\in[\delta+1,\gamma]} G_\alpha \restriction f(\alpha)] \models$ "The subsets of σ are the same as those in V". Thus, any new subsets of σ in $V[\prod_{\alpha\leqslant\delta} G_\alpha \restriction f(\alpha)]$ are generated by forcing over $V[\prod_{\alpha\in[\delta+1,\gamma]} G_\alpha \restriction f(\alpha)]$ with $\prod_{\alpha\leqslant\delta} \mathscr{P}_\alpha \restriction f(\alpha)$, i.e., since

$$V\Big[\prod_{\alpha\in[\delta+1,\gamma]} G_\alpha \restriction f(\alpha)\Big]\Big[\prod_{\alpha\leqslant\delta} G_\alpha \restriction f(\alpha)\Big] = V\Big[\prod_{\alpha\leqslant\delta} G_\alpha \restriction f(\alpha)\Big]\Big[\prod_{\alpha\in[\delta+1,\gamma]} G_\alpha \restriction f(\alpha)\Big],$$

forcing over $V[\prod_{\alpha\leqslant\delta} G_\alpha \restriction f(\alpha)]$ with $\prod_{\alpha\in[\delta+1,\gamma]} \mathscr{P}_\alpha \restriction f(\alpha)$ adds no new subsets of σ. Thus, $x \in V[s^\delta_n]$. This proves Lemma 4.1. ■

LEMMA 4.2. *For $\sigma = \varkappa_\alpha$ or $\sigma = (\bigcup_{\alpha<\lambda} \varkappa_\alpha)^+$, λ a limit ordinal, $N_3 \models$ "σ is a cardinal".*

Proof. For δ as in Lemma 4.1, since $N_3 \subseteq M$ and $\sigma < \alpha_0$, Lemma 4.1 shows that if $x \subseteq \sigma$ and $x \in N_3$, then $x \in V[s^\delta_n]$ for some $n < \omega$. Let f be as in Lemma 4.1. By the identification of $V[s^\delta_n]$ with $V[\prod_{\alpha\leqslant\delta} G_\alpha \restriction f(\alpha)]$, view $V[s^\delta_n]$ as $V[G_\delta \restriction f(\delta)][\prod_{\alpha<\delta} G_\alpha \restriction f(\alpha)]$.

If $\sigma = \varkappa_\alpha$, then $\delta = \alpha+1$ and $\mathscr{P}_\delta \restriction f(\delta)$ is a modified Prikry ordering on $P_{\varkappa_\alpha}(f(\delta))$. This means that $V[G_\delta \restriction f(\delta)] \models$ "$\varkappa_\alpha$ is a cardinal and $|\prod_{\beta<\delta} \mathscr{P}_\beta \restriction f(\beta)| < \varkappa_\alpha$", so $V[s^\delta_n] \models$ "$\varkappa_\alpha$ is a cardinal". Thus, no subset of $\varkappa_\alpha$ in N_3 can code a collapsing map of $\varkappa_\alpha$, i.e., $N_3 \models$ "$\varkappa_\alpha$ is a cardinal". If $\sigma = (\bigcup_{\alpha<\lambda} \varkappa_\alpha)^+$, then $\delta = \lambda$ and $\mathscr{P}_\delta \restriction f(\delta)$ is $\mathrm{Col}((\bigcup_{\alpha<\lambda} \varkappa_\alpha)^+, f(\lambda))$. Therefore, by the definition of $\mathscr{P}$ and f, $V[G_\lambda \restriction f(\lambda)] \models$ "σ is a regular cardinal and for each $\alpha < \lambda$, $\mathscr{P}_\alpha \restriction f(\alpha)$ is $\varkappa(\alpha)$-c.c. for some $\varkappa(\alpha) < \bigcup_{\alpha<\lambda} \varkappa_\alpha$ which depends on $\mathscr{P}_\alpha \restriction f(\alpha)$", so by the definition of $\mathscr{P}_\alpha \restriction f(\alpha)$, $V[G_\lambda \restriction f(\lambda)] \models$ "All antichains in $\prod_{\alpha<\lambda} \mathscr{P}_\alpha \restriction f(\alpha)$ have size $\leqslant \bigcup_{\alpha<\lambda} \varkappa_\alpha$". This means that $x \in V[s^\delta_n] \models$ "σ is a cardinal". The exact same reasoning as before shows $N_3 \models$ "σ is a cardinal". This proves Lemma 4.2. ■

LEMMA 4.3. *Every successor cardinal in N_3 is either a $\varkappa_\alpha$ or a $(\bigcup_{\alpha<\lambda} \varkappa_\alpha)^+$ for some limit ordinal $\lambda < \alpha_0$.*

Proof. Since $N_3 = R(\alpha_0)^M$, it suffices to show that any successor cardinal $\varkappa^+$ in M below α_0 is either $\varkappa_\alpha$ or $(\bigcup_{\alpha<\lambda} \varkappa_\alpha)^+$ for some limit $\lambda < \alpha_0$. To show this, we argue by contradiction. Asume $\varkappa^+$ is the least successor cardinal in M below α_0 which does not satisfy this property, and consider two cases:

Case 1: $\varkappa = (\delta^+)^M$ for some cardinal $\delta < \alpha_0$. By the leastness of $\varkappa^+$, $\varkappa$ is either a $\varkappa_\alpha$

or a $(\bigcup_{\alpha<\lambda}\varkappa_\alpha)^+$ for some limit $\lambda<\alpha_0$. If $\varkappa=\varkappa_\alpha$ for some $\alpha<\alpha_0$, then by the definition of M, for each $n<\omega$, $V[r_{\alpha+1}\restriction(r^n_{\alpha+1}\cap\varkappa_{\alpha+1})]\subseteq M$. As in Lemma 2.4, by the fact that $r_{\alpha+1}$ is generated by a modified Prikry forcing using a strongly compact measure on $P_{\varkappa_\alpha}(\varkappa_{\alpha+1})$, $V[r_{\alpha+1}\restriction(r^n_{\alpha+1}\cap\varkappa_\alpha)]\models$ "There are no cardinals in the interval $(\varkappa_\alpha,(r^n_{\alpha+1}\cap\varkappa_{\alpha+1})]$". Since $\langle(r^n_{\alpha+1}\cap\varkappa_{\alpha+1})\colon n<\omega\rangle$ is cofinal in $\varkappa_{\alpha+1}$, $M\models$ "There are no cardinals in the interval $(\varkappa_\alpha,\varkappa_{\alpha+1})$". By Lemma 4.2, $M\models$ "$\varkappa_{\alpha+1}$ is a cardinal", so $M\models$ "$\varkappa_{\alpha+1}=\varkappa_\alpha^+$". If $\varkappa=(\bigcup_{\alpha<\lambda}\varkappa_\alpha)^+$ for some limit $\lambda<\alpha_0$, then again by the definition of M, for each $n<\omega$, $V[r_\lambda\restriction(r^n_\lambda\cap\varkappa_\lambda)]\subseteq M$. Since r_λ is generic for $\mathrm{Col}((\bigcup_{\alpha<\lambda}\varkappa_\alpha)^+,\varkappa_\lambda)$ and $\langle r^n_\lambda\cap\varkappa_\lambda\colon n<\omega\rangle$ is cofinal in $\varkappa_\lambda$, $V[r_\lambda\restriction(r^n_\lambda\cap\varkappa_\lambda)]\models$ "There are no cardinals in the interval $((\bigcup_{\alpha<\lambda}\varkappa_\alpha)^+,(r^n_\lambda\cap\varkappa_\lambda))$" and $M\models$ "There are no cardinals in the interval $((\bigcup_{\alpha<\lambda}\varkappa_\alpha)^+,(r^n_\lambda\cap\varkappa_\lambda))$". By Lemma 4.2, $M\models$ "$\varkappa_\lambda$ is a cardinal", so $M\models$ "$\varkappa_\lambda=((\bigcup_{\alpha<\lambda}\varkappa_\alpha)^+)^+$". Thus, if $\varkappa=(\delta^+)^M$ for some cardinal $\delta<\alpha_0$, then $(\varkappa^+)^M=\varkappa_\alpha$ for some $\alpha<\alpha_0$.

Case 2: $\varkappa<\alpha_0$ is a limit cardinal in M. There must be unboundedly many $\varkappa_\alpha$'s below $\varkappa$, for if $\sigma<\varkappa$ is a bound on the $\varkappa_\alpha$'s, then the cardinal (in V or M) $(\sigma^{++})^M$ is below $\varkappa$ and is neither a $\varkappa_\alpha$ nor a $(\bigcup_{\alpha<\lambda}\varkappa_\alpha)^+$, contradicting the leastness of $\varkappa^+$. We can thus write, in V, $\varkappa=\bigcup_{\alpha<\lambda}\varkappa_\alpha$ for some $\lambda<\alpha_0$. By Lemma 4.2, $(\bigcup_{\alpha<\lambda}\varkappa_\alpha)^+$ remains a cardinal in M. Since $V\subseteq M$, $M\models$ "$(\bigcup_{\alpha<\lambda}\varkappa_\alpha)^+=\varkappa^+$". Thus, if $\varkappa$ is a limit cardinal in M and $\varkappa<\alpha_0$, $M\models$ "There is a limit ordinal $\lambda<\alpha_0$ so that $\varkappa^+=(\bigcup_{\alpha<\lambda}\varkappa_\alpha)^{+V}$". This fact, together with Case 1, proves Lemma 4.3. ∎

LEMMA 4.4. $N_3\models$ "SP(ω), *for every* $\alpha<\alpha_0$, SP$(\varkappa_\alpha)$, *and for every limit ordinal* $\lambda<\alpha_0$, SP$((\bigcup_{\alpha<\lambda}\varkappa_\alpha)^+)$".

Proof. Let us first consider the cardinal $\varkappa_\alpha$. Working in N_3, let $X_n=\{x\subseteq\varkappa_\alpha\colon x$ is definable using $s_n^{\alpha+1}\}$. By Lemma 4.1 and the fact that $N_3=R(\alpha_0)^M$, $X_n\in V[s_n^{\alpha+1}]$, and the cardinality of X_n in $V[s_n^{\alpha+1}]$ is some ordinal $\delta<(r^{n+1}_{\alpha+1}\cap\varkappa_{\alpha+1})<\varkappa_{\alpha+1}$. Since in $V[s^{\alpha+1}_{n+1}]\subseteq M$, δ is collapsed to $\varkappa_\alpha$ (this is shown by the same argument as in Lemma 2.4), $\varkappa_\alpha$ is a cardinal in $V[s^{\alpha+1}_{n+1}]$, and $X_n\in V[s^{\alpha+1}_{n+1}]$ (this follows from $V[s_n^{\alpha+1}]\subseteq[s^{\alpha+1}_{n+1}]$), $V[s^{\alpha+1}_{n+1}]\models$ "$|X_n|=\varkappa_\alpha$". Finally, as Lemma 4.1 and the fact that $N_3=R(\alpha_0)^M$ show that any $x\subseteq\varkappa_\alpha$ so that $x\in N_3$ is in X_n for some n, $N_3\models\mathrm{SP}(\varkappa_\alpha)$.

Turning now to the cardinal $\delta_\lambda=(\bigcup_{\alpha<\lambda}\varkappa_\alpha)^+$, we can again define, in N_3, $X_n=\{x\subseteq\delta_\lambda\colon x$ is definable using $s_n^\lambda\}$. As before, Lemma 4.1 implies that $X_n\in V[s_n^\lambda]$, and $|X_n|$ in $V[s_n^\lambda]$ is some $\delta<(r_\lambda^{n+1}\cap\varkappa_\lambda)<\varkappa_\lambda$. Again, since in $V[s^\lambda_{n+1}]\subseteq M$, δ is collapsed to δ_λ by the Lévy collapse map generated by $r_\lambda\restriction(r_\lambda^{n+1}\cap\varkappa_\lambda)$, $V[s^\lambda_{n+1}]\models$ "δ_λ is a cardinal", and $V[s_n^\lambda]\subseteq V[s^\lambda_{n+1}]$, $V[s^\lambda_{n+1}]$ and M all satisfy "$|X_n|=\delta_\lambda$". Lemma 4.1 and the fact that $R(\alpha_0)^{V[s^\lambda_{n+1}]}\subseteq R(\alpha_0)^M=N_3$ then again yield that $N_3\models\mathrm{SP}(\delta_\lambda)$.

The proof of Lemma 4.4 is completed by noting that the argument for SP$(\varkappa_\alpha)$ works for ω by letting $\varkappa_{-1}=\omega$. ∎

Since we have already stated that the proof that $N_3\models\mathrm{ZF}$ can be found in [AG], Lemmas 4.1–4.4 complete the proof of Theorem 3. ∎

We observe that AC_ω fails in N_3, since the presence of even one cardinal $\varkappa$ such that SP$(\varkappa)$ holds ensures the failure of AC_ω.

In conclusion, we remark that Theorems 1 and 2 provide upper bounds in

consistency strength for the theories "ZF + $\varkappa$ is a Rowbottom, strong limit cardinal of cofinality ω + $\varkappa^+$ is a measurable cardinal which carries a normal measure" and "ZF + $\aleph_\omega$ is a Rowbottom, strong limit cardinal + $\aleph_{\omega+1}$ is a measurable cardinal which carries a normal measure", namely, the existence of cardinals $\varkappa<\lambda$ such that $\varkappa$ is λ strongly compact and λ is measurable. Theorem 3 provides an upper bound in consistency strength for the theory "ZF + For every successor ordinal α, SP$(\aleph_\alpha)$", namely, a regular limit of strongly compact cardinals.

It is particularly interesting to note that since "$\varkappa$ is a singular Rowbottom cardinal of cofinality ω + $\varkappa^+$ is a measurable cardinal which carries a normal measure" and "$\aleph_{\omega+1}$ is a measurable cardinal which carries a normal measure" are both consequences of AD, the results of Martin, Steel, and Woodin provide another upper bound in consistency strength for these theories, namely, ω-many Woodin cardinals. Which of these provides the weaker upper bound is currently unknown.

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Received 18 July 1990

Added in proof (September 1991). Dehornoy has pointed out to the second author that Section 4 of [H] is essentially covered in Dehornoy's paper *Iterated ultrapowers and Prikry forcing*, Ann. Math. Logic 15 (1978), 109–160. The second author is embarrasssed at not having credited Dehornoy in his earlier paper [H], specifically with the prior development of modified Prikry forcing.