By the continuity of $f'$, we have $|f'|_{\alpha\beta} - \delta_{\alpha\beta}|_{\alpha\beta} < \eta$. Hence $f'|_{\alpha\beta}: A_{\alpha\beta} \to A_{\alpha\beta}$ has a fixed point in $V_{\alpha\beta}(C_2)$. Thus, $f'$ has a fixed point in $V_{\alpha\beta}(C_2) \subset U_{\alpha\beta}(C)^*$. From Cases (ii)(i) and (ii) it follows that every mapping $f': Z \to Z$ with $|f'|_{\alpha\beta} < \min\{\delta_{\alpha\beta}, \delta_{\beta\alpha}\}$ has a fixed point in $U_{\alpha\beta}(C)^*$. This completes the proof.

Remark. While $Z$ has $f^*_{\alpha\beta}$-p.p., the cone over $Z$ does not have $f_{\alpha\beta}$-p.p. (see [3]).

Addendum. We can construct an example of a locally connected continuum which has f.p.p. but does not have $f^*_{\alpha\beta}$-p.p. Define

$$B_\alpha = \{x : |x - 1/2n^2| + y^2 < 1/(2n^2)\};$$

$$Y = \{(0,0) \times I \cup \bigcup_{n=0}^{\infty} (B_n \times I) \cup \bigcup_{n=0}^{\infty} (B_n \times \{0\})\}.$$

By a similar argument to that of Theorem 1, we can prove that $Y$ has f.p.p. but does not have $f^*_{\alpha\beta}$-p.p. Another similar example corresponding to Theorem 2 can also be easily constructed.

References


SCHOOL OF SCIENCE
KWANSEI GAKUIN UNIVERSITY
Nishinomiya 662, Japan

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Torsion free types

by

John Dauns (New Orleans, La.)

Abstract. It is shown how the known classification of nonsingular, injective modules $M$ into Types I, II, and III as well as corresponding direct sum decompositions $M = M_1 \oplus M_2 \oplus M_3$ are merely special cases of a more general phenomenon. There is a functor $\mathcal{E}$ from rings $R$ to complete Boolean lattices (equivalently Boolean rings) $\mathcal{E}(R)$, where each point of $\mathcal{E}(R)$ is a class of similar nonsingular modules. Types I, II, III, continuous, discrete, and certain other classes of modules correspond to unique elements of $\mathcal{E}(R)$. Appropriate finite sets of disjoint classes of modules induce direct sum decompositions of $\mathcal{E}$ as a direct sum of subfunctors. The latter give rise to corresponding direct sum decompositions of nonsingular injective modules $M$, such as $M = M_1 \oplus M_2 \oplus M_3$.

Introduction. This article will show how the classification of certain torsion free modules into Types I, II, and III ([MN], [K], [B], and [GB]) is a special case of a classification scheme developed in [D]. If $A$ is a unital right $R$-module and $ZA$ its singular submodule, then the second singular submodule $ZA \subseteq ZA \subseteq A$ is defined by $Z(A/ZA) = (Z_2A)/ZA$. A module is torsion free if $ZA = Z_2A = A$ and torsion if it equals its torsion submodule $Z_2A = A$. This is a contraction of [D] where the following was shown. There exists a contravariant functor $\mathcal{E}$ applicable to any associative ring $R$ with identity. The result is a complete Boolean lattice $\mathcal{E}(R)$. The functor $\mathcal{E}$ classifies or partitions the class of all torsion free right $R$-modules $\{A, B, \ldots\}$ into equivalence classes $\mathcal{E}(R) = \{[A], [B], \ldots\}$ where $A \in [A], [A]$ consists of a class of modules that are similar, or are like $A$. An appropriate ring homomorphism $R \to S$ induces a lattice (equivalently Boolean ring) homomorphism $\mathcal{E}(S) \to \mathcal{E}(R)$.

Goodearl and Boyle ([GB]) extended the classification of operator algebras, $W^*$-algebras, and Baer $*$-rings into Types I = $I_1 \uplus I_2$, II = $I_2 \uplus I_3$, III, abelian, directly finite, and purely infinite to all torsion free injective modules. Here this latter theory is extended to all torsion free modules over any ring by defining $M$ to belong to any of the latter classes if and only if its injective hull $EM$ does (e.g. $M \in III \iff EM \in III$). In order to obtain necessary and sufficient conditions for $M$ (as opposed to $EM$) to be of Types I, II, III, abelian etc. (4.2, 4.4, and 4.5), the usual definitions are reformulated without reference to idempotents (3.3, 3.4).

It is shown that there exist unique largest elements [1], [II], [III] $\in \mathcal{E}(R)$ which determine Type I, II, and III modules (Corollary 3.16). More specifically, [III] consists
1. Complements and full invariance. Some facts, needed in later proofs, have been isolated out of the specialized contexts of later theorems, and proved here in a general module setting, because these facts may be of independent interest. Moreover, many of the subsequent proofs will just amount to quoting the lemmas of this section in the right order.

1.1. Notation. Modules $M$ are right unital over a ring $R$. Denote submodules by $<0$, or $<\leq$, and large or essential submodules by $\leq$. The symbol $A < B$ means that $A \subset B$ but that $A$ is not large in $B$. For $K < M$ and $m \in M$, $m^* = \{r \in R | mr \neq 0\} < R$, while $m^{-1}K = \{r \in R | mr \in K\} < R$. For a subset $Y \subset M$, $Y^* = \{r \in R | y \in Y, yr = 0\} = (r | yr = 0)$. Ideals (two-sided) in $R$ or other rings are denoted by "$+$"; thus $M^+ \leq R$.

The singular submodule $ZM$ of $Z(M) = ZM = \{m \in M | m^* = R\}$. Define the second singular submodule in $ZM$ over $Z_2 \subset M$ by $Z[M/ZM] = (Z_2/M)ZM$. Denote $Z_2/M$ to be the torsion submodule of $M$. Consequently $M$ is torsion if $M = Z_2M$, which holds if $ZM \leq M$, and $M$ is torsion free, abbreviated $tf$, if $Z_2M = 0$, i.e., if $ZM = 0$. Right R-injective hulls of right $R$-modules are denoted by "$\ast$" and "$\ast'$" as $M = E=Mw$, where $E$ is used for quotient modules or when $M$ is given by a complex formula.

For those $K < M$ with $ZM \leq K$ and only for these $\leq$, define the complement closure $K$ of $K$ in $K < K = K(ZM/K)$. Various facts and properties of $K$ will be used without further mention, see [D]; pp. 51–55, 1.1–1.11. The symbols $\prec, \subset, \leq, <, \ast, \ast'$, $Z, Z_2, \ast, \ast'$ and $\ast$ refer to right $R$-modules over $R$, and never to other rings $S$ with $1 \in S$ which also will be used.

For a module map $f: A \to B$, and for $C < A$, denote the restriction of $f$ to $C$ by $f|C: C \to B$. The trace of a module $A$ in a module $B$ is defined as $tr_A = \{a \in A | h \in Hom_B(A, B)\}$. Dually the reject of $A$ inside $B$ is $rj_A = \{a \in A | h \in Hom_B(B, A)\}$.

For any set $X$, $\partial(X) = \langle \partial(X), \cup, \cap, \setminus, \times, \emptyset \rangle$ is the usual Boolean lattice of the power set of $X$; also $\partial(X)$ denotes the associated Boolean ring. Denote the cardinality of any set $X$ by $|X|$.

The following simple construction will be used repeatedly.

1.2. Projection argument. Let $\{A_\gamma | \gamma \in \Gamma\}$ be any family of modules. View $E(\bigoplus A_\gamma) = \bigoplus A_\gamma$. For any $0 \neq \xi = (\xi_\gamma) \in E(\bigoplus A_\gamma)$, choose $r_\gamma \in R$ such that $0 \neq \xi_\gamma = a_\gamma + \cdots + a_\delta A_{\delta(\gamma)} \cdots \oplus A_{\delta(0)}$ with all $0 \neq a_\delta A_{\delta(0)}$, and with the length $n \neq 0$ of $\xi_\gamma$ minimal. Then $\langle \xi_\gamma \rangle = a_\gamma + \cdots + a_\delta R$. Thus $r_\gamma A_\delta \supseteq a_\delta R + A_{\delta(\gamma)}$. In particular, every nonzero submodule of $E(\bigoplus A_\gamma)$ contains an isomorphic copy of a nonzero submodule of some $A_\gamma$.

1.3. Trace projection argument. Let $\{A_\gamma | \gamma \in \Gamma\}$ be any family of torsion free modules and $B$ any torsion free module. Suppose that $0 \neq \xi \in E[\bigoplus A_\gamma]$. Then there exists a $\xi \in \Gamma$ and elements $0 \neq a_\delta A_{\delta(0)}$, $0 \neq b \in B \cap V$ such that $a_\delta R \supseteq bR$ and $a_\delta^* = b_\delta$. 

entirely of Type III modules, and determines the class III as $III = \{M | [M] \leq [III]\}$. Or equivalently, the class III is the union of the interval $[0, [III]] \leq \mathbb{Z}(R)$, $III = \bigcup_{x \leq [III]} \{x \leq [III]\}$. Verbatim identical statements hold for Types I, II and [III]. It is shown that this latter phenomenon is not limited to Types I, II, or III, but holds for any so-called saturated class of modules (see 3.5). Types I, II, III are examples of saturated classes. Four other examples are discrete, continuous, molecular, and bottomless torsion free modules (see 3.1, 3.2). It is shown in Section 3 (Theorem 3.9) that there is a bijective correspondence between saturated classes of torsion free $R$-modules $\Delta$ and elements of $\mathbb{Z}(R)$ given by $\Delta = [C | [C] \leq [\Delta]]$, where $[\Delta]$ is the supremum of $[CC]$ where $\Delta$. Conversely, any $[\Delta] \in \mathbb{Z}(R)$ determines a saturated class $\Delta$ by $\Delta = [C | [C] \leq [\Delta]]$. 

Suppose that for each associative ring $R$ with identity, there is a saturated class $\Delta(R)$ of torsion free $R$-modules. Then the assignment $\Delta: R \to \Delta(R)$ is called a universal saturated class provided one additional property holds (3.12 and 3.13). The above-mentioned seven saturated classes are universal saturated classes. Theorem 3.14 shows how a universal saturated class $\Delta$ induces a decomposition of the functor $\mathbb{Z}$ as a direct sum of two subfunctors $\mathbb{Z} = \mathbb{Z}_c \oplus \mathbb{Z}_d$, where $\mathbb{Z}_d$ is a universal saturated class complementary to $\Delta$. Such decompositions extend to certain finite number of disjoint universal classes. Thus Types I, II, and III induce a decomposition $\mathbb{Z} = \mathbb{Z}_c \oplus \mathbb{Z}_d \oplus \mathbb{Z}_m$ (Corollary 3.16); or the continuous molecular, continuous bottomless, and discrete classes induce another decomposition $\mathbb{Z} = \mathbb{Z}_c \oplus \mathbb{Z}_d \oplus \mathbb{Z}_m$ (Corollary 3.15). Such direct sum decompositions of $\mathbb{Z}$ induce unique direct sum decompositions of injective modules as finite direct sums (Main Corollary 3.17). Furthermore, the Module Decomposition Theorem 2.3 generalizes the latter even further, among other things, to full direct product representations of injective modules.

The lattice $\mathbb{Z}(R)$ is isomorphic to the lattice of all fully invariant complement right ideals $J$ of $R$ with $ZR \leq J$. For this reason alone fully invariant complement submodules are important. Section 1 investigates such submodules, and particularly how they are mapped under module homomorphisms. This section involves no special hypotheses and should by itself be of independent interest. Section 2 reformulates some of $[\Delta]$ in a form in which it will be used later, as well as derives some corollaries from the results of $[\Delta]$. Thus Corollary 2.7 gives the new result that if $\Gamma \leq \mathbb{Z}(R)$ is any pairwise disjoint subset with supremum $\Gamma = 1$, then it induces a Boolean lattice (equivalently ring) isomorphism $\mathbb{Z}(R) \cong \prod \{\mathbb{Z}_c | \gamma \in \Gamma\}$ where each $\mathbb{Z}_c$ is a convex sublattice of $\mathbb{Z}_c \leq \mathbb{Z}(R)$ with $\mathbb{Z}_c \cong \mathbb{Z}(R/R_c)$ for a certain unique ideal $R_c$. Section 5 shows how some of the criteria of Section 4 can be used to construct various Type I, III, and bottomless modules. It also begins the problem of relating the Type I, II, and III classification to the discrete, continuous, molecular, and bottomless ones (5.3).

The author hopes to show in a subsequent paper that functors similar to $\mathbb{Z}$ can be defined for bigger ring categories, i.e. $\mathbb{Z}$ would be applicable to more ring homomorphisms; and that there also is an analogous theory for the torsion modules. In summary, among other things, this article puts the previously well developed theory of torsion free injective modules over a single fixed ring $R$ in a broader functorial context where many open questions remain.
Proof. By [D; p. 55, Theorem 2.1] there are $0 \neq \xi \in E(\bigoplus_{i=1}^{n} A_i)$, $0 \neq v \in B \cap V$ with $\xi = a^v$. Choose $r_0 \in K$ such that $0 \neq \xi_0 = a_1 + \ldots + a_n$, $0 \neq q_0 \in A_{\gamma_0}$ with $A_{\gamma_0} \oplus \ldots \oplus A_{\gamma_n}$ a direct sum, and where $n$ is minimal. Then $(x_{\gamma_0})^2 = a_1^2$, where $x_{\gamma_0} = y((1))$. To avoid the case that $n = 1$, let $a_1 = A_{\gamma_1}$ and $b = r_0 \in B \cap V$. Then $b^2 = (x_{\gamma_0})^2 = (x_{\gamma_1})^2 = a_1^2$.

1.4. Lemma. Suppose that $ZA \subseteq K \subseteq A$ and $B$ are modules and that $f : A \rightarrow B$ is a right $R$-module map with $ZB \subseteq fK$. Then

(i) $fK < fK \subseteq (fK)^{\perp}$.

(ii) In particular, $\forall L \subseteq A$, $K \subseteq L$ also $fK < fL$.

Proof. If $x \in fK \subseteq A$, then $x \in fK \subseteq K$ and $x^{-1} \in (fK)^{\perp}$. Since $ZB \subseteq fK$, we have $x \in (fK)^{\perp}$. A consequence $\Rightarrow K \subseteq (fK)^{\perp}$. But $(fK) \subseteq (fK)^{\perp}$. Hence $fK \subseteq fK \subseteq (fK)^{\perp}$.

From [D; p. 56, 24], the result below will be needed. The fact that an arbitrary intersection of injective submodules remains injective may lead the reader to conjecture that below, the "$E$" may be omitted from in front of the next summand. A counterexample (5.4) shows that this is not so.

1.5. For an any torsion free modules $A$ and $B$, $E = E(\times_{x \in A} A) \cap \rho_{x \in E} EB$ is a direct sum of two fully invariant submodules.

1.6. Remarks. Let $K < M$ be any modules. Form $\hat{K} = \hat{K} \uplus \hat{C}$. Then

$\hat{K} < \hat{M}$ is fully invariant $\Rightarrow \text{Hom}_M(\hat{K}, \hat{C}) = 0$.

In particular, if $M$ is torsion free, then

$\hat{K} < \hat{M}$ is f.i. $\Rightarrow \hat{C} < \hat{M}$ is f.i.

Note that in case $M$ is torsion free, Lemma 1.4 immediately proves the next lemma, but not in general, for there is no guarantee that $ZM \subseteq fK$ below.

1.7. Lemma. For any modules $ZM \subseteq K < M$,

$K < M$ is f.i. $\Rightarrow \hat{K} < \hat{M}$ is also f.i.

Proof. Let $f \in \text{Hom}_M(M, M)$ and $\xi \in \hat{K}$ be arbitrary. Then since $\hat{K} = \{ x \in M | x^{-1} \subseteq K \subseteq K \}$ (by [D; p. 53, 1.3(1)]), $\xi^{-1} \subseteq K \subseteq K$. But then $(f\xi) \hat{K} \subseteq fK \subseteq K$. Hence $\xi^{-1} \subseteq (f\xi)^{-1} \subseteq K \subseteq K$ and also $f\xi \in \hat{K}$. Thus $f\hat{K} \subseteq \hat{K}$.

1.8. Proposition. Let $K = \hat{K} < \hat{M}$ be a complement submodule with $ZM \subseteq K$. Then

(i) $K < \hat{M}$ is f.i. $\Rightarrow K < M$ is also fully invariant.

(ii) $K < \hat{M}$ is f.i. $\Rightarrow \hat{K} < \hat{M}$ is fully invariant.

Proof. (i) Since $K \subseteq \hat{K} \cap M$, $M = \hat{K} \cap M$. For any $\phi \in \text{Hom}_M(M, M)$, extend $\phi$ to $\hat{K} \rightarrow \hat{M}$. By hypotheses, $\phi \hat{K} \subseteq \hat{K}$. But then $\phi \hat{K} \subseteq \phi \hat{K} \cap M \subseteq \hat{K} \cap M = K$.

(ii) Let $\psi \in \text{Hom}_M(M, M)$. Write $M = \hat{K} \oplus U$ for some $U \subseteq M$. For $\xi \in \hat{K}$, suppose that $\psi \xi \in \hat{K}$. Then $\psi \xi = x + y$, $x \in \hat{K}$, $0 \neq y \in U$. Set $L = x^{-1} \cap \hat{K} \cap U$. For any $r_0 \in L$, $\psi r_0 \in \hat{K}$, and hence also $\psi r_0 \in \hat{K}$. Thus $L \subseteq \hat{K} < K \subseteq \hat{K}$ shows (by [D; p. 53, 1.3(1)]) that $y \in \hat{K}$, a contradiction.

1.9. Corollary. If $ZM \subseteq K < M$, then

$K < \hat{M}$ is f.i. $\Rightarrow \hat{K} < \hat{M}$ is f.i.; $K < M$ is f.i. $\Rightarrow \hat{K} < \hat{M}$ is f.i.

Note that in the next proposition, $K < M$ need not be a complement necessarily.

1.10. Proposition. Let $ZM \subseteq K < M$ and assume that $\hat{K} < \hat{M}$ is fully invariant. Let $\hat{C}$ be any

(a) complement submodule $\hat{C} < M$ such that

(b) $K \oplus \hat{C} < M$.

Then

(i) $\hat{C} < \hat{M}$ is fully invariant.

(ii) Unique: If $C_1 < M$ satisfies (a) and (b), then $C_1 = \hat{C}$.

Proof. Write $M = \hat{K} \oplus \hat{C}$. By 1.6 it suffices to show that $\text{Hom}_M(\hat{K}, \hat{C}) = 0$. If not, let $0 \neq \phi : \hat{C} \rightarrow \hat{K}$. Write $\hat{C} = U \oplus \ker \phi$. Then

$\hat{C} > U = \phi U < \hat{K}$

contradicts the fact that $\hat{K} < \hat{M}$ is fully invariant.

1.11. Lemma. If $ZM \subseteq K < K < M$, then below (i) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (iv), (iii) $\Rightarrow$ (ii), and (iv) $\Rightarrow$ (ii):

(i) $K < M$ is fully invariant.

(ii) $K/ZM < M/ZM$ is fully invariant.

(iii) $K < \hat{M}$ is fully invariant.

(iv) $K = (K/ZM) < M = (M/ZM)$ fully invariant.

Proof. Conclusions (iii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (ii) were shown in 1.8. It will be shown that (ii) $\Rightarrow$ (iv). The proof of (ii) $\Rightarrow$ (i) is similar and is omitted.

Take any $D \subseteq M$ such that $ZM \cap D < M$, and hence $ZM \cap (K \cap D) \subseteq D$. Then since $ZM < M$ is a complement,

$ZM/\cap D \subseteq M/ZM < ZM/\cap D \subseteq M/ZM < \cap D$.

Since $EZM = ZEM < EM$ is fully invariant it follows that:

$\hat{K} < \hat{M}$ is fully invariant $\Rightarrow (K \cap D) < \cap D$ is fully invariant

$\Rightarrow (K/ZM) < (M/ZM)$ is fully invariant.

1.12. Lemma. For any modules $L < K < M$ assume that $ZM \subseteq L$ and $L \subseteq K$. Then

(i) $L < \hat{M}$ is a complement $\Rightarrow \hat{K} < \hat{L}$ is a complement.

Proof. (i) If $m + L \subseteq Z(\hat{M}/L)$ then $(m + L)^2 = m^{-1} L \subseteq K$, and $m \subseteq L \subseteq K$ by [D; p. 53, 1.3(1)].
(ii) \( \Rightarrow \) If not, \( K \preceq B \preceq M \). By hypothesis, \( K/L \nless B/L \). Hence there exists a cyclic submodule of \( B/L \) such that \( (b_0 + L)R \cap K/L = 0 \), \( (b_0 + L)R \cap K = L \). The latter implies \( b_0 \preceq b_0 + L \). But \( K \preceq B \), and hence \( b_0 \preceq L \). Thus \( b_0 + L \nless B \) and \( b_0 \in L \preceq K \). Hence \( K \nless M \) is a contradiction. Hence \( K \preceq M \) is a cyclomorphism.

(ii) \( \Rightarrow \) This proof will only require that \( Z(M) \nless K \). By way of contradiction, suppose that \( K(L) \nless B \) for some \( K < B \). Then \( K \nless B \), and hence \( K \nless B \). Hence \( Z(M) \nless K \). By [D; p. 53, 1.30] \( Z(M) \nless K \) we conclude that \( b_0 \in K = K \). A contradiction. Thus \( K(L) \nless B \) is a cyclomorphism.

1.1.3. Corollary. If \( Z(M) \nless K \), then \( K \preceq M \) is a cyclomorphism \( K/Z(M) \nless M \) is a cyclomorphism.

1.1.4. Corollary. Suppose that \( Z(M) \nless K \) and \( E(K/Z(M)) \nless E(M/Z(M)) \) is a fully invariant complement. Let \( L, C \nless M \) be any submodules satisfying the following:

\( \Phi \nless C \nless M \) and

(a) \( L \nless C \) is a complement with \( L \cap C \nless Z(M) \); and

(b) \( K + L \nless M \).

Then

(i) \( K/Z(M) \nless L/Z(M) \nless M \) and also \( L/Z(M) \) is a fully invariant.

(ii) \( K \cap C \nless M \) satisfies (a) and (b), then \( L \cap C \nless L \); Consequently,

(iii) \( L = (Z(M) + C) \nless M \).

1.1.5. Lemma. Suppose that \( \{A_i \} \in I \) is a family of torsion free modules such that

\[ \text{Hom}_K(A_i, A) = 0 \text{ for all } i \neq j. \]

Then

\[ E(\Phi/A) = \bigcap_{i \in I} A_i. \]

Proof. Reverses to show that for any \( 0 \neq \xi \in (\Phi) \nless 1 \), and any \( \xi \nless 0 \), there exists an \( R \nless 0 \) with \( \bar{0} \neq \hat{\xi} \cdot \bar{R} \nless 1 \).

Suppose \( \xi \nless 0 \) in \( K \). Take any \( \xi \preceq R \). Take any \( 0 \nless R \) with \( \xi \preceq \xi \cdot \bar{R} \nless 1 \).

Hence for any \( 0 \nless R \) in \( \Phi \) we have \( \bar{0} \neq \hat{\xi} \cdot \bar{R} \nless 1 \).

2. The functor \( S \). This section reformulates, and generalizes some of the facts about the lattice \( S(R) \) from \([D]\).

2.1. Types. On the class of all torsion free right \( R \)-modules, \( A, B, C, \ldots \) define a quasi-order \( \preceq \) by \( A \preceq B \) if there is an embedding \( A \in E(B) \) for some (arbitrary) index set \( J \). Then \( A \preceq B \) if and only if \( A \preceq B \). The equivalence classes \( [A] \in [C] \) of all equivalence classes \( [A] \) of \( A \) have the property that \( [A] \preceq [B] \) if and only if \( A \preceq B \).

The lattice \( S(R) = ([A], \{[A], [B], [C], \ldots \}) \) for all equivalence classes \( [A] \) is called types. The class \( S(R) = ([A], [E(R)], \{[C] \}) \) of all equivalence classes \( [A] \) of \( A \) is a sublattice of \( S(R) \).

Next some facts from \([D]\) are summarized and notation is established which later will be used.

2.2. Type lattice. Let \( E(R) = ([A], [B], [C], \ldots) \) be the partially ordered class of isomorphism classes of torsion free right \( R \)-modules, \( A, B, \ldots \). Set \( \bar{R} = R/Z(R) \). Let \( ([A], [B]) \ preceq \bar{R} \) be any indexed subset. Then

(1) \( E(R) \) is a set. \( \{E(R) \} \subseteq [E(R)] \).

(2) \( \sup \{([A], [B]) \preceq \bar{R} \} = \bigvee \{[A], [B] \in \{E(R) \} \} \), and \( E(R) \) is a complete Boolean lattice (distributive, with 0 and 1).

(3) \( \{E(R) \} \) is a complete Boolean lattice (distributive, with 0 and 1).

2.3. Module decomposition theorem. Let \( M \) be any torsion free module and \( a \in [M] \in E(R) \). Define \( M = \bigcup \{M \} = \bigcup \{M \} \in E(R) \). Then

(i) \( [M] = a \); hence \( M_a \) is the unique submodule of \( M \) of type \( a \);

(ii) \( M_a \nless M \); both \( M_a \nless M \) and \( E(M_a) \) are fully invariant.

2.4. Definition. Let \( A \) be the category whose objects are all rings \( R \) and \( S \) with identity, and morphisms \( \phi : R \to S \) are ring homomorphisms satisfying:

(a) \( \phi 1 = 1 \);

(b) \( \phi R = S \);

(c) \( I \) is a monoid, \( I \); and \( \phi R \) is a right complement, i.e. with closed kernels.

Dually, the category \( B \) is the category of all complete Boolean lattices \( L_1 \) and \( L_2 \) whose morphisms \( \phi^* : L_1 \to L_2 \) satisfy:

\( (\phi^*) 0 = 0 \);\( (\phi^*) \phi \) is monic; and \( (\phi^*) \phi L_1 \subseteq L_2 \) is convex, i.e. with closed images.

2.5. Functoriality. For any morphism \( \phi \subseteq A \subseteq B \), for simplicity, take \( R \to S = R/1 \) to be the natural projection with \( I = \ker \phi - R \).

For every \( S \)-module \( N \), \( N \subseteq N \subseteq E(R) \) denotes the induced \( R \)-module, i.e. \( N \subseteq N \).

If \( N \subseteq S \), then \( [N] \subseteq E(R) \), and define \( \phi^* : E(S) \to E(S) \) by \( \phi^*([N]) = [N] \subseteq E(R) \).

For a proof that \( \phi^* \) is a well defined lattice homomorphism, see \( [D]; p. 65-68, 68 \).

Define \( \phi^* : E(S) \to E(R) \).

(i) \( S : A \subseteq B \) is a contravariant functor; \( E(S) \subseteq E(R) \) is a zero preserving monic lattice homomorphism which preserves arbitrary suprema and infima.

(ii) \( \phi^* : E(S) \subseteq E(R) \) is a convex (and complete) sublattice, and hence \( E(S) \subseteq E(R) \) is a lattice direct sum \( E(S) \subseteq \phi^*(E(S)) \).

For a proof that \( \phi^* \) is a well defined lattice homomorphism, see \( [D]; p. 65-68, 68 \).

Among other things, the next theorem is a generalization of \( [D]; p. 69, 12 \).

In the latter the results were proved for \( a \) equal to one of four special elements of \( E(R) \), while here below for every \( a \subseteq E(R) \). The proof proceeds as follows. The previous decomposition (2.3) is applied to \( M = R/Z(R) \), and then with the aid of (1.11 and 1.13) the results lifted back up to \( R \). Below (3) follows by use of (1.14). Since the ideal lattice \( \mathcal{J}(R) \) defined by \( \mathcal{J}(R) \) in the simplest and in many cases the only way to explicitly compute, describe, or determine \( \mathcal{J}(R) \) for examples of rings, or for concrete classes of rings, several descriptions or characterizations are given for the elements \( R \subseteq \mathcal{J}(R) \subseteq E(R) \).
2.6. Ideal Lattice Theorem. For \( a \in \mathcal{E}(R) \) define \( R_a \subseteq Z_2 R \subseteq R \) by \((R_a/Z_2 R) = R_a/Z_2 R\). Let \( c(a) \in \mathcal{E}(R) \) be the complement of \( a \), i.e. \( a \wedge c(a) = 1, a \wedge c(0) = 0 \). Define \( \mathcal{I}(R) = \{ R_a \mid a \in \mathcal{E}(R) \} \). Then

1. \( \mathcal{I}(R) \) consists exactly of all \( I \) with \( Z_2 R \subseteq I \subseteq R \) such that \( J = R \) is a right complement and \( E \subseteq R \) is fully invariant. Hence \( J = R \). For \( J \in \mathcal{I}(R) \) define \( J^* \) to be maximal with respect to the properties

\[
Z_2 R \subseteq J^* \subseteq R, \quad J \wedge J^* = Z_2 R.
\]

Then \( J^* \in \mathcal{I}(R) \) and \( \mathcal{I}(R) \) is a Boolean lattice under

\[
J_1 \wedge J_2 = J_1 \wedge J_2, \quad J_1 \vee J_2 = (J_1 \vee J_2)^-;
\]

\[
J \vee J^* = R = 1, \quad J \wedge J^* = Z_2 R = 0, \quad J, J_1, J_2 \in \mathcal{I}(R).
\]

Then there is an isomorphism of Boolean lattices

\[
(2) \quad \mathcal{E}(R) \rightarrow \mathcal{I}(R), \quad a \rightarrow R_a,
\]

where

(i) \( R_a \wedge R_b = R_a \cap R_b, \quad R_a \vee R_b = (R_a + R_b)^- = R_a \cap R_b \wedge R_b \quad \text{and} \quad R_a = Z_2 R, \quad R_a \cap R_{c(a)} = Z_2 R \).

(ii) \( (R_a/Z_2 R) = [R_a/Z_2 R], \quad (R/b) = [R/b]_a \quad \text{and} \quad R_a \cap R_{c(a)} = Z_2 R \).

(iii) For any \( a \in \mathcal{E}(R) \) let \( c(a) \in \mathcal{E}(R) \) be right ideal maximal with respect to \( C_a \cap Z_2 R = 0, \quad x = [C_a] \).

Then \( R_a = (Z_2 R \cap C_a^-), \quad Z_2 R \cap C_a \cap C_{c(a)} \subseteq R \).

(4) \( \forall \in \mathcal{E}(R), R_{c(a)} \subseteq R \) is the unique smallest right ideal of \( R \) with respect to the following properties:

(i) \( Z_2 R \subseteq R_{c(a)} \subseteq R \); (ii) \( [R/R_{c(a)}] = a \).

(5) \( R_a \cap R_{c(a)} = Z_2 R, \quad R_a + R_{c(a)} \subseteq R \).

The next corollary is obtained by combining the ideal lattice \( \mathcal{I}(R) \) of the last theorem with the functoriality of \( \mathcal{E} \). Later, 2.7(ii) below will be used to show that \( \mathcal{E} = \mathcal{E}(S) \cap \mathcal{E}(S) \) is a direct sum of subfunctors corresponding to Types I, II, and III.

2.7. Corollary. For any \( a \in \mathcal{E}(R) \), let \( (a), R_a, R_{c(a)} \subseteq R \) be as in 2.6, and let \( \mathcal{E}(R) \) be the Boolean lattice \( \mathcal{E}(R) = \{ b \in \mathcal{E}(R) \mid b \leq a \} \) with largest element \( a \) and relative complementation \( b \rightarrow c(b) \wedge a \). Let \( \pi_{c(a)} : R \rightarrow R/c(a) \) be the natural projection and form the monic lattice homomorphism \( \pi_{c(a)}^* : \mathcal{E}(R/c(a)) \rightarrow \mathcal{E}(R) \). Then

(i) \( \mathcal{E}(R/c(a)) \subseteq \mathcal{E}(R) \) is image \( \pi_{c(a)}^* \); i.e., the restriction of \( \pi_{c(a)}^* \) to its image is a component preserving isomorphism of Boolean lattices.

Now suppose that \( I \subseteq \mathcal{E}(R) \) is any pairwise disjoint subset whose supremum is \( \sqrt{I} \in \mathcal{E}(R) \). Then there is a canonical isomorphism of Boolean lattices

\[
(6) \quad \mathcal{E}(R) \cong \bigoplus_{a \in I} \mathcal{E}(R/a), \quad \text{if} \quad |I| < \infty, \quad \text{then} \quad \mathcal{E}(R) = \bigoplus_{a \in I} \mathcal{E}(R/a).
\]

(iii) \( \forall \in \mathcal{T} \) injective \( M = M_R \)

\[
M = E(tr_m R_a) \oplus (tr_m R_{c(a)});
\]

\[
[M] \wedge x = E(tr_m R_a), \quad [M] \wedge c(a) = [tr_m R_{c(a)}];
\]

\[
E(tr_m R_{c(a)}) = [R_{c(a)}]^E.
\]

Proof. (i) and (ii). The proof of these conclusions can be modeled after [D; p. 69, 4.12], and is omitted.

(iii) From \( (R_a/Z_2 R) \oplus (R_{c(a)}/Z_2 R) \not\subseteq R/Z_2 R \) it follows that

\[
M = E(tr_m R_a) = E(tr_m R_a/Z_2 R) = E(tr_m R_a/Z_2 R) \oplus (tr_m R_{c(a)}/Z_2 R) = E(tr_m R_a) \oplus (tr_m R_{c(a)}).
\]

But in 1.5, the second complementary summand is unique because of full invariance, and hence we also have

\[
M = E(tr_m R_a) \oplus (tr_m R_{c(a)}).
\]

and so \( E(tr_m R_a) = [tr_m R_{c(a)}] \). The rest is clear.

3. Classes of modules. This section relates universal saturated classes of modules to certain points of \( \mathcal{E}(R) \), and shows how such classes induce decompositions of \( S \) as a direct sum of subfunctors. Type I, II, and III modules, and continuous, discrete, molecular, and bottommost modules are examples of universal saturated classes. This makes it possible to apply all the results of Section 2 and this section to these seven classes of modules. The definitions of Types I, II, III are reformulated without reference to idempotents so that later they can be applied to not necessarily injective torsion free modules.

3.1. Definition. Let \( C = C_R \) and \( D = D_R \); \( C \) is continuous if \( C \) contains no uniform submodules. \( D \) is discrete if \( D \) contains an essential direct sum of uniform submodules. Let \( A = A_R \) and \( B = B_R \) be t.f. continuous. \( A \) is atomic if \( [A] \subseteq \mathcal{E}(R) \) is an atom. More generally, \( A \) is molecular if every nonzero submodule of \( A \) contains an atom. At the other end of the continuous module spectrum, \( B \) is bottomless if \( B \) contains no atomic submodules.

3.2. Definition. Let \( A, B, C, D \) generically represent t.f. modules. Define \( \mathcal{E}(R) = ([D], D \text{ is discrete}), \mathcal{E}(R) = ([C], C \text{ is continuous}), \mathcal{E}(R) = ([A], A \text{ is continuous molecular}), \text{and} \mathcal{E}(R) = ([B], B \text{ is bottomless}).

3.3. Definition. Let \( M \) be an injective torsion free module, and throughout this definition let \( N \) represent any arbitrary nonzero direct summand of \( M \), i.e. \( M = N \oplus N \) for some \( N \neq M \), \( N \neq 0 \). Then \( M \) is directly finite if for any such \( M, N \) is not isomorphic to \( N \oplus N \) ([GB; p. 16, 3.1(c)]). The module \( M \) is abelian if any isomorphic direct summands of \( M \) are equal ([GB; p. 12, 2.1]). Thus abelian implies directly finite.

The module \( M \) is Type I if every \( N \) contains a nonzero abelian submodule ([GB; p. 30, 5.1]). Secondly, \( M \) is of Type II if for any \( N \), \( N \) is not abelian, and if every \( N \) contains a nonzero directly finite direct summand ([GB; p. 34, 5.5]). Thirdly, \( M \) is Type III if for any \( N \), we have \( N = N \oplus N \) ([GB; p. 38, 5.9]).

Our module \( M \) is purely infinite if \( M \) has no directly finite invariant direct summands ([GB; p. 40, p. 41, 6.2]). Clearly Type III implies purely infinite. Next, \( M \) is Type I (or Type II) if \( M \) is Type I (or Type II respectively) and \( M \) is purely infinite ([GB; p. 42]). A module \( M \) is Type I (or Type II) if \( M \) is Type I (or Type II respectively) and directly finite.
3.4. DEFINITION. A torsion free module $M$ has any one of the properties in the last definition, if $M$ is a torsion free module $M$ has any one of the properties in the last definition, if $M$ does (directly finite, abelian, I, II, III, purely infinite, $I_{0}$, $I_{1}$, and $I_{2}$). Consequently, by 1.7, if $M$ is purely infinite then $M$ contains no directly finite fully invariant submodules.

Our next objective is to show that there is a simple and natural characterization of the elements of $\Sigma(R)$ in terms of saturated classes, and to determine which of the above ten classes of modules in the last definition in some sense correspond to elements of $\Sigma(R)$.

3.5. DEFINITION. For any nonempty class of modules $\Delta$, define a complementary class $c(\Delta)$ by $c(\Delta) = \{W \mid W = W_\Lambda, \forall \Lambda \neq V \in \Delta, \forall \lambda \}$. Furthermore, if $M$ is a torsion free module (and in this case define an intrinsic submodule $M_{\Delta} \leq M$ by $M_{\Delta} = \sum \{U \mid U \leq M, U \notin \Delta \}$, and similarly $M_{\Delta} \leq M$. A class $\Delta$ is said to be closed under essential extensions if $V \in \Delta, \forall \lambda \notin \Delta \Rightarrow \forall \lambda \in \Delta$; and closed under essential submodules if $W \in \Delta, \forall \lambda \in \Delta$.

A nonempty class of torsion free modules $\Delta$ is said to be saturated if $\Delta$ is closed under direct sums, injective hulls, submodules, and isomorphic copies (i.e., $V \in \Delta, V \cong W \Rightarrow W \in \Delta$).

3.6. CONSEQUENCES. For any class $\Delta$ of modules, closed under submodules, and any torsion free module $M$, the following hold:

(i) $c(\Delta)$ is closed under submodules, extensions, and, in particular, under injective hulls.

(ii) $\Delta \cap c(\Delta) = \{0\}; \Delta \subseteq c(\Delta)$.

(iii) If $\Delta$ is closed under isomorphic copies, so is $c(\Delta)$.

(iv) $M_{\Delta} \cap c(M_{\Delta}) = 0 \Rightarrow M_{\Delta} \subseteq c(M_{\Delta})$.

Proof. Suppose that $M_{\Delta} \cap c(M_{\Delta}) \subseteq W \subseteq M$ for some $0 \neq W \subseteq M$. Since $M_{\Delta} \cap c(M_{\Delta}) \subseteq W = 0$, $W \in c(M_{\Delta})$, and hence there exists a $0 \neq \lambda \in W, \lambda \in \Delta$. But then $0 \neq \lambda \subseteq M_{\Delta}, \lambda \subseteq W = 0$ is a contradiction.

3.7. LEMMA. Suppose that $\Delta$ is a saturated class and $M$ a torsion free module. Then

(i) $c(\Delta$) is closed under torsion free homomorphic images and arbitrary (not necessarily direct) sums.

(ii) If $\Delta$ is a saturated class.

(iii) $M_{\Delta} \subseteq M; M_{\Delta} \subseteq c(\Delta)$.

(iv) $c(\Delta) \subseteq \Delta$.

Proof. (i) The proof of (i) is not difficult and is omitted.

(iii) In view of 3.6(ii), if there is a class $c(\Delta)$ closed under direct sums, if $A_{e} \in \Delta, \gamma \subseteq \Gamma$, then there exists a $0 \neq \lambda \subseteq \Gamma, A_{e}$, where $\forall \lambda \in \Delta$. By 1.2, both $\lambda$ and some one single $A_{e}$ contain nonzero submodules which are isomorphic, a contradiction. Hence $c(\Delta)$ is saturated.

(ii) By 3.7(iii) above, $M_{\Delta} \subseteq M, M_{\Delta} \subseteq c(\Delta)$, and $M_{\Delta} \cap c(M_{\Delta}) \subseteq d \cap c(\Delta) = \{0\}$. It now follows from 3.6(iv), 3.7(ii), that $M_{\Delta} \subseteq c(M_{\Delta}) \subseteq M$ where both submodules are fully invariant complements.

(iv) Always for any class $\Delta$ closed under submodules, $c(\Delta) = \{A \mid A \in \Delta, V \neq B \subseteq A, 0 \neq C \subseteq B, C \in \Delta\}$, and hence $\Delta \subseteq c(\Delta)$. By use of Zorn's lemma and the fact that $\Delta$ is closed under direct sums, injective hulls, and submodules, it now follows that $c(\Delta) = \Delta$.

3.8. REMARK. Except for 3.7(iii), the proofs of 3.6 and 3.7 did not use the torsion free hypothesis in Definition 3.5.

The next theorem says among other things that the saturated classes are precisely the unions of intervals $\bigcup \{x \mid 0 \leq x \leq y, y \in \Sigma(R)\}$ of the lattice $\Sigma(R)$.

3.9. THEOREM. Over any ring $R$, suppose that $\Delta$ is any (nonempty, well defined in ZFC) class of torsion free right $R$-modules which is closed under isomorphic copies (3.5). Let $c(\Delta), M_{\Delta}, M_{\Delta}, \Sigma(R)$ be as in 3.5 and 2.1. Let $A$ be any torsion free module defined by $\bigcup \{C \mid C \subseteq A\} = \{A \in \Sigma(R)\}$.

(a) $\Delta = \{C \mid C \subseteq A\}$.

(b) $\Delta$ is a saturated class.

(c) $\forall$ torsion free module $M, M_{\Delta} \subseteq c(M_{\Delta}) \subseteq M$ with $M_{\Delta} \subseteq A, M_{\Delta} \subseteq c(c(M_{\Delta}))$; and

(d) $\Delta$ is closed under injective hulls and essential submodules.

(2) Conversely, for any $[D] \in \Sigma(R), \bigcup \{C \mid C \subseteq [D]\} = \bigcup \bigcup \{C \mid C \subseteq [D]\}$ is a saturated class.

Proof. Conclusions (2) and (1)(a) imply (b) is obvious.

(1)(b) $\Rightarrow$ (a). Since $\Sigma(R)$ is a set, there is an indexed set of torsion free modules $C_{e} \in \Gamma, \gamma \subseteq \Gamma$, such that $\Delta = \bigcup \{C \mid C \subseteq \bigcup \{C \mid C \subseteq \Delta\}\}$. Since $\Delta$ is saturated, we infer that $A \subseteq \Delta$. From the latter it follows that $\Delta = \{C \mid C \subseteq \bigcup \{C \mid C \subseteq \Delta\}\}$. From this we infer that $\Delta = \{C \mid C \subseteq \bigcup \{C \mid C \subseteq \Delta\}\}$.

(1)(b) $\Rightarrow$ (c). This was shown in 3.7(iii).

(1)(c) $\Rightarrow$ (b). First, for $W \in \Delta$ and $0 \neq V \subseteq W$, it has to be shown that $W \subseteq V$. By (1)(c), $W_{\Delta} \subseteq W_{\Delta}$, and from $W \subseteq W$, we have $W = W_{\Delta}$. Finally, since $W \subseteq W_{\Delta}$, we have $V_{\Delta} \subseteq W_{\Delta} = 0$. By hypothesis (1)(c), secondly, $V \subseteq V_{\Delta}$; thirdly, $V \subseteq EV = E(V_{\Delta})$; and fourthly, $V \subseteq V_{\Delta}$.

Next, suppose that $A_{e} \in \Delta, \gamma \subseteq \Gamma$, is an indexed set of modules. Let $M = E(\bigcup A_{e}, A_{e})$, and $M_{\Delta} \subseteq c(M_{\Delta}) \subseteq M$. If $M_{\Delta} \neq 0$ then by 1.2 we have for some $W \neq 0$ and $V \subseteq W$ and some $\gamma \subseteq \Gamma$

$$M_{\Delta} \cap c(M_{\Delta}) = W \subseteq \gamma \subseteq A_{e},$$

a contradiction. Hence $M_{\Delta} \subseteq M$. We now invoke the two hypotheses that $M_{\Delta} \subseteq M$ and that $E(M_{\Delta}) \subseteq \Delta$ to conclude that $M \subseteq \Delta$. Thus $\Delta$ is saturated.

3.10. MAIN COROLLARY. For any ring $R$, let $\mathcal{S}$ denote the class consisting of all saturated classes of torsion free right $R$-modules. Furthermore, let $\Delta$ be any one of the following seven classes of right $R$-modules: Type I, II, III, continuous, discrete, continuous molecular, or bottomless (3.4 and 3.1). Then there exists a $\mathcal{S} \subseteq \mathcal{R}$ such that the following hold.

(i) $\mathcal{S}$ is a set; $|\mathcal{S}| \leq 2^{2^{(R)}}$.

(ii) $\mathcal{R} \to \mathcal{S}$, $x \to \bigcup \{y \mid 0 \leq y \leq x\}$, is a bijection.
(3) A is a saturated class; there exists a $J \leq R$ such that in (2) $[J/Z_2 R] \to \Delta$ and such that the following hold:

4) (i) \( Z_2 R \subset J, J \subset R \) is a right complement, \( J/Z_2 R \in \mathcal{Y} \).
(ii) \( \Delta = [C/C] \subseteq \{ [J/Z_2 R] \} \).
(iii) \( \text{Unique}: \text{if} \ I < R \text{ satisfies} (i) \text{ and} (ii), \text{ then} I \subset J. \)
(iv) \( J < R \) and \( J < K \) are fully invariant, in particular, \( J \preceq R \).

3.11. Observations. (i) Purely infinite modules need not be closed under submodules, and hence they do not form a saturated class.

2) By 3.10 and 3.7(i) it follows that if \( A \) is any saturated class, and \( \{ A_\gamma, \gamma \in \Gamma \} \subseteq A \) a subset, then \( \sum_{\gamma \in \Gamma} A_\gamma \subseteq A \). In particular, this holds for Types I, II, and III, and generalizes three separate proofs in [GB] for merely direct sums of Type I (GB; pp. 31–32, 5.2), Type II ([GB; p. 35, 5.6]), and Type III ([GB; pp. 38–39, 5.10]).

Our next objective is to show that certain saturated classes induce decompositions of the functor \( E \) as a direct sum of subfunctors.

3.12. Suppose that for any ring \( R \) with identity there is a saturated class \( \Delta(R) \) of torsion free right \( R \)-modules. Define \( \delta(R) = \bigvee \{ [C/C] \subseteq \Delta(R) \} \) as long as \( \Delta(R) \neq \{ [C/C] \} \subseteq [C/C] \). Let \( \phi: R \to S \) in the category \( A \), and thus \( \phi^*: \mathcal{E}(S) \to \mathcal{E}(R) \). Then the following are equivalent:

(i) \( \forall S \)-torsion free \( N = N_{\phi}, \) if \( N \in \mathcal{E}(S) \), then \( N_{\phi} \in \mathcal{E}(R) \).
(ii) \( \phi^* \delta(S) \subseteq \delta(R) \).

Proof. (i) \( \Rightarrow \) (ii). The proof requires the use of the fact from [D; p. 68, 4.10] that \( \phi^* \) commutes with arbitrary suprema.

\[ \phi^* \delta(S) = \bigvee \{ \phi^*[N_{\phi}] \subseteq [N_{\phi}] \} \subseteq \delta(R) \]

because \( N_{\phi} \in \delta(R) \) implies that \( \phi^*[N_{\phi}] = [N_{\phi}] \subseteq \delta(R) \) for all \( N \).

(ii) \( \Rightarrow \) (i). Since \( \phi^* \) is order preserving we have

\[ \bigvee \{ [N_{\phi}] \} = \bigvee \{ N_{\phi} \} \subseteq \phi^* \delta(S). \]

If \( \phi^* \delta(S) \subseteq \delta(R) \), then for any \( N_{\phi} \), \( [N_{\phi}] \subseteq \delta(R) \) and hence \( N_{\phi} \in \delta(R) \).

3.13. Definition. If for every ring \( R \) with identity \( \Delta \subseteq A \) is a saturated class of torsion free right \( R \)-modules, and if for all \( R, S \) and all \( \phi: R \to S \) in the category \( A \) the above two conditions 3.12(i),(ii) hold, then \( \delta \), \( \Delta \), \( \delta \) is called a universal saturated class. Note that \( \delta \) is completely determined by \( \delta \). In this case \( \delta(A) \subseteq A \) is another universal saturated class, where \( \delta(A) = \phi^*(\delta(R)) \) is a subfunctor in \( \mathcal{E}(R) \).

3.14. Theorem. Let \( R \) denote any ring with \( 1 \in R \). Suppose that \( (\Delta, \delta, (\Delta, \delta, \delta), \ldots, (\Delta, \delta, \delta) \ldots) \) is a universal saturated class, and for any \( \Delta, (\Delta, \delta, \delta) \) is as in 3.13. Define \( \Sigma R \subseteq \{ [C/C] \subseteq \Delta(R) \} \), and \( \Sigma R \subseteq \mathcal{E}(R) \) be as in 2.7. Let \( Z_2 R = R Z_2 R \) be as in 2.6(2) with \( \delta(R) = [R Z_2 R] \). Then

1) \( \Sigma(R) = \Sigma R \subseteq R \).
2) \( \Sigma = \Sigma R \) is a direct sum of subfunctors of \( \mathcal{E} \).

3.15. The following three conditions are equivalent:

(i) \( E \subseteq \Delta \subseteq \Gamma \subseteq \Delta \subseteq \Sigma \) is a direct sum of subfunctors.
(ii) \( \forall A \subseteq \mathcal{E}(R), M_{\Delta(A)} \subseteq \mathcal{E}(R) \subseteq M \).
(iii) \( \forall A \subseteq \mathcal{E}(R), \delta(A) \wedge \delta(A) = 1, \delta(A) \wedge \delta(A) = 0, i \neq j \).

4) Now assume that the above three conditions (3)(i)-(iii) hold. Then for any \( 1 \leq j \leq n \),

\[ R_{\Delta(R)} = \left[ \sum_{i=1}^n R_{\Delta(R)} \right] \subseteq \mathcal{E}(R), \quad R_{\Delta(R)} \subseteq \mathcal{E}(R) \subseteq R. \]

There is a canonical natural lattice (equivalently Boolean ring) isomorphism

\[ \mathcal{E}(R) \cong \mathcal{E}(R/\Delta(R)). \]

Proof. (i) First upon combining 3.9(i)(a) and 3.12 we get that \( \mathcal{E}(R) \cong \mathcal{E}(R/\Delta(R)) \) is actually a Boolean lattice. If \( \phi: R \to S \) is any ring morphism in \( A \), then by 3.12(i), \( \phi^* \mathcal{E}(S) \cong \mathcal{E}(R) \). Thus \( \mathcal{E} \) is a subfunctor of \( \mathcal{E} \).

(ii) In view of (1) above, and by 2.7(iii) (with \( \Gamma = \delta(R), (\delta(R)) \)), we have \( \mathcal{E}(R) \cong \mathcal{E}(R/\Delta(R)) \cong \mathcal{E}(R) \). By 3.12, 3.7(iii), and 3.13 it follows that \( \phi^* \mathcal{E}(S) \subseteq \mathcal{E}(R) \), and thus \( \mathcal{E} \cong \mathcal{E}(R/\Delta(R)) \) is a direct sum of subfunctors.

(iii) \( \Rightarrow \) (ii). By 3.14(i) above, \( \mathcal{E}(R/\Delta(R)) \subseteq \mathcal{E}(R) \) is a direct sum. If \( \delta(R) \wedge \delta(R) = 0 \) for some \( i \neq j \), it is easily seen that this sum would not be direct.

(iii) \( \Rightarrow \) (i). By the definitions of \( M_{\Delta(R)} \) and \( M_{\Delta(R)} \) in 3.12(i) and 3.15, from 2.7(ii) it follows that \( M_{\Delta(R)} = M_{\Delta(R)} \) for any \( (\Delta, \delta) \). With \( \Gamma = (\delta(R), \delta(R)) \), in 2.3 we get \( M_{\Delta(R)} \subseteq \mathcal{E}(R/\Delta(R)) \).

Thus by 2.6(2),

\[ R_{\Delta(R)} \subseteq J/Z_2 R \quad \text{and} \quad [J/Z_2 R] \cong [J/Z_2 R] \]

By 2.3(iii) applied to \( M = R \), we get first \( J \subseteq R \). Let \( J \subseteq C \subseteq \mathcal{E}(R) \) for some torsion free \( C \subseteq R \). It then follows from the above that \( \delta(R)^* \mathcal{E}(R) \subseteq \Sigma \). Thus \( C \subseteq \mathcal{E}(R) \) and hence \( C \subseteq \mathcal{E}(R) \). By 1.13, \( J \subseteq \mathcal{E}(R) \) is a right complement. Hence \( J = R_{\Delta(R)} \).
The next corollary generalizes [D; p. 62, 3.15(iv)] where it was shown that we get lattice direct sums if we apply the functors below to any ring $R$.

**3.15. Corollary.** $E = E_G \oplus E_P = E_{C_1} \oplus E_{C_2} \oplus E_B$ is a direct sum of subfunctors of $E$ (see 3.2).

In view of their importance, it seems useful to summarize what the previous theory says about Types I, II, and III. The next corollary shows among other things that every associative ring with identity $R$ contains certain unique ideals. It would be interesting to investigate and relate the algebraic properties of $R$ to those six ideals.

**3.16. Corollary.** Let $R$ be any ring with $1 \in R$. For $(\alpha, \beta)$ equal to any one of Types I, II, or III, denote the corresponding $R_{\alpha\beta}$ by $R_{\alpha\beta}$ given by 3.14(4) by $R_{\alpha\beta}$, $R_{\alpha\beta}$, $R_{\alpha\beta}$, etc. For $A_\alpha, B_\beta, C_\gamma \in I, II, III$, let $E = E_{C_1} \oplus E_{C_2} \oplus E_B$ denote the decomposition of $E$ as a direct sum of subfunctors. Then $E(R) = \{M \mid M \text{ is Type I}\}$ for each $1 \leq i \leq 3$.

Then the following hold for any one of $I, II, III, or III$.

1. (i) $Z/R \cong R_{\alpha\beta}$ is a right complement, $R_{\alpha\beta}/Z/R_{\alpha\beta}$.
   (ii) $\forall t.f. M \in I$, there exists an embedding $M \cong E(R_{\alpha\beta})$ for some $\gamma$.

2. (i) If $L < R$ is a right complement, $R_{\alpha\beta}/Z/R_{\alpha\beta}$.
   (ii) $\forall t.f. M \in I$, there exists an embedding $M \cong E(R_{\alpha\beta})$ for some $\gamma$.

3. (i) $Z_1/R \cong R_{\alpha\beta}$ is a unique smallest ideal with respect to:
   (ii) $R_{\alpha\beta}/Z/R_{\alpha\beta}$.
   (iii) $R_{\alpha\beta}/Z/R_{\alpha\beta}$.

We combine 2.3, 3.7, and 3.14 to show how the classical direct sum decomposition into Types I, II, and III is a very special case of a general phenomenon.

**3.17. Main Corollary.** Suppose that $(\alpha_1, \delta_1), \ldots, (\alpha_n, \delta_n)$ are any saturated universal classes satisfying the conditions 3.14(3) and that $M$ is any $R$-module. Then there exist submodules $M_i \subset M$ such that

1. (a) $M_i \ominus \cdots \ominus M_i \subset M$; (b) $M_i \cong M_i$.
2. $M_i \cong M_i$.
3. $M = M_1 \oplus \cdots \oplus M_n$.
4. $M_i = E(M_{\alpha_i})$ for $i = 1, 2, \ldots, n$.

We prove this by induction on $n$.

**4. Characteristic Types.** Various internal module-theoretic conditions are found which tell us whether a module is atomic, continuous module, or torsion-free, or has some of the eight properties listed in Section 3.4.

**4.1. Atomic.** For a t.f. module $A, (A) \subset (R) = \{0\}$. homomorphisms $E(R), E(R) = \{0\} = 0$.

**Proof.** $A$ is clear.

1. Choose any index set $I$ of cardinality $|A| > |A|$. We will show that for any $0 \neq A \neq A_0$, there exists an embedding $M = (E(R) \oplus A)$.
2. Define $K = (K_0, K_1, K_2, K_3, \ldots)$.
3. Let $K \neq A$.
4. Then $K \oplus (R_0, R_1, R_2, R_3, \ldots)$.
5. Define $K = f(K_0, K_1, K_2, K_3, \ldots)$.
6. Then $K \neq A$.
7. Thus $K \neq A$. Since $K = f(K_0, K_1, K_2, K_3, \ldots)$, we have $K \neq A$. Therefore $K \neq A$. This completes the proof.

**4.2. Abelian.** Let $A$ be torsion free continuous. Then $A$ is abelian $\Leftrightarrow A = 0$.

**Proof.** If $A$ is torsion free and continuous, then $A$ is abelian.

**Note:** The following says that $M$ is a direct sum of a Type II and Type III module.

**4.3. Continuous module.** Let $M$ be a torsion free continuous $R$-module. Then

1. $\forall \neq 0 < M$, $3P \oplus Q < V$ with $P \cong Q$.
2. $M$ does not contain any nonzero Type I submodules.

**Proof.** By Zorn’s lemma, there exists a maximal independent family $\{P_i \oplus Q_i\}$ of submodules of $V$ with $P_i \cong Q_i$. Let $M \cong \bigoplus (P_i \oplus Q_i) \oplus D \cong V$ and set $P = \ell \oplus P_i \oplus \ell \oplus Q_i \cong 0$.

If $D \neq 0$, there exists a continuous atomic submodule $0 \neq A \neq D$. Let $A = \{0\}$. Then $A = \{0\}$.

**Proof.** By 4.3(i) and 4.2, $M$ cannot contain a nonzero abelian submodule, and hence no Type I submodules.
5. Examples. The next two examples illustrate how our characterizations of Type I and III modules can be applied in concrete cases.

5.1. EXAMPLE. Suppose that \( R_j, j \in J \), is a family of torsion free rings and \( R = \prod R_j \). Then

(i) \( \forall j, R_j \subseteq R \Rightarrow R \in \text{End} \).

(ii) \( \forall j, R_j \in \text{End} \Rightarrow R \in \text{End} \).

Proof. (i) For any \( 0 \neq w \in R, w = (w_j) \in R \), there is a cyclic \( R \)-abelian submodule \( A_j \subseteq \prod R_j \), such that \( a \neq 0 \) whenever \( w_j = 0 \). Then \( A = (a_j) \in R \).

Suppose that \( 0 \neq w \in R \) with \( w \neq 0 \) and \( w_j \neq 0 \) for all \( j \). Then \( A \neq 0 \) and \( A \neq 0 \).

Any \( j \) with \( x_j \neq 0 \). Since \( xR_j = x_j R \subseteq A_j, \) necessarily \( x_j R_j \neq 0 \), and also \( 0 \neq x_j R_j \subseteq A_j \). Thus \( A \) is isomorphic to \( R \)-or \( R \)-modules. This contradicts that \( A \) is an \( R \)-abelian module. Therefore \( A \neq 0 \).

(ii) Let \( x_j R_j \rightarrow R_j \) be the projection. Take \( 0 \neq W \subseteq R \). For any \( j, \pi_j W \neq 0 \) is a directly infinite \( R \)-module. There exist two large \( R \)-submodules of \( \pi_j W \) as in 4.5.3, where \( B_j \subseteq B_j \subseteq \pi_j W \) and \( C_j \subseteq \pi_j W \) are both \( R \)-large, with \( B_j \subseteq B_j \subseteq C_j \). If \( \pi_j W \neq 0 \), also \( C_j \neq 0 \). Otherwise when \( \pi_j W = 0 \), also \( C_j = 0 \). Set \( B_j \subseteq B_j \subseteq C_j \subseteq \pi_j W \), with \( B_j \subseteq B_j \subseteq C_j \subseteq \pi_j W \). By 4.5.3, \( R \) is Type III.

A class of continuous bottomless rings is constructed below, and it is also determined when these are Type I or III. It is known ([D]: p. 75, 6.3) that a domain \( R \) that is not right Ore is continuous atomic, and it is easy to see that ([D]: p. 75, 6.4) that then \( R \) is Type III. Atomless Boolean rings are examples of torsion free continuous bottomless Type I rings ([D]: p. 77, 6.9). Concrete atomless Boolean rings are given in [P]: p. 895, Thm. 3.3; [S]: pp. 105-107, [A, B], [H]: p. 96, (4); and [GHKLMS] p. 113, (3). By use of the above in the new construction, it is easy to construct very elementary examples of Type I and Type III rings.

5.2. Construction. For any family of rings with identity \( R_j, j \in J \), form \( P = \prod R_j, S = \otimes R_j \), and set \( R = P/S \).

(i) \( R \) is torsion free.

(ii) \( R \) is continuous bottomless. If now in addition all the \( R_j, j \in J \), are countable domains, then

(iii) \( \langle \{ j \in J | R_j \notin \text{End} \} \rangle \neq \emptyset \Rightarrow R \notin \text{End} \).

(iv) \( \langle \{ j \in J | R_j \notin \text{End} \} \rangle \neq \emptyset \Rightarrow R \notin \text{End} \).

Proof. First it will be shown that for any \( \hat{a} + S, \hat{b} \in R, \) a \( = (a_j) \in P, \) and any \( R \)-map \( f: \hat{a} \rightarrow R \) with \( \hat{b} = f(\hat{a}) \), there exists a coset representative \( b = (b_j) \in P \) with \( \hat{b} = f(\hat{a}) \). Then \( f = \hat{b} \). For any subset \( Y \subseteq J, \chi_Y \) will denote the characteristic function of \( Y \). First take any \( b \in P \) with \( \hat{b} = f(\hat{a}) \), and set \( f = \hat{b} = f((\hat{a} - \chi_Y \hat{a})) \). Consequently \( \hat{b} = f((\hat{a} - \chi_Y \hat{a})) \).
(i) For $0 \not= a \in R$, define $a^{-1}0 = \{p \in P | ap = 0\}$. Since $a^2 = (a^{-1}0 + S)/S = \{1 - (1 - a^{-1}0 + S)/S\}$, and since $a^2 \oplus \{E \cup S) + (S + P)/S\} = R$, we have $a^2 \not= R$ iff $|\text{supp}a| < \infty$. Thus $\text{Z}(R) = 0$.

(ii) If $R_{\alpha}$ is not continuous bottomless, then there exists a continuous atomic submodule $0 \not= \alpha R$ for some $a = a + S \in R$. Partition $\text{supp} a = Y \cup Z$, $Y \cap Z = \emptyset$, and $|Y| = |Z| = \infty$. Since $0 \not= \alpha Y$, $0 \not= \alpha X \alpha Z$ is an atomic, and it follows that $\alpha Y \alpha Z \subseteq E(\alpha Y \alpha Z)R$ for some $f$, and by 1.2, that $(\alpha Y \alpha Z) = (\alpha X \alpha Z) = 0$. But then $\alpha Y \alpha Z = 0$ and $\alpha X \alpha Z = 0$ is a contradiction. Hence $R_{\alpha}$ is continuous bottomless.

(iii) Set $Y = \{x \in J | R_x \neq 1\}$. If $R_{\alpha} \neq 1$, then there exist $\alpha = a + S$, $\beta = b + S \in R$ with $0 \not= \alpha \beta \not\subseteq \text{Z}(R)$ and with $\alpha R \cap \beta R = 0$. In general

$\alpha R \cap \beta R = 0 \Rightarrow \{x \in J | R_x \neq 1\} = \varnothing$.

By a change of coset representatives $a$ and $b$ of $\alpha$ and $\beta$ we may assume that the latter set is empty. By the observation at the beginning of the proof, we may also assume that $\text{supp} a = \text{supp} b$. Thus, for each $j \in \text{supp} a \cup \text{supp} b$, $R_j \not= a_j R_j \not= b_j R_j$ and also $R_j \cap R_j = 0$. This implies that $R_{\alpha} \not= 1$. Hence $\alpha \not= \beta$. This contradicts that $\alpha \not= \beta$. Thus $R_{\alpha} \not= 1$.

(iii) Suppose that $|Y| = \infty$. For each $x \in Y$ choose $0 \not= a_x \not= b_x \not= 0$. Let $x \in \text{supp} a \cup \text{supp} b$. Let $R_j \not= a_j R_j \not= b_j R_j$. Let $x_j = 0$. Then $a_x R \cap b_x R = 0$, $\alpha R \cap \beta R = 0$, and $0 \not= \alpha R \not\subseteq \text{Z}(R)$ because $a^2 = b^2 = (\alpha Y \alpha Z)R$. Thus $R_{\alpha} \neq 1$, a contradiction. Hence $\alpha \not= \beta$.

(iv) It suffices to show that any cyclic submodule of $R_{\alpha}$ is directly infinite. Let $0 \not= c = c + S \in R$, $c = (c_j) \in P, S$. For each $n$, there exists

$\sum_{i=1}^{n} z_i R_{\alpha} = \sum_{i=1}^{n} z_i P \subseteq c_j P = c_j R_{\alpha}$,

where the latter is a large extension of right $P$-modules, and where we view all $z_i \in R_{\alpha} \subseteq S \subseteq P$, and also $c_j \in R_{\alpha} \subseteq P$. If $n \not= c_j \subseteq R_{\alpha}$, all $0 \not= z_j \subseteq R_{\alpha}$. Then $c_j \subseteq R_{\alpha} = 0$, let all $z_j = 0$. Then $z_i = (z_{\alpha} + 1) \subseteq c_j P$. Then $\sum_{i=1}^{n} z_i P \subseteq c_j P$ is a large extension of right $P$-modules. Moreover, also $\sum_{i=1}^{n} z_i (\alpha R \cap \beta R) \subseteq R_{\alpha}$, where $\alpha = z_i = z_i + S \subseteq \text{Z}(R)$.

$0 \not= c \subseteq c \subseteq R \subseteq R_{\alpha} \subseteq \text{Z}(R)$ shows that $E(R) \cong E(cR) \oplus E(cR)$). Hence $R$ is a continuous bottomless module of type III. This is similar to (iii) and is omitted.

5.3. The classes of discrete, continuous, continuous molecular, and bottomless torsion free modules are denoted by $D$, $C$, $CA$, and $CB$ respectively; while the classes of all modules of Types I, II, and III are abbreviated by $I$, $II$, and $III$. By [D: p. 75, 6.4], $CA \cap III = \{0\}$; while [D: p. 77, 6.5] shows that $CB \cap III = \{0\}$. By 4.3, $D = I$ (and hence $D \cap II = D \cap III = \{0\}$). In the table below,

"0" means that the intersection of the two classes (the one in the row and the second in the column) is $\{0\}$; and "yes" means that there exist nonzero modules in the intersection of the two classes, and "no" means that so far the answer is not known.

<table>
<thead>
<tr>
<th>I</th>
<th>II</th>
<th>III</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>yes</td>
<td>0</td>
</tr>
<tr>
<td>CA</td>
<td>0</td>
<td>?</td>
</tr>
<tr>
<td>CB</td>
<td>yes</td>
<td>?</td>
</tr>
</tbody>
</table>

For any torsion free injective modules $A$ and $B$, the reject $	ext{tr}_{A,B}$ is injective. The next counterexample shows that the dual of this for the trace fails—$	ext{tr}_{A,B}$ need not be injective.

5.4. COUNTEREXAMPLE. For an infinite-dimensional right vector space $V$ over a division ring $F$, take $R$ to be the ring $R = \text{End}_{F}V$, where $R$ acts on the left of $V$. Then the ideal $H \subseteq R$ of finite rank transformations is a simple ring with $R / H \cong R$. Since $R$ is a right self-injective regular ring ([G, p. 53, 2.21]), for any $0 \not= x \in H$, $xR$ is injective because $xR = eR$ for some $e^2 = eR$. Then the trace of $xR$ in $R$ is $\text{tr}_{xR} R = xR = H$ and is not injective.

References


Department of Mathematics
Tulane University
New Orleans, Louisiana 70118, U.S.A.

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