Linear topologies on sesquilinear spaces of uncountable dimension*

by

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Abstract. Appenzeller [A] introduced an invariant Γ for orthosymmetric sesquilinear spaces of regular uncountable dimension $\varkappa$ which takes its values in some Boolean algebra $\mathscr{D}(\varkappa)$. Constructively he shows that Γ maps onto $\mathscr{D}(\omega_1)$. We show that this is not true for $\varkappa > \omega_1$. Orthosymmetric sesquilinear spaces naturally bear linear topologies defined by the form (see [O], [B]). There are various relations between the arithmetic, geometric and topological properties of such spaces. E.g. Bäni [B] characterizes γ-diagonal spaces using the notions of γ-compactness and continuous bases. We present an alternative characterization: existence of a convergent algebraic basis. In [B], the question is asked whether there exist γ-compact spaces without continuous bases for arbitrary regular γ. We give a positive answer by showing that the spaces defined in [A] are examples of this.

0. Introduction. Modifying a concept of Eklof ([E1], [E2]), Appenzeller [A] introduced an invariant Γ which assigns to every orthosymmetric sesquilinear space ([1]) (X, Φ) over a fixed *-field k a value in $\mathscr{D}(\varkappa)$, where $\varkappa$ is the dimension of X (which is supposed to be a regular uncountable cardinal) and $\mathscr{D}(\varkappa)$ is the Boolean algebra $\mathscr{P}(\varkappa)$ modulo closed unbounded sets. The invariant Γ reflects some geometrical properties of (X, Φ), but it does not classify up to isometry. E.g. the following holds: $\Gamma(X, \Phi) = 0$ iff (X, Φ) is $\varkappa$-diagonal ([2]). Using a method developed by Shelah (see [Sh, Thm. 1.2]), Appenzeller has constructed spaces (which are called ladder spaces in the sequel) whose invariant ranges over a certain nontrivial interval of $\mathscr{D}(\varkappa)$. In case $\varkappa = \omega_1$, this interval equals $\mathscr{D}(\omega_1)$. It follows that Γ maps onto $\mathscr{D}(\omega_1)$. We show that this is not true for $\varkappa > \omega_1$ (Chapter 1).

As Ogg [O] and Bäni [B] pointed out, any orthosymmetric sesquilinear space of dimension $\varkappa$ bears linear topologies σ_γ (for any infinite cardinal $\gamma \leqslant \varkappa$) which are closely related to the form. Using the notions of γ-compactness and continuous bases, Bäni [B] gives a topological characterization of γ-diagonal spaces. Namely: A nondegenerate space is γ-diagonal iff, with respect to the topology σ_γ, it is γ-compact and it has a continuous basis. We give an alternative characterization: If the space has dimension $\varkappa = \gamma$ then it is $\varkappa$-diagonal iff it has an algebraic basis which is convergent with respect to $\sigma_\varkappa$ (Chapter 2).

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([1]) The definition, which generalizes symmetric bilinear spaces, is given in 1.2 after Lemma 1.

([2]) I.e. it is an orthogonal sum of subspaces whose dimension is $<\varkappa$.

In Chapter 3, we recall the definition of Appenzeller's ladder spaces and some essential properties of them.

In Chapter 4, we look at convergent sequences in ladder spaces and obtain a direct proof for the fact that such a space has no convergent basis if its invariant Γ is > 0.

In [B], the question arises whether there exist γ-compact spaces (in some dimension) without continuous bases. For certain γ (e.g. $\gamma = (2^{\omega})^{+}$), such a space has been found in [B]. Particularly, the question remained open whether there is one for $\gamma = \omega_1$. We show that there is one for every regular $\gamma > \omega$ (even $2^{\varkappa}$ many in every regular dimension $\varkappa \geqslant \gamma$, which are nondegenerate ε-Hermitian). Namely, in Chapter 5 we show that Appenzeller's ladder spaces, if endowed with σ_{γ}, are γ-compact for every uncountable $\gamma \leqslant$ the dimension of the space. It is well known that if σ_{ω} is complete, then the space is of finite dimension. So this result cannot be improved.

From the result in Chapter 5 and Bäni's characterization of γ-diagonal spaces it follows that a ladder space with its $\Gamma \neq 0$ cannot have a continuous basis. In Chapter 6, we give a direct combinatorial proof for this fact.

1. The Γ-invariant of a sesquilinear space

1.1. *The Boolean algebra* $\mathscr{D}(\mu)$. For any limit ordinal μ with $\mathrm{cf}(\mu) > \omega$ (cf is the cofinality function) a set $C \subseteq \mu$ is called *cub* (closed unbounded) if for all limit $\delta < \mu$, if $C \cap \delta$ is unbounded in δ then $\delta \in C$, and if C is unbounded in μ. The set of all cubs in μ generates the $\mathrm{cf}(\mu)$-complete cub-filter $\mathscr{F}(\mu)$ on μ. The dual ideal $\mathscr{J}(\mu)$ contains the *thin* sets. A set $E \subseteq \mu$ is called *stationary* if $E \notin \mathscr{J}(\mu)$.

The following well-known theorem ("pressing-down lemma") which is due to Fodor will be frequently used in in this paper. A map $f\colon E \to \mu$, where $E \subseteq \mu$, is called *regressive* if $(\forall \alpha \in E - \{0\})\, f(\alpha) < \alpha$.

THEOREM 1 (e.g. [Ku], p. 80; the additional statement can be proved analogously). *Let* $\mathrm{cf}(\mu) > \omega$, *E a stationary subset of* μ, *and* $f\colon E \to \mu$ *regressive. Then for some* $\gamma < \mu$, $\{\alpha \in E\colon f(\alpha) \leqslant \gamma\}$ *is stationary. If* μ *is a regular cardinal we can even find* $\gamma < \mu$ *such that* $f^{-1}\{\gamma\}$ *is stationary.*

Remark. It is not difficult to see that Theorem 1 characterizes stationary sets; moreover: *Let* $E \subset \mu$ *be unbounded and thin. Then there exists a regressive* $f\colon E \to \mu$ *such that* $\lim_{\alpha \in E} f(\alpha) = \mu$, *i.e.* $(\forall \gamma < \mu)(\exists \alpha_{\gamma} < \mu)(\forall \alpha \in E)\, \alpha \geqslant \alpha_{\gamma} \to f(\alpha) \geqslant \gamma$.

For $E, F \subseteq \mu$ let us define

$$E \sim F \;:\leftrightarrow \text{ there exists a cub } C \subseteq \mu \text{ such that } E \cap C = F \cap C.$$

Then $\sim$ is an equivalence relation on $\mathscr{P}(\mu)$. If we set $\mathscr{D}(\mu) := \mathscr{P}(\mu)/\sim$, $E^{\sim} \vee F^{\sim} := (E \cup F)^{\sim}$ and $E^{\sim} \wedge F^{\sim} := (E \cap F)^{\sim}$ for $E^{\sim}, F^{\sim} \in \mathscr{D}(\mu)$, then $(\mathscr{D}(\mu), \vee, \wedge)$ becomes a Boolean algebra with $\mathscr{J}(\mu)$ as its least and $\mathscr{F}(\mu)$ as its greatest element.

From Solovay's theorem on partitioning stationary subsets of a regular cardinal $\varkappa$ one obtains the following fact:

THEOREM 2 [Sh]. *For every* $e \in \mathscr{D}(\varkappa) - \{0\}$, $|\{f \in \mathscr{D}(\varkappa)\colon f \leqslant e\}| = 2^{\varkappa}$.

For a regular cardinal $\gamma < \varkappa$ we set $E_{\gamma}(\varkappa) := \{\lambda < \varkappa\colon \mathrm{cf}(\lambda) = \gamma\}$ and $\mathscr{D}_{\gamma}(\varkappa)$ $:= \{e \in \mathscr{D}(\varkappa)\colon e \leqslant E_{\gamma}(\varkappa)^{\sim}\}$. As is easily seen, $E_{\gamma}(\varkappa)$ is stationary, hence by Theorem 2 we have $|\mathscr{D}_{\gamma}(\varkappa)| = 2^{\varkappa}$.

1.2. *The invariant* Γ. The following concept is due to Eklof (see [E1], [E2]) who has designed it for Abelian groups. Appenzeller has adapted it to the context of sesquilinear spaces (see [A]).

Throughout this second part, k will be a skew field of any characteristic endowed with an involutory antiautomorphism * and ε a central element subject to $\varepsilon\varepsilon^* = 1$. (Thus, if * is the identity, k must be commutative.)

Given a k-linear space X of dimension $\varkappa$, a family $(X_{\nu})_{\nu<\varkappa}$ of subspaces is called a *$\varkappa$-filtration* of X if

(1) $(\forall \nu < \varkappa) \dim X_{\nu} < \varkappa$,

(2) $(\forall \lambda \in \lim(\varkappa))\, X_{\lambda} = \bigcup_{\nu<\lambda} X_{\nu}$,

(3) $X = \bigcup_{\nu<\varkappa} X_{\nu}$.

For the rest of this section we assume $\varkappa$ to be regular uncountable.

The following (easy) lemma is crucial for the definition of Γ.

LEMMA 1 ([A], p. 691). *Let* $(X_{\nu})_{\nu<\varkappa}$ *and* $(Y_{\nu})_{\nu<\varkappa}$ *be $\varkappa$-filtrations of* X. *Then* $\{\nu < \varkappa\colon X_{\nu} = Y_{\nu}\}$ *is a cub.*

A biadditive function $\Phi\colon X \times X \to k$ such that $\Phi(\lambda x, y) = \lambda \Phi(x, y)$, $\Phi(x, \lambda y) = \Phi(x, y)\lambda^*$ and $\Phi(x, y) = 0$ implies $\Phi(y, x) = 0$ for all $x, y \in X$ and $\lambda \in k$ is called an *orthosymmetric sesquilinear form* (3). If in addition Φ satisfies $\Phi(y, x) = \varepsilon\Phi(x, y)^*$ it is called *ε-Hermitian* (3). For a subspace $Y \subseteq X$ the orthogonal complement is defined by $Y^{\perp} = \{x \in X\colon (\forall y \in Y)\, \Phi(x, y) = 0\}$. We say that Y is a *summand* if $X = Y \oplus Y^{\perp}$. By Lemma 1 it follows that

$$\Gamma(X, \Phi) = \{\nu < \varkappa\colon X_{\nu} \text{ is no summand}\}^{\sim} \in \mathscr{D}(\varkappa)$$

is an invariant of the space (X, Φ) and does not depend upon the particular filtration $(X_{\nu})_{\nu<\varkappa}$ of X. If no confusion about the form is possible we write $\Gamma(X)$ instead of $\Gamma(X, \Phi)$.

The following theorem characterizes the cases where $\Gamma(X) = 0$ and $\Gamma(X) = 1$.

THEOREM 3 ([A], p. 691). (1) $\Gamma(X) = 0$ *iff* X *is $\varkappa$-diagonal.*

(2) $\Gamma(X) \neq 1$ *iff for every $\varkappa$-filtration* $(X_{\nu})_{\nu<\varkappa}$ *of* X *there exists a normal* (4) *function* $f\colon \varkappa \to \varkappa$ *such that for all* $\nu \in \mathrm{succ}(\varkappa)$, $X_{f(\nu)}$ *is a summand in* X.

Until Appenzeller's work, no sesquilinear spaces were known with invariant Γ different from 0 or 1. In [A], for arbitrary $e \in \mathscr{D}_{\omega}(\varkappa)$ such a space (even $2^{\varkappa}$ non-isometric ones) has been constructed with e as its invariant. Since clearly $\mathscr{D}_{\omega}(\omega_1) = \mathscr{D}(\omega_1)$ it follows that Γ maps onto $\mathscr{D}(\omega_1)$. The question remained open whether this is true for $\varkappa > \omega_1$, as one could have expected; for this holds (for every $\varkappa < \omega_{\omega}$) in the context of Abelian groups (see [E2], p. 62). Nevertheless, the following lemma shows that the situation is different with sesquilinear spaces.

(3) See [G1] for further details.

(4) I.e., strictly increasing and continuous at limit points.

LEMMA 2. *Assume $\varkappa > \omega_1$, and let (X, Φ) be a $\varkappa$-dimensional orthosymmetric sesquilinear space over $k, *$ with a $\varkappa$-filtration $(X_\nu)_{\nu<\varkappa}$. If $\mu < \varkappa$ is a limit with $\mathrm{cf}(\mu) \geqslant \omega_1$ and X_μ is no summand, then the set $\{\nu < \mu: X_\nu$ is no summand$\}$ contains a cub in μ.*

Proof. Assume by way of contradiction that the set $E = \{\nu < \mu: X_\nu$ is a summand$\}$ is stationary in μ. Then $E \cap \lim(\mu)$ is also stationary. Because X_μ is no summand there exists $x \in X$ such that $x - y \notin X_\mu^\perp$, for all $y \in X_\mu$. On the other hand, for every $\nu \in E$ there exists a unique $y_\nu \in X_\nu$ such that $x - y_\nu \in X_\nu^\perp$. Let $f(\nu)$ denote the least ξ such that $y_\nu \in X_\xi$. Then clearly f is regressive on $E \cap \lim(\mu)$. By Fodor's Theorem there exists $\gamma < \mu$ such that $E' = \{\nu \in E: f(\nu) \leqslant \gamma\}$ is stationary, thus unbounded in μ. Let $\nu_1, \nu_2 \in E'$ with $\gamma \leqslant \nu_1 < \nu_2$. We conclude $x - y_{\nu_2} \in X_{\nu_2}^\perp \subseteq X_{\nu_1}^\perp$ and $y_{\nu_2} \in X_\gamma \subseteq X_{\gamma_1}$, hence $y_{\nu_1} = y_{\nu_2}$ by uniqueness. Let us call this vector y. Then $x - y \in X_\nu^\perp$ for all $\nu \in E'$ with $\nu \geqslant \gamma$, thus $x - y \in X_\mu^\perp$. A contradiction.

Lemma 2 implies that in case $\varkappa > \omega_1$ the image of Γ is not all of $\mathscr{D}(\varkappa)$. As an example we note:

COROLLARY. *Let $\omega_1 \leqslant \gamma < \varkappa$ be uncountable regular cardinals. There exists no orthosymmetric sesquilinear space (X, Φ) of dimension $\varkappa$ such that $0 < \Gamma(X) \leqslant E_\gamma(\varkappa)^\sim$.*

Proof. Suppose by way of contradiction that $(X_\nu)_{\nu<\varkappa}$ is a $\varkappa$-filtration of the space (X, Φ) such that $0 < E^\sim \leqslant E_\gamma(\varkappa)^\sim$, where $E = \{\nu < \varkappa: X_\nu$ is no summand$\}$. By hypothesis, there exists a stationary $E_1 \subseteq E_\gamma(\varkappa)$ and a thin $F \subset \varkappa$ such that $E = E_1 \cup F$. It is not difficult to see that $E_1 - \{\nu < \varkappa: F \cap \nu$ is thin in $\nu\}$ is thin. Hence, we can choose $\lambda \in E_1 - F$ such that $\lambda \cap F$ is thin in λ. Clearly $E_\omega(\varkappa) \cap E \subseteq F$, and we conclude that X_ν is a summand for all $\nu \in (E_\omega(\varkappa) - F) \cap \lambda$, which set is stationary in λ. This contradicts Lemma 2.

Remark. Lemma 2 has been observed independently by J. E. Baumgartner in [Ba]. Furthermore, under the assumption of the Continuum Hypothesis, for every $E \subseteq \varkappa$ with the property that for every limit $\mu \in E$ with $\mathrm{cf}(\mu) \geqslant \omega_1$ a cub in μ is included in E he constructs a space (X, Φ) such that $\Gamma(X) = E^\sim$. His construction is a variation of the spaces found by Appenzeller. It is an open question whether CH is necessary for this result.

2. Linear topologies on sesquilinear spaces

2.1. *γ-Compactness and continuous bases.* We recall that a topological vector space $(X, \mathscr{T})$ over $k, *$ is said to be *linearly* topologized if k bears the discrete topology and the filter of zero-neighbourhoods possesses a basis consisting of linear subspaces of X. Thus, to give a linear topology $\mathscr{T}$ on the vector space X is the same as to give a filter $\mathscr{U}$ in the lattice of subspaces. Elements of $\mathscr{U}$ are then open and closed and $\mathscr{T}$ is separated iff $\bigcap \mathscr{U} = \{0\}$. The basic facts about linear topologies can be found in [B] and [K]. A filter $\mathscr{F}$ is called *linear* if it admits a basis consisting of affine subspaces (= linear varieties) of X, and it is called a *γ-filter* (where γ is an infinite cardinal) if every intersection of $< \gamma$ filter-elements belongs again to the filter. If the neighbourhood filter is a γ-filter, we simply call X a *γ-space* or $\mathscr{T}$ a *γ-topology*.

DEFINITION 1. We say that $(X, \mathscr{T})$ is *γ-compact* (more exactly: *linearly γ-compact*) if every linear γ-filter on X has a cluster point.

LEMMA 1 ([B], p. 1565). *Let γ be an infinite regular cardinal.*

(a) *Any γ-compact γ-space is complete, and thus*

(b) *any γ-compact subspace of a separated γ-space is closed.*

(c) *A closed subspace of a γ-compact space is again γ-compact.*

(d) *The image of a γ-compact space by a linear continuous map is γ-compact.*

(e) *Let γ be regular. A discrete space X is γ-compact iff $\dim X < \gamma$.*

The well known notion of summability of infinite families of elements applies to arbitrary commutative topological groups, in particular to linearly topologized spaces. From [B], [K] we recall the concept of continuous bases.

DEFINITION 2. A family $(x_\iota)_{\iota\in I}$ of vectors in X is called a *continuous basis* of $(X, \mathscr{T})$ if

(i) for every $x \in X$ there is exactly one family $(\xi_\iota)_{\iota\in I}$ of elements of k with $x = \sum_{\iota\in I} \xi_\iota x_\iota$,

(ii) all "coordinate functions" $p_\iota: X \to k$, $x \mapsto \xi_\iota$ are continuous.

For regular $\gamma > \omega$ the following holds:

THEOREM 1 ([B], Prop. 2, p. 1568). *Every continuous basis of a γ-space is also a basis in the algebraic sense.*

2.2. *The topologies σ_γ.* Let $\gamma \leqslant \varkappa$ be infinite cardinals and (X, Φ) an orthosymmetric sesquilinear space of dimension $\varkappa$. The set $\{Y^\perp: Y \subset X$ is a linear subspace with $\dim Y < \gamma\}$ is the neighbourhood filter of a linear topology on X which is denoted by $\sigma(\Phi, \gamma)$. If Φ is clear from the context we write σ_γ. If γ is regular, then σ_γ is a γ-topology; σ_γ is separated iff Φ is nondegenerate.

One main result in [B] is the following topological characterization of γ-diagonal spaces:

THEOREM 2 ([B], p. 1576). *Let $\omega < \gamma \leqslant \varkappa$ be regular cardinals. For any orthosymmetric sesquilinear space (X, Φ) of dimension $\varkappa$ the following statements are equivalent:*

(i) *(X, Φ) is γ-diagonal and nondegenerate.*

(ii) *(X, σ_γ) is γ-compact and has a continuous basis.*

We give an alternative characterization in case of $\gamma = \varkappa$:

THEOREM 3. *Let $\varkappa > \omega$ be a regular cardinal. For any space (X, Φ) as above, the following statements are equivalent:*

(i) *(X, Φ) is $\varkappa$-diagonal.*

(ii) *X has an (algebraic) basis $(x_\iota)_{\iota\in\varkappa}$ such that $\lim_{\iota\to\varkappa} x_\iota = 0$ with respect to the topology $\sigma_\varkappa$.*

For the proof of this statement we need the following generalized version of Theorem 1 in [G2], p. 99. It can be proved analogously ($\varkappa$ is supposed to be regular).

LEMMA 2. *If X has an (algebraic) basis $(x_\iota)_{\iota\in\varkappa}$ with the property that for every $\iota < \varkappa$ we have $|\{\nu < \varkappa: \Phi(x_\iota, x_\nu) \neq 0\}| < \varkappa$, then (X, Φ) is $\varkappa$-diagonal.*

Proof (of Theorem 3). (i)→(ii). Let $X = \bigoplus^{\perp}_{\iota<\varkappa} X_\iota$ such that $(\forall \iota < \varkappa) \dim X_\iota < \varkappa$. Let $\mathscr{B}_\iota$ be a basis of X_ι and $(x_\iota)_{\iota\in\varkappa}$ an enumeration of $\bigcup_{\iota\in\varkappa} \mathscr{B}_\iota$. Then clearly $\lim_{\iota\to\varkappa} x_\iota = 0$; for $\{\bigoplus_{\nu<\iota<\varkappa} X_\iota\colon \nu < \varkappa\}$ is a basis of the zero-neighbourhood filter of $\sigma_\varkappa$.

(ii)→(i). Let $(x_\iota)_{\iota<\varkappa}$ be a basis of X such that $\lim_{\iota<\varkappa} x_\iota = 0$. Set $X_\nu := \bigoplus_{\iota<\nu} kx_\iota$ for every $\nu < \varkappa$. Then $\{X_\nu^\perp\colon \nu < \varkappa\}$ is a basis of the neighbourhood filter of $\sigma_\varkappa$. Let $\iota < \varkappa$. By hypothesis there exists $\nu_\iota < \varkappa$ such that $(\forall \nu < \varkappa)\, \nu \geqslant \nu_\iota \to x_\nu \in X_{\iota+1}^\perp$. Hence $\{\nu\colon \Phi(x_\iota, x_\nu) \neq 0\} \subseteq \nu_\iota$. By Lemma 2 we conclude that (X, Φ) is $\varkappa$-diagonal.

Remark. It is not difficult to see that from the existence of a convergent basis of X with arbitrary limit it follows that there exists a basis with limit 0. Thus we arrive at:

THEOREM 3′. *Let $\varkappa > \omega$ be a regular cardinal. For any space (X, Φ) as above the following statements are equivalent:*

(i) *(X, Φ) is $\varkappa$-diagonal.*

(ii) *X has a basis which is convergent with respect to the topology $\sigma_\varkappa$.*

3. Appenzeller's ladder spaces. Let $\varkappa$ be a regular uncountable cardinal. If $E \subseteq E_\omega(\varkappa)$ (see 1.1), a *ladder system* on E is a family $\eta = (\eta_\lambda)_{\lambda\in E}$ such that every η_λ is a *ladder* on λ, i.e. a strictly increasing function $\omega \to \lambda$ having $\sup_{n<\omega} \eta_\lambda(n) = \lambda$; we shall additionally require that each *round* $\eta_\lambda(n)$ is a successor ordinal. Given η on E, $X(E, \eta)$ is the k-linear space spanned by a basis

$$\mathscr{B} = \{x_\tau^{s,n}\colon s\in\{0, 1\},\ n<\omega,\ \tau\in\operatorname{succ}(\varkappa)\} \cup \{y_\lambda^t\colon t\in\{0, 1\},\ \lambda\in E\},$$

endowed with an ε-Hermitian form over k, $*$ defined as follows ($s, t\in\{0, 1\}$; $m, n\in\omega$; $\tau, \sigma\in\operatorname{succ}(\varkappa)$; $\lambda, \mu\in E$):

$$\Phi(x_\tau^{s,n}, x_\sigma^{t,m}) := \begin{cases} \varepsilon^s & \text{if } s\neq t \wedge m = n \wedge \tau = \sigma, \\ 0 & \text{otherwise,} \end{cases}$$

$$\Phi(x_\tau^{s,n}, y_\lambda^t) := \begin{cases} 1 & \text{if } s = t \wedge \tau = \eta_\lambda(n), \\ 0 & \text{otherwise,} \end{cases}$$

$$\Phi(y_\mu^s, y_\lambda^t) := \begin{cases} \varepsilon^s\cdot|\{m<\omega\colon \eta_\mu(m) = \eta_\lambda(m)\}| & \text{if } s\neq t \wedge \mu\neq\lambda, \\ 0 & \text{otherwise.} \end{cases}$$

For each function $\phi\colon E\to\{\{0\}, \{1\}, \{0, 1\}\}$ let $X(E, \eta, \phi) \subseteq X(E, \eta)$ be the subspace spanned by

$$\mathscr{B}_\phi = \{x_\tau^{s,n}\colon s\in\{0, 1\},\ n<\omega,\ \tau\in\operatorname{succ}(\varkappa)\} \cup \{y_\lambda^t\colon \lambda\in E,\ t\in\phi(\lambda)\},$$

endowed with the restricted form. If $\phi(\lambda) = \{0, 1\}$ for all $\lambda\in E$, then clearly $X(E, \eta, \phi) = X(E, \eta)$.

If we let $X_\nu(E, \eta, \phi)$ be the span of

$$\{x_\tau^{s,n}\colon s\in\{0, 1\},\ n<\omega,\ \tau\in\operatorname{succ}(\nu)\} \cup \{y_\lambda^t\colon \lambda\in E\cap\nu,\ t\in\phi(\lambda)\},$$

then $(X_\nu(E, \eta, \phi))_{\nu<\varkappa}$ is a $\varkappa$-filtration of $X(E, \eta, \phi)$.

We quote the following results in [A]. We keep η and ϕ fixed, writing X instead of $X(E, \eta, \phi)$ and X_ν instead of $X_\nu(E, \eta, \phi)$. By $^\perp$ we mean orthogonality in X, and we use $I(\nu, \lambda) = \{n<\omega\colon \eta_\lambda(n) < \nu\}$.

THEOREM ([A], pp. 694–695). *For every $\nu < \varkappa$ we have:*

(1) $X_\nu^\perp = \operatorname{span}\{x_\tau^{s,n}\colon s\in\{0, 1\},\ n<\omega,\ \tau\in\operatorname{succ}(\varkappa),\ \nu\leqslant\tau\} \oplus \operatorname{span}\{y_\lambda^t - \varepsilon^t\sum_{m\in I(\nu,\lambda)} x_{\eta_\lambda(m)}^{1-t,m}\colon \lambda\in E,\ \nu<\lambda,\ t\in\phi(\lambda)\}$; *consequently, X is nondegenerate.*

(2) $\nu\in E \to X = X_\nu \oplus \operatorname{span}\{y_\nu^t\colon t\in\phi(\nu)\} \oplus X_\nu^\perp \to X_\nu$ *is no summand.*

(3) $\nu\notin E\to X_\nu$ *is a summand; thus* $\nu\in\operatorname{succ}(\varkappa)\to X_\nu$ *is a summand.*

(4) $\Gamma(X) = E^\sim$.

4. Convergent sequences in ladder spaces. Let $X(E, \eta, \phi)$ be a ladder space over $(k, *, \varepsilon)$ endowed with the $\sigma_\varkappa$-topology. Again, we write X, X_ν instead of $X(E, \eta, \phi)$, $X_\nu(E, \eta, \phi)$ respectively.

If E is thin, then by the Theorem in Chapter 3, $\Gamma(X) = 0$. Hence by Theorem 3, 1.2 and Theorem 3′, 2.1, X has a convergent basis. Using Remark 1.1, we can specify such a basis if $|E| = \varkappa$ (unless $\mathscr{B}_\phi$ is sufficient). Choose a regressive function $f\colon E\to\varkappa$ such that $\lim_{\lambda\in E} f(\lambda) = \varkappa$ and let $\mathscr{C}$ be an enumeration of all $x_\tau^{s,n}$ ($s\in\{0, 1\}$, $n<\omega$, $\tau\in\operatorname{succ}(\varkappa)$) and all $y_\lambda^t - \varepsilon^t\sum_{m\in I(f(\lambda),\lambda)} x_{\eta_\lambda(m)}^{1-t,m}$ ($t\in\{0, 1\}$, $\lambda\in E$). It is not difficult to verify that $\lim\mathscr{C} = 0$.

If E is stationary, then by the theorems referred to above, X cannot have a convergent basis. This follows directly from the following lemma. For any $\varkappa$-sequence $(x_\iota)_{\iota<\varkappa}$ in X set

$$E((x_\iota)_{\iota<\varkappa}) = \{\lambda\in E\colon (\exists\iota<\varkappa)(\exists t\in\{0, 1\})\, p_{y_\lambda^t}(x_\iota) \neq 0\},$$

where $p_{y_\lambda^t}$ is the coordinate function (see Def. 2, 2.1) of the vector y_λ^t with respect to the basis $\mathscr{B}_\phi$.

LEMMA. *Let $(x_\iota)_{\iota<\varkappa}$ be a $\varkappa$-sequence in X such that $\lim_{\iota<\varkappa} x_\iota = 0$. Then the set $E((x_\iota)_{\iota<\varkappa})$ is thin.*

Proof. For every $\lambda\in E((x_\iota)_{\iota<\varkappa})$ choose $\iota(\lambda)<\varkappa$ such that $p_{y_\lambda^t}(x_{\iota(\lambda)}) \neq 0$ for a $t\in\{0, 1\}$. By nondegeneracy of X and the fact that for $\lambda\in\lim(\varkappa)$ we have $X_\lambda^\perp = \bigcap_{\nu<\lambda} X_\nu^\perp$ there exists a greatest $\nu<\varkappa$ such that $x_{\iota(\lambda)}\in X_\nu^\perp$; denote this ν by $\nu(\iota(\lambda))$. Now consider the function

$$f\colon E((x_\iota)_{\iota<\varkappa}) \to \varkappa, \qquad \lambda\mapsto \nu(\iota(\lambda)).$$

By clause (1) of the Theorem, Chapter 3, f is regressive.

Assume that $E((x_\iota)_{\iota<\varkappa})$ is stationary. Hence, by Theorem 1, Chapter 1, there exists a stationary $S\subseteq E((x_\iota)_{\iota<\varkappa})$ and $\gamma<\varkappa$ such that $(\forall\lambda\in S-\{0\})\, f(\lambda) = \gamma$. So clearly $|\{\iota(\lambda)\colon \lambda\in S\}| = \varkappa$ and for every $\nu>\gamma$ and $\lambda\in S$, $x_{\iota(\lambda)}\notin X_\nu^\perp$. This contradicts the hypothesis.

COROLLARY 1. *Let $(x_\iota)_{\iota<\varkappa}$ be a convergent $\varkappa$-sequence in X. Then $E((x_\iota)_{\iota<\varkappa})$ is thin.*

Proof. Let $x = \lim_{\iota<\varkappa} x_\iota$. Set $y_\iota := x - x_\iota$ for all $\iota < \varkappa$. Hence $\lim_{\iota<\varkappa} y_\iota = 0$. In the representation of x by $\mathscr{B}_\phi$, let $y_{\lambda_1}^{t_1}, \ldots, y_{\lambda_n}^{t_n}$ be the finitely many y_λ^t with a nonzero coordinate. Then clearly $E((x_\iota)_{\iota<\varkappa}) \subseteq E((y_\iota)_{\iota<\varkappa}) \cup \{\lambda_1, \ldots, \lambda_n\}$. By the Lemma, this set is thin.

Corollary 2. *Let $(x_\iota)_{\iota<\varkappa}$ be a basis of a ladder space X with $\Gamma(X) > 0$. Then $(x_\iota)_{\iota<\varkappa}$ does not converge.*

Proof. Clearly, we must have $E((x_\iota)_{\iota<\varkappa}) = E$. By hypothesis, E is stationary; so, by Corollary 1 we are done.

Remark. Let $(x_\iota)_{\iota<\varkappa}$ be a Cauchy sequence in $X(E, \eta, \phi)$. It is not difficult to see that $(x_\iota)_{\iota<\varkappa}$ has a subsequence $(y_\iota)_{\iota<\varkappa}$ such that $E((y_\iota)_{\iota<\varkappa}) \sim E((z_\iota)_{\iota<\varkappa})$, where $z_\iota = y_{\iota+1} - y_\iota$. Since $(x_\iota)_{\iota<\varkappa}$ is Cauchy $(z_\iota)_{\iota<\varkappa}$ is convergent with limit 0. Hence by the Lemma, $E' := E((y_\iota)_{\iota<\varkappa})$ is thin. But clearly we have $(y_\iota)_{\iota<\varkappa} \subseteq X(E', \eta', \phi') =: X' \subseteq X(E, \eta, \phi)$, where $\eta' = (\eta_\lambda)_{\lambda\in E'}$, $\phi' = \phi | E'$. By Theorem 3, Chapter 1, X' is $\varkappa$-diagonal and so $\varkappa$-compact by Theorem 2, Chapter 2. Furthermore it is easy to see that on X', the topology $\sigma_\varkappa$ defined by $\Phi | X' \times X'$ coincides with the topology induced by $\sigma_\varkappa$ on $X(E, \eta, \phi)$. It follows that $(y_\iota)_{\iota<\varkappa}$ is Cauchy in X' and thus converges in X' and hence in $X(E, \eta, \phi)$.

In order to prove completeness of $X(E, \eta, \phi)$ (for arbitrary $E \subseteq E_\omega(\varkappa)$) we still have to show that convergence of arbitrary Cauchy sequences implies convergence of Cauchy nets, which is true in a more general context. We do not prove this here since in the following chapter we show that ladder spaces are even γ-compact for arbitrary regular $\gamma > \omega$.

5. Ladder spaces are compact. Let $\omega < \gamma \leqslant \varkappa$ be regular cardinals and let X be a ladder space of dimension $\varkappa$, endowed with the topology σ_γ.

A directed set $(D, \leqslant)$ is called *γ-directed* iff $(\forall A \subseteq D)\, |A| < \gamma \to (\exists d \in D)(\forall a \in A)\, d \geqslant a$. If D is γ-directed, so is every cofinal $D' \subset D$.

The following lemma is the key element in the proof of γ-compactness.

Lemma 1. *Let $t \in \{0, 1\}$ and let $N = \langle y_{\lambda(d)}^t, d \in D\rangle$ be a net in X such that D is γ-directed. Then N has a cluster point.*

Proof. Successively, we distinguish several cases:

Case 1: There exists a cofinal $D' \subseteq D$ and $\lambda \in E$ such that $(\forall d \in D')\, \lambda(d) = \lambda$. Then clearly $\lim \langle y_{\lambda(d)}^t, d \in D'\rangle = y_\lambda^t$, and so y_λ^t is a cluster point of N.

Case 2 ($\neg$Case 1): For every $\lambda \in E$ there exists $d_\lambda \in D$ such that $(\forall d \in D)$ $d \geqslant d_\lambda \to \lambda(d) \neq \lambda$. We choose $\mu \leqslant \varkappa$ minimal such that there exists a cofinal $D' \subseteq D$ with the property $(\forall d \in D')\, \lambda(d) < \mu$. Then for every $\alpha < \mu$ there exists $d_\alpha \in D'$ such that $(\forall d \in D')\, d \geqslant d_\alpha \to \alpha \leqslant \lambda(d) < \mu$. We abbreviate this by writing $\lim_{d\in D'} \lambda(d) = \mu$. By γ-directedness of D' we conclude $\mathrm{cf}(\mu) \geqslant \gamma$ $(> \omega)$.

Case 2.1: There exists a subnet $\langle y_{\lambda(da)}^t, a \in A\rangle$ of $\langle y_{\lambda(d)}^t, d \in D'\rangle$ such that A is γ-directed and $\lim_{a\in A} \eta_{\lambda(da)}(0) = \mu$. This is a subcase of Case 2.2 … 2.1 which will be treated below.

Case 2.2 (Case 2 $\wedge$ $\neg$Case 2.1): In this case there exists $\alpha_0 < \mu$ and $d_0 \in D'$ such that $(\forall d \in D')\, d \geqslant d_0 \to \eta_{\lambda(d)}(0) < \alpha_0$; for otherwise, for every $\alpha < \mu$ and $d \in D'$ there exists $d' \in D'$ such that $d' \geqslant d$ and $\eta_{\lambda(d')}(0) \geqslant \alpha$. If we endow $\mathscr{D} := \{\langle d, \alpha\rangle \in D' \times \mu\colon\ \eta_{\lambda(d)}(0) \geqslant \alpha\}$ with the product ordering, then $\mathscr{D}$ is γ-directed, and the map $f\colon \mathscr{D} \to D'$, $\langle d, \alpha\rangle \mapsto d$, is cofinal; hence $\langle y_{\lambda(f\langle d,\alpha\rangle)}^t, \langle d, \alpha\rangle \in \mathscr{D}\rangle$ is a subnet of $\langle y_{\lambda(d)}^t, d \in D'\rangle$ and it is easily seen that $\lim_{\langle d,\alpha\rangle\in\mathscr{D}} \eta_{\lambda(f\langle d,\alpha\rangle)}(0) = \mu$. A contradiction to Case 2.2.

We proceed analogously for $\eta_{\lambda(d)}(1)$: Distinguish

Case 2.2.1: There exists a subnet $\langle y_{\lambda(da)}^t, a \in A\rangle$ of $\langle y_{\lambda(d)}^t, d \in D'\rangle$ such that A is γ-directed and $\lim_{a\in A} \eta_{\lambda(da)}(1) = \mu$.

Case 2.2.2: As in Case 2.2, one shows that there exists $\alpha_1 < \mu$ and $d_1 \in D'$ such that $(\forall d \in D')\, d \geqslant d_1 \to \eta_{\lambda(d)}(1) < \alpha_1$.

We claim that after finitely many steps we arrive at Case 2.2 … 2.1. Otherwise, there exist $\alpha_0, \alpha_1, \ldots < \mu$ and $d_0, d_1 \ldots \in D'$ such that $(\forall n < \omega\, \forall d \in D')$ $d \geqslant d_n \to \eta_{\lambda(d)}(n) < \alpha_n$. By $\mathrm{cf}(\mu) \geqslant \gamma > \omega$, we have $\alpha_\omega := \sup_{n<\omega} \alpha_n < \mu$, and by γ-directedness of D' there exists $d_\omega \in D'$ such that $(\forall n < \omega)\, d_\omega \geqslant d_n$. Consequently $(\forall d \in D'\, \forall n < \omega)\, d \geqslant d_\omega \to \eta_{\lambda(d)}(n) < \alpha_\omega$ and hence $(\forall d \in D')\, d \geqslant d_\omega \to \lambda(d) \leqslant \alpha_\omega$. This contradicts the minimality of μ (see Case 2).

By what we have shown so far, there exist a subnet $\langle y_{\lambda(da)}^t, a \in A\rangle$ of $\langle y_{\lambda(d)}^t, d \in D'\rangle$ with γ-directed A and $n < \omega$ such that $\lim_{a\in A} \eta_{\lambda(da)}(n) = \mu$. We write $\lambda(a)$ instead of $\lambda(da)$.

Now consider $\langle \eta_{\lambda(a)}(0), a \in A\rangle$.

We distinguish two cases:

Case a: There exists $\beta_0 < \mu$ and a cofinal $A_0 \subseteq A$ such that $(\forall a \in A_0)\, \eta_{\lambda(a)}(0) = \beta_0$.

Case b: For every $\beta < \mu$ there exists $a_\beta \in A$ such that $(\forall a \in A)\, a \geqslant a_\beta \to \eta_{\lambda(a)}(0) \neq \beta$. In this case set $A_0 := A$.

Proceeding analogously, either choose a cofinal $A_1 \subseteq A_0$ and $\beta_1 < \mu$ such that $(\forall a \in A_1)\, \eta_{\lambda(a)}(1) = \beta_1$ (Case a) or otherwise (Case b) set $A_1 := A_0$. Finally, we choose $A_{n-1} \subseteq A_{n-2}$ and, possibly, $\beta_{n-1} < \mu$.

If in the ith step we can choose $\beta_i < \mu$ (Case a), set $z_i := \varepsilon^t x_{\beta_i}^{1-t,i}$, $z_i := 0$ otherwise (Case b), and let $z := \sum_{i=0}^{n-1} z_i$. We claim that $\lim \langle y_{\lambda(a)}, a \in A_{n-1}\rangle = z$ and thus, z is a cluster point of N.

Let $U \subseteq X$ be a subspace such that $\dim U < \gamma$. We may certainly assume that U is spanned by vectors in $\mathscr{B}_\phi$. By $\dim U < \gamma$ and $\mathrm{cf}(\mu) \geqslant \gamma$ there exists $\alpha < \mu$ such that $(\forall \langle n, \tau\rangle \in \omega \times \mathrm{succ}(\varkappa))\, x_\tau^{t,n} \in U \to \tau \notin \{\xi\colon \alpha \leqslant \xi < \mu\}$, and $(\forall \lambda \in E)\, y_\lambda^{1-t} \in U \to (\forall n < \omega)$ $\eta_\lambda(n) \notin \{\xi\colon \alpha \leqslant \xi < \mu\}$. Now since $\lim_{a\in A_{n-1}} \eta_{\lambda(a)}(n) = \mu$ there exists $a_0 \in A_{n-1}$ such that $(\forall a \in A_{n-1})(\forall m \geqslant n)\, a \geqslant a_0 \to \alpha \leqslant \eta_{\lambda(a)}(m) < \mu$. Hence, by construction of $A_{n-1} \subseteq A_{n-2} \subseteq \ldots \subseteq A_0 \subseteq A$ and β_i and γ-directedness of A_{n-1}, there exists $a_1 \in A_{n-1}$, $a_1 \geqslant a_0$, such that for all $a \in A_{n-1}$ with $a \geqslant a_1$ the following two statements hold:

(1) $(\forall \langle j, \tau\rangle \in \omega \times \mathrm{succ}(\varkappa))\, [(x_\tau^{t,j} \in U \wedge \eta_{\lambda(a)}(j) = \tau) \to (\exists\, 0 \leqslant i < n)\, \langle j, \tau\rangle = \langle i, \beta_i\rangle]$,

(2) $(\forall \lambda \in E\, \forall j < \omega)\, [(y_\lambda^{1-t} \in U \wedge \eta_\lambda(j) = \eta_{\lambda(a)}(j)) \to (\exists\, 0 \leqslant i < n)\, \langle j, \eta_\lambda(j)\rangle = \langle i, \beta_i\rangle]$.

We conclude $(\forall a \in A_{n-1})\, a \geqslant a_1 \to y_{\lambda(a)}^t - z \in U^\perp$, and hence, $\langle y_{\lambda(a)}, a \in A_{n-1}\rangle$ eventually remains in $z + U^\perp$; for let $a \in A_{n-1}$, $a \geqslant a_1$. If there exists $x_\tau^{t,m} \in U$ such that

$\Phi(x_\tau^{t,m}, y_{\lambda(a)}^t) \neq 0$, then by (1) there exists $0 \leqslant i < n$ such that $x_\tau^{t,m} = x_{\beta_i}^{t,i}$, and we compute

$$\Phi(x_{\beta_i}^{t,i}, y_{\lambda(a)}^t - z) = \Phi(x_{\beta_i}^{t,i}, y_{\lambda(a)}^t) - \Phi(x_{\beta_i}^{t,i}, \varepsilon^t x_{\beta_i}^{1-t,i}) = 1 - (\varepsilon^t)^* \varepsilon^t = 0.$$

If $\Phi(x_\tau^{t,m}, y_{\lambda(a)}^t) = 0$, then clearly $\Phi(x_\tau^{t,m}, z) = 0$. Finally, if $y_\lambda^{1-t} \in U$, then by (2) we have

$$\Phi(y_\lambda^{1-t}, y_{\lambda(a)}^t - z) = \varepsilon^{1-t} |\{0 \leqslant i < n\colon \eta_\lambda(i) = \beta_i\}| - (\varepsilon^t)^* |\{0 \leqslant i < n\colon \eta_\lambda(i) = \beta_i\}| \varepsilon = 0.$$

LEMMA 2. *Let $t \in \{0, 1\}$ and let $N = \langle x_{\tau(d)}^{t,n(d)}, d \in D\rangle$ be a net in X such that D is γ-directed. Then N has a cluster point.*

Proof. We distinguish two cases:

Case 1: There exist a cofinal $D' \subseteq D$ and $\tau \in \mathrm{succ}(\varkappa)$ such that $(\forall d \in D') \tau(d) = \tau$. Then by γ-directedness of D' and $\gamma > \omega$, there exist $n \in \omega$ and a cofinal $D'' \subseteq D'$ such that $(\forall d \in D'') n(d) = n$. We conclude $\lim \langle x_{\tau(d)}^{t,n(d)}, d \in D''\rangle = x_\tau^{t,n}$.

Case 2: For every $\tau \in \mathrm{succ}(\varkappa)$ there exists $d_\tau \in D$ such that $(\forall d \in D)\, d \geqslant d_\tau \to \tau(d) \neq \tau$. We claim that in this case we have $\lim N = 0$.

Let $U \subset X$ be a subspace spanned by vectors in $\mathscr{B}_\Phi$ with $\dim U < \gamma$. By γ-directedness of D, there exists $d_0 \in D$ such that $(\forall \langle n, \tau\rangle \in \omega \times \mathrm{succ}(\varkappa))\, x_\tau^{1-t,n} \in U \to d_\tau \leqslant d_0$ and $(\forall \lambda \in E)\, y_\lambda^t \in U \to (\forall n \in \omega) d_{\eta_\lambda(n)} \leqslant d_0$. Consequently, $(\forall d \in D)\, d \geqslant d_0 \to x_{\tau(d)}^{t,n(d)} \in U^\perp$.

THEOREM. *Let $\omega < \gamma \leqslant \varkappa$ be regular cardinals. For any ladder space X of dimension $\varkappa$, (X, σ_γ) is γ-compact.*

Proof. Let $N = \langle z_d, d \in D\rangle$ be a net in X such that D is γ-directed. We have to show that the linear γ-filter $\mathscr{F}$ which has as base the linear manifolds generated by the final segments of N has a cluster point.

By γ-directedness of D, there exist $n < \omega$ and $D_1 \subseteq D$ cofinal such that every z_d with $d \in D_1$ has at most n nonzero coordinates with respect to the $x_\tau^{0,n}$'s, $x_\tau^{1,n}$'s, y_λ^0's and y_λ^1's. Then every such z_d can be written as

$$z_d = \sum_{t=0}^{1} \sum_{i=1}^{n} \big(\alpha(d,t,i) x(d,t,i) + \beta(d,t,i) y(d,t,i)\big),$$

each $x(d,t,i)$ being one of the $x_\tau^{t,n}$'s, each $y(d,t,i)$ one of the y_λ^t's.

Using Lemmata 1 and 2, we successively choose convergent subnets as follows: Choose a convergent subnet $N_1 = \langle x(da_1, 0, 1), a_1 \in A_1\rangle$ of $\langle x(d,0,1), d \in D_1\rangle$ such that A_1 is γ-directed and let $u(0,1) := \lim N_1$. Then choose a convergent subnet $N_2 = \langle x(da_1 a_2, 0, 2), a_2 \in A_2\rangle$ of $\langle x(da_1, 0, 2), a_1 \in A_1\rangle$ with γ-directed A_2 and let $u(0,2) := \lim N_2$. In the $2n$th step the net $\langle x(da_1 \ldots a_{2n-1}, 1, n), a_{2n-1} \in A_{2n-1}\rangle$ has a convergent subnet $\langle x(da_1 \ldots a_{2n}, 1, n), a_{2n} \in A_{2n}\rangle$ such that A_{2n} is γ-directed. Let $u(1,n) := \lim N_{2n}$.

Now, analogously handle the $y(d,t,i)$'s: Choose a convergent subnet $N_{2n+1} = \langle y(da_1 \ldots a_{2n+1}, 0, 1), a_{2n+1} \in A_{2n+1}\rangle$ of $\langle y(da_1 \ldots a_{2n}, 0, 1), a_{2n} \in A_{2n}\rangle$ such that A_{2n+1} is γ-directed and let $v(0,1) := \lim N_{2n+1}$. In the $4n$th step we arrive at a convergent subnet $N_{4n} = \langle y(da_1 \ldots a_{4n}, 1, n), a_{4n} \in A_{4n}\rangle$ of $\langle y(da_1 \ldots a_{4n-1}, 1, n), a_{4n-1} \in A_{4n-1}\rangle$ such that A_{4n} is γ-directed. Let $v(1,n) := \lim N_{4n}$.

For simplicity, let us write A instead of A_{4n}, a instead of a_{4n} and $d(a)$ instead of $da_1 \ldots a_{4n}$.

By construction, we have $\forall t \in \{0,1\}\, \forall 1 \leqslant i \leqslant n$

(3) $$\lim_{a \in A} \big\langle x(d(a), t, i), a \in A\big\rangle = u(t,i), \quad \lim_{a \in A} \big\langle y(d(a), t, i), a \in A\big\rangle = v(t,i).$$

Now consider the net

$$\big\langle \alpha(d(a),0,1), \ldots, \alpha(d(a),0,n), \alpha(d(a),1,1), \ldots, \alpha(d(a),1,n),$$
$$\beta(d(a),0,1), \ldots, \beta(d(a),0,n), \beta(d(a),1,1), \ldots, \beta(d(a),1,n), a \in A\big\rangle.$$

By Lemma 1(e), 2.1, the k-space k^{4n}, endowed with the discrete topology, is even ω-compact. Hence, there exist scalars $\alpha(t,i), \beta(t,i) \in k$, where $t \in \{0,1\}, 1 \leqslant i \leqslant n$, such that for every $a \in A$ there exist finitely many $a_j \in A$ with $a_j \geqslant a$ and $\delta_j \in k$ with $\sum_j \delta_j = 1$ such that for every $t \in \{0,1\}$ and for all $i \in \{1, \ldots, n\}$ the following equations hold:

(4) $$\sum_j \delta_j \alpha(d(a_j), t, i) = \alpha(t,i), \quad \sum_j \delta_j \beta(d(a_j), t, i) = \beta(t,i).$$

We claim that the vector

$$z = \sum_{t=0}^{1} \sum_{i=1}^{n} \big(\alpha(t,i) u(t,i) + \beta(t,i) v(t,i)\big)$$

is a cluster point of $\mathscr{F}$. So let $U \subset X$ be a subspace such that $\dim U < \gamma$, and $a \in A$. We have to show that $z + U^\perp$ contains an affine combination of $z_{d(a'')}$'s with $a'' \geqslant a$.

By (3), there exists $a' \in A$ such that for every $a'' \geqslant a'$, $t \in \{0,1\}$ and $i \in \{1, \ldots, n\}$ we have $x(d(a''), t, i) \in u(t,i) + U^\perp$ and $y(d(a''), t, i) \in v(t,i) + U^\perp$. Now choose finitely many $a_j \in A$ with $a_j \geqslant a, a'$ and $\delta_j \in k$ with $\sum_j \delta_j = 1$ such that the equations (4) hold. Then for all $t \in \{0,1\}$ and $i \in \{1, \ldots, n\}$ we have

$$\sum_j \delta_j \alpha(d(a_j), t, i) x(d(a_j), t, i) \in \sum_j \delta_j \alpha(d(a_j), t, i) u(t,i) + U^\perp = \alpha(t,i) u(t,i) + U^\perp,$$

$$\sum_j \delta_j \beta(d(a_j), t, i) y(d(a_j), t, i) \in \sum_j \delta_j \beta(d(a_j), t, i) v(t,i) + U^\perp = \beta(t,i) v(t,i) + U^\perp,$$

and hence

$$\sum_j \delta_j z_{d(a_j)} = \sum_j \delta_j \sum_{t=0}^{1} \sum_{i=1}^{n} \alpha(d(a_j),t,i) x(d(a_j),t,i) + \sum_j \delta_j \sum_{t=0}^{1} \sum_{i=1}^{n} \beta(d(a_j),t,i) y(d(a_j),t,i)$$
$$= \sum_{t=0}^{1} \sum_{i=1}^{n} \sum_j \delta_j \alpha(d(a_j),t,i) x(d(a_j),t,i) + \sum_{t=0}^{1} \sum_{i=1}^{n} \sum_j \delta_j \beta(d(a_j),t,i) y(d(a_j),t,i)$$
$$\in \sum_{t=0}^{1} \sum_{i=1}^{n} \big(\alpha(t,i) u(t,i) + \beta(t,i) v(t,i)\big) + U^\perp.$$

6. Ladder spaces and continuous bases. Assume that $E \subset \varkappa$ is thin. Then, by the Theorem in Chapter 3, the ladder space $X = X(E, \eta, \Phi)$ satisfies $\Gamma(X) = E^\sim = 0$. So, by

Theorem 3, 1.2, and Theorem 2, 2.2, $(X, \sigma_\varkappa)$ has a continuous basis. It is not difficult to see that the basis $\mathscr{C}$ given at the beginning of Chapter 4 is continuous.

If E is stationary, we have $\Gamma(X) = E^\sim > 0$. By the theorems referred to above and the result in the previous chapter, for all regular $\omega < \gamma \leqslant \varkappa$, (X, σ_γ) has no continuous basis. In the sequel, we give a direct proof for this fact. We need the following combinatorial lemma:

LEMMA. *Let $E \subseteq \varkappa$ be stationary and $(\eta_\lambda)_{\lambda\in E}$ a ladder system. Then there exists $\tau \in \mathrm{succ}(\varkappa)$ such that for every $\nu < \varkappa$ with $\nu > \tau$ the set*

$$A_\tau^\nu(E) = \{\lambda \in E\colon \eta_\lambda \text{ has no rounds in } \{\xi < \varkappa\colon \tau \leqslant \xi < \nu\}\}$$

is stationary.

Proof. Assume by way of contradiction that for every $\tau \in \mathrm{succ}(\varkappa)$ there exists $\nu_\tau < \varkappa$ with $\nu_\tau > \tau$ such that $A_\tau^{\nu_\tau}(E)$ is thin. We may certainly assume $\nu_\tau \in \mathrm{succ}(\varkappa)$ for all $\tau \in \mathrm{succ}(\varkappa)$. Inductively, define an ω-sequence: Let $\tau_0 \in \mathrm{succ}(\varkappa)$ be arbitrary and $\tau_{n+1} := \nu_{\tau_n}$. Then clearly the set

$$A := \bigcup_{n<\omega} A_{\tau_n}^{\tau_{n+1}}(E)$$

is thin (as the union of $< \varkappa$ thin sets). It follows that $E - A$ is stationary and hence unbounded in $\varkappa$. Choose $\lambda \in E - A$ such that $\lambda > \sup_{n<\omega}\tau_n$. By construction, then the ladder η_λ has rounds in $\{\xi \in \varkappa\colon \tau_n \leqslant \xi < \tau_{n+1}\}$ for every $n < \omega$. This contradicts the fact that η_λ is strictly increasing and $\sup_{n<\omega}\eta_\lambda(n) = \lambda$.

THEOREM. *Let X be a ladder space (from Chapter 3) of regular dimension $\varkappa$. If $\Gamma(X) > 0$ then $(X, \sigma_\varkappa)$ has no continuous basis.*

Proof. Assume by way of contradiction that $(X, \sigma_\varkappa)$ has a continuous basis $(x_\iota)_{\iota\in I}$. By Theorem 1, 2.1, $(x_\iota)_{\iota\in I}$ is an algebraic basis of X. Thus, we may assume $I = \varkappa$.

There exists $t \in \{0, 1\}$, $n < \omega$ and a stationary $E' \subseteq E$ such that for all $\lambda \in E'$ we have $t \in \phi(\lambda)$ and, in its representation by $(x_\iota)_{\iota<\varkappa}$, y_λ^t has n nonzero coordinates.

Consider the function

$$f_0\colon E' \to \varkappa, \qquad \lambda \mapsto \eta_\lambda(0).$$

By definition, f_0 is regressive. So, by Fodor's Theorem, there exists a stationary $E'' \subseteq E'$ and $\gamma_0 \in \mathrm{succ}(\varkappa)$ such that $\eta_\lambda(0) = \gamma_0$ for all $\lambda \in E''$.

By the Lemma there exists $\tau_1 \in \mathrm{succ}(\varkappa)$ such that $A_{\tau_1}^\nu(E'')$ is stationary for all $\nu < \varkappa$ with $\nu > \tau_1$. Then clearly $\tau_1 > \gamma_0$. There exists $I_1 \subset \varkappa$ with $|I_1| < \varkappa$ such that

(1) $$X_{\tau_1} \subseteq \bigoplus_{\iota\in I_1} kx_\iota.$$

By continuity of $(x_\iota)_{\iota<\varkappa}$ and regularity of $\varkappa$, there exists $\nu_1 < \varkappa$ such that for all $\iota \in I_1$

(2) $$p_\iota(X_{\nu_1}^\perp) = \{0\},$$

where p_ι is the ιth coordinate function (see Definition 2, 2.1). We may assume $\nu_1 > \tau_1$. Then $A_{\tau_1}^{\nu_1}(E'')$ is stationary.

Once more using the fact that the union of $< \varkappa$ thin sets is thin, we find $m_1 < \omega$, $\gamma_0 < \gamma_1 < \ldots < \gamma_{m_1} < \tau_1$ and a stationary $E_1 \subseteq A_{\tau_1}^{\nu_1}(E'')$ such that for all $\lambda \in E_1$, for all $0 \leqslant i \leqslant m_1$ and for all $m_1 + 1 \leqslant j < \omega$ we have $\eta_\lambda(i) = \gamma_i$ and $\eta_\lambda(j) \geqslant \nu_1$.

Consequently, for any choice of $\lambda \in E_1$,

(3) $$\sum_{m\in I(\nu_1,\lambda)} x_{\eta_\lambda(m)}^{1-t,m} = \sum_{i=0}^{m_1} x_{\gamma_i}^{1-t,i} =: u_1 \in X_{\tau_1},$$

and

(4) $$y_\lambda^t - u_1 \in X_{\nu_1}^\perp.$$

With respect to $(x_\iota)_{\iota<\varkappa}$, u_1 has a representation

(5) $$u_1 = \sum_{i=1}^{q_1} \alpha_i x_{\iota_i},$$

where $q_1 > 0$ and $\alpha_i \neq 0$ for all $1 \leqslant i \leqslant q_1$.

From (1)–(5) we conclude $p_{\iota_i}(y_\lambda^t - u_1) = p_{\iota_i}(y_\lambda^t) - \alpha_i = 0$ for all $1 \leqslant i \leqslant q_1$ and hence $p_{\iota_i}(y_\lambda^t) \neq 0$. By construction, every y_λ^t has n nonzero coordinates (with respect to $(x_\iota)_{\iota<\varkappa}$). But now, we already know q_1 of them; so clearly $q_1 \leqslant n$.

Now we repeat the procedure above. The function

$$f_{m_1+1}\colon E_1 \to \varkappa, \qquad \lambda \mapsto \eta_\lambda(m_1+1),$$

is regressive on the stationary set E_1. Again, by Fodor's Theorem there exists a stationary set $E_1' \subseteq E_1$ and $\gamma_{m_1+1} \in \mathrm{succ}(\varkappa)$ such that $\eta_\lambda(m_1+1) = \gamma_{m_1+1}$ for all $\lambda \in E_1'$. Then clearly $\nu_1 \leqslant \gamma_{m_1+1}$.

By the Lemma, there exists $\tau_2 \in \mathrm{succ}(\varkappa)$ such that for all $\nu < \varkappa$ with $\nu > \tau_2$ the set $A_{\tau_2}^\nu(E')$ is stationary. Again, there exists $I_2 \subset \varkappa$ with $|I_2| < \varkappa$ such that

(6) $$X_{\tau_2} \subseteq \bigoplus_{\iota\in I_2} kx_\iota.$$

As above, we find $\nu_2 < \varkappa$ with $\nu_2 > \tau_2$ such that for all $\iota \in I_2$

(7) $$p_\iota(X_{\nu_2}^\perp) = \{0\}.$$

Again there exist $m_2 < \omega$, $\nu_1 \leqslant \gamma_{m_1+1} < \gamma_{m_1+2} < \ldots < \gamma_{m_1+m_2} < \tau_2$ and a stationary $E_2 \subseteq A_{\tau_2}^{\nu_2}(E_1')$ such that for all $\lambda \in E_2$, for all $m_1 + 1 \leqslant i \leqslant m_1 + m_2$ and for all $m_1 + m_2 + 1 \leqslant j < \omega$ we have $\eta_\lambda(i) = \gamma_i$ and $\eta_\lambda(j) \geqslant \nu_2$.

For any choice of $\lambda \in E_2$ we conclude

$$\sum_{m\in I(\nu_2,\lambda)} x_{\eta_\lambda(m)}^{1-t,m} = u_1 + \sum_{i=m_1+1}^{m_1+m_2} x_{\gamma_i}^{1-t,i} =: u_1 + u_2,$$

(8) $$y_\lambda^t - u_1 - u_2 \in X_{\nu_2}^\perp$$

and furthermore

(9) $$u_2 \in X_{\tau_2}.$$

Let

(10) $$u_2 = \sum_{i=q_1+1}^{q_1+q_2} \alpha_i x_{\iota_i},$$

such that $q_2 > 0$ and $\alpha_i \neq 0$ for all $q_1+1 \leqslant i \leqslant q_1+q_2$. Putting together (1)–(9), we conclude

$$p_{\iota_i}(y^t_\lambda - u_1 - u_2) = 0$$

for all $1 \leqslant i \leqslant q_1+q_2$. Furthermore, $\iota_i \notin \{\iota_1, \ldots, \iota_{q_1}\}$ for every $i \in \{q_1+1, \ldots, q_1+q_2\}$; for otherwise we would have $p_{\iota_i}(y^t_\lambda - u_1) = p_{\iota_i}(y^t_\lambda - u_1 - u_2) = 0$, thus $p_{\iota_i}(u_2) = 0$. This contradicts (10). We conclude $p_{\iota_i}(y^t_\lambda) \neq 0$ for all $1 \leqslant i \leqslant q_1+q_2$. Hence $q_1+q_2 \leqslant n$.

After finitely many steps we must arrive at $\sum_i q_i > n$. A contradiction.

COROLLARY. *Let $\omega < \gamma \leqslant \varkappa$ be regular, X a ladder space of dimension $\varkappa$. If $\Gamma(X) > 0$, then (X, σ_γ) has no continuous basis.*

Proof. By Theorem 1, 2.1, a continuous basis $(x_\iota)_{\iota\in I}$ of (X, σ_γ) is an algebraic basis. Because σ_γ is coarser than $\sigma_\varkappa$, the coordinate functions p_ι are continuous on $(X, \sigma_\varkappa)$; and hence, $(x_\iota)_{\iota\in I}$ would be a continuous basis of $(X, \sigma_\varkappa)$. This contradicts the Theorem.

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Relative consistency results via strong compactness

by

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Abstract. We show in this paper that certain relative consistency proofs which had originally been done using supercompactness can be recast, using Henle's notion of modified Prikry forcing, in terms of strong compactness.

The notion of strongly compact cardinal is perhaps the most peculiar in the entire litany of large cardinal axioms. The well known results of Magidor [M] and Kimchi and Magidor [KM] show that strongly compact cardinals suffer from a severe identity crisis: The least strongly compact cardinal can be either the least measurable cardinal or the least supercompact cardinal, and the class of strongly compact cardinals can coincide precisely with the class of measurable cardinals or with the class of supercompact cardinals (except at limit points). It is further the case that the consistency strength of strongly compact cardinals is still unknown. Guesses on their consistency strength range from equiconsistent with supercompacts to a consistency strength far below that of supercompactness.

One of the most frustrating aspects of working with strongly compact cardinals is their intractability in forcing constructions due to a lack of the normality and closure properties associated with supercompactness. Very few forcing proofs for this reason have been done using strongly compact cardinals. A notable exception is Gitik's construction of [G1] in which, starting from a class of strongly compact cardinals, a model in which all uncountable cardinals are singular is constructed.

In [H], a notion of modified Prikry forcing in which normal measures are not used was developed. We adapt this forcing construction to show that certain theorems originally proven using supercompactness can be reproven using strong compactness. Specifically, we establish the following results.

THEOREM 1. Con (ZFC + *There exist cardinals $\varkappa < \lambda$ so that $\varkappa$ is λ strongly compact and λ is measurable*) $\Rightarrow$ Con(ZF + *$\varkappa$ is a strong limit cardinal of cofinality ω carrying a Rowbottom filter* + *$\varkappa^+$ is a measurable cardinal which carries a normal measure*).

THEOREM 2. Con (ZFC + *There exist cardinals $\varkappa < \lambda$ so that $\varkappa$ is λ strongly compact and λ is measurable*) $\Rightarrow$ Con(ZF + *$\aleph_\omega$ is a strong limit cardinal carrying a Rowbottom filter* + *$\aleph_{\omega+1}$ is a measurable cardinal which carries a normal measure*).

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