By the continuity of f', we have $|f'|_{A_m} - g_m|_{A_m}|^* < \varrho$. Hence $f'|_{A_m}: A_m \to A_m$ has a fixed point in $V_{\varrho_0}(C_m^*)$. Thus, f' has a fixed point in $V_{\varrho_0}(C_m^*) \subset U_\varepsilon(C^*)$.

From Cases 2(b)(i) and (ii) it follows that every mapping $f': Z \to Z$ with $|f'-f| < \min\{\delta_A, \delta_B\}$ has a fixed point in $U_\varepsilon(C^*)$. This completes the proof.

Remark. While Z has f.*p.p., the cone over Z does not have f.p.p. (see [3]).

Addendum. We can construct an example of a locally connected continuum which has f.p.p. but does not have f.*p.p. Define

$$B_n = \{x \mid (x - 1/2^n)^2 + y^2 \leqslant 1/(3 \cdot 2^n)^2\},$$

$$Y_5 = (\{(0,0)\} \times I) \cup \bigcup_{n=0}^{\infty} (\partial B_n \times I) \cup \bigcup_{n=0}^{\infty} (B_n \times \{0\}).$$

By a similar argument to that of Theorem 1, we can prove that Y_5 has f.p.p. but does not have f.*p.p. Another similar example corresponding to Theorem 2 can also be easily constructed.

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Torsion free types

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Abstract. It is shown how the known classification of nonsingular, injective modules M into Types I, II, and III as well as corresponding direct sum decompositions $M = M_{\mathrm{I}} \oplus M_{\mathrm{II}} \oplus M_{\mathrm{III}}$ are merely special cases of a more general phenomenon. There is a functor Ξ from rings R to complete Boolean lattices (equivalently Boolean rings) $\Xi(R)$, where each point of $\Xi(R)$ is a class of similar nonsingular modules. Types I, II, III, continuous, discrete, and certain other classes of modules correspond to unique elements of $\Xi(R)$. Appropriate finite sets of disjoint classes of modules induce direct sum decompositions of Ξ as a direct sum of subfunctors. The latter give rise to corresponding direct sum decompositions of nonsingular injective modules M, such as $M = M_{\mathrm{I}} \oplus M_{\mathrm{II}} \oplus M_{\mathrm{III}}$.

Introduction. This article will show how the classification of certain torsion free modules into Types I, II, and III ([MN], [K], [B], and [GB]) is a special case of a classification scheme developed in [D]. If A is a unital right R-module and ZA its singular submodule, then the second singular submodule $ZA \subseteq Z_2A \subseteq A$ is defined by $Z[A/ZA] = (Z_2A)/ZA$. A module is *torsion free* if $ZA = Z_2A = 0$, and *torsion* if it equals its *torsion submodule* $Z_2A = A$. This is a continuation of [D] where the following was shown. There exists a contravariant functor Ξ applicable to any associative ring R with identity. The result is a complete Boolean lattice $\Xi(R)$. The functor Ξ classifies or partitions the class of all torsion free right R-modules $\{A, B, \ldots\}$ into equivalence classes $\Xi(R) = \{[A], [B], \ldots\}$ where $A \in [A]$, and $[A]$ consists of a class of modules that are similar, or are like A. An appropriate ring homomorphism $R \to S$ induces a lattice (equivalently Boolean ring) homomorphism $\Xi(S) \to \Xi(R)$.

Goodearl and Boyle ([GB]) extended the Murray–von Neumann–Kaplansky ([MN], [K], and [B]) classification of operator algebras, W^*-algebras, and Baer $*$-rings into Types $\mathrm{I} = \mathrm{I}_f \cup \mathrm{I}_\infty$, $\mathrm{II} = \mathrm{II}_f \cup \mathrm{II}_\infty$, III, abelian, directly finite, and purely infinite to all torsion free injective modules. Here this latter theory is extended to all torsion free modules over any ring by defining M to belong to any of the latter classes if and only if its injective hull EM does (e.g. $M \in \mathrm{III}$ iff $EM \in \mathrm{III}$). In order to obtain necessary and sufficient conditions for M (as opposed to EM) to be of Types I, II, III, abelian etc. (4.2, 4.4, and 4.5), the usual definitions are reformulated without reference to idempotents (3.3, 3.4).

It is shown that there exist unique largest elements [I], [II], [III] $\in \Xi(R)$ which determine Type I, II, and III modules (Corollary 3.16). More specifically, [III] consists entirely of Type III modules, and determines the class III as III $= \{M \mid [M] \leqslant [\mathrm{III}]\}$. Or equivalently, the class III is the union of the interval $[0, [\mathrm{III}]] \subset \Xi(R)$, III $= \bigcup\{x \mid 0 \leqslant x \leqslant [\mathrm{III}]\}$. Verbatim identical statements hold for Types I, II and [I], [II]. It is shown that this latter phenomenon is not limited to Types I, II, or III, but holds for any so-called saturated class of modules (see 3.5). Types I, II, III are examples of saturated classes. Four other examples are discrete, continuous, molecular, and bottomless torsion free modules (see 3.1, 3.2). It is shown in Section 3 (Theorem 3.9) that there is a bijective correspondence between saturated classes of torsion free R-modules Δ and elements of $\Xi(R)$ given by $\Delta = \{C \mid [C] \leqslant [A]\}$, where $[A]$ = supremum $\{[C] \mid C \in \Delta\}$. Conversely, any $[D] \in \Xi(R)$ determines a saturated class Δ by $\Delta = \{C \mid [C] \leqslant [D]\}$.

Suppose that for each associative ring R with identity, there is a saturated class $\Delta(R)$ of torsion free R-modules. Then the assignment $\Delta\colon R \to \Delta(R)$ is called a *universal saturated class* provided one additional property holds (3.12 and 3.13). The above-mentioned seven saturated classes are universal saturated classes. Theorem 3.14 shows how a universal saturated class Δ induces a decomposition of the functor Ξ as a direct sum of two subfunctors $\Xi = \Xi_\Delta \oplus \Xi_{c\Delta}$, where $c\Delta$ is a universal saturated class complementary to Δ. Such decompositions extend to certain finite number of disjoint universal classes. Thus Types I, II, and III induce a decomposition $\Xi = \Xi_{\mathrm{I}} \oplus \Xi_{\mathrm{II}} \oplus \Xi_{\mathrm{III}}$ (Corollary 3.16); or the continuous molecular, continuous bottomless, and discrete classes induce another decomposition $\Xi = \Xi_{CA} \oplus \Xi_{CB} \oplus \Xi_D$ (Corollary 3.15). Such direct sum decompositions of Ξ induce unique direct sum decompositions of injective modules as finite direct sums (Main Corollary 3.17). Furthermore, the Module Decomposition Theorem 2.3 generalizes the latter even further, among other things, to full direct product representations of injective modules.

The lattice $\Xi(R)$ is isomorphic to the lattice of all fully invariant complement right ideals J of R with $ZR \subseteq J$. For this reason alone fully invariant complement submodules are important. Section 1 investigates such submodules, and particularly how they are mapped under module homomorphisms. This section involves no special hypotheses and should by itself be of independent interest. Section 2 reformulates some of [D] in a form in which it will be used later, as well as derives some corollaries from the results of [D]. Thus Corollary 2.7 gives the new result that if $\Gamma \subset \Xi(R)$ is any pairwise disjoint subset with supremum $\Gamma = 1$, then it induces a Boolean lattice (equivalently ring) isomorphism $\Xi(R) \cong \prod\{\Xi_\alpha(R) \mid \alpha \in \Gamma\}$ where each $\alpha \in \Gamma$ defines a convex sublattice $\Xi_\alpha(R) \subset \Xi(R)$ with $\Xi_\alpha(R) \cong \Xi(R/R_\alpha)$ for a certain unique ideal $R_\alpha \lhd R$. Section 5 shows how some of the criteria of Section 4 can be used to construct various Type I, III, and bottomless modules. It also begins the problem of relating the Type I, II, and III classification to the discrete, continuous, molecular, and bottomless ones (5.3).

The author hopes to show in a subsequent paper that functors similar to Ξ can be defined for bigger ring categories, i.e. Ξ would be applicable to more ring homomorphisms; and that there also is an analogous theory for the torsion modules. In summary, among other things, this article puts the previously well developed theory of torsion free injective modules over a single fixed ring R in a broader functorial context where many open questions remain.

1. Complements and full invariance. Some facts, needed in later proofs, have been isolated out of the specialized contexts of later theorems, and proved here in a general module setting, because these facts may be of independent interest. Moreover, many of the subsequent proofs will just amount to quoting the lemmas of this section in the right order.

1.1. *Notation.* Modules M are right unital over a ring R. Denote submodules by $<$ or $\leqslant$, and large or essential submodules by $\ll$. The symbol $A \not\ll B$ means that $A < B$ but that A is not large in B. For $K < M$ and $m \in M$, $m^\perp = \{r \in R \mid mr = 0\} < R$, while $m^{-1}K = \{r \in R \mid mr \in K\} < R$. For a subset $Y \subset M$, $Y^\perp = \{r \in R \mid \forall y \in Y,\ yr = 0\} = \{r \mid Yr = 0\}$. Ideals (two-sided) in R or other rings are denoted by "$\lhd$"; thus $M^\perp \lhd R$.

The singular submodule ZM of M is $Z(M) = ZM = \{m \in M \mid m^\perp \ll R\}$. Define the second singular submodule in $ZM \subset Z_2 M \subset M$ by $Z[M/(ZM)] = (Z_2 M)/ZM$. Define $Z_2 M$ to be the *torsion submodule* of M. Consequently M is *torsion* if $M = Z_2 M$, which holds iff $ZM \ll M$, and M is *torsion free*, abbreviated t.f., if $Z_2 M = 0$, i.e. iff $ZM = 0$. Right R-injective hulls of right R-modules M are denoted by both "^" and "E" as $\hat{M} = E(M) = EM$, where E is used for quotient modules or when M is given by a complex formula.

For those $K < M$ with $ZM \subset K$ — and only for these —, define the *complement closure* $\bar{K}$ of K in $K \ll \bar{K} \leqslant M$ as $\bar{K}/K = Z(M/K)$. Various facts and properties of $\bar{K}$ will be used without further mention, see [D; pp. 51–55, 1.1–1.11]. The symbols $<, \leqslant, \ll, \not\ll, {}^\perp, {}^{-1}, Z_1 Z_2$, ^, E, and ¯ refer to right R-modules over R, and never to other rings S with $1 \in S$ which also will be used.

For a module map $f\colon A \to B$, and for $C < A$, denote the restriction of f to C by $f|C\colon C \to B$. The *trace* of a module A in a module B is defined as $\mathrm{tr}_B A = \sum\{fA \mid f \in \mathrm{Hom}_R(A, B)\}$. Dually the *reject* of A inside B is $\mathrm{rej}_A B = \bigcap\{\ker h \mid h \in \mathrm{Hom}_R(B, A)\}$. A submodule $K < M$ is *fully invariant*—abbreviated f.i.—if $\mathrm{Hom}_R(M, M)K \leqslant K$.

For any set X, $\mathscr{P}(X) = \langle \mathscr{P}(X), \cup, \cap, \backslash, X, \varnothing \rangle$ is the usual Boolean lattice of the power set of X; also $\mathscr{P}(X)$ denotes the associated Boolean ring. Denote the cardinality of any set X by $|X|$.

The following simple construction will be used repeatedly.

1.2. Projection argument. *Let $\{A_\gamma \mid \gamma \in \Gamma\}$ be any family of modules. View $E(\bigoplus A_\gamma) \subset \prod \hat{A}_\gamma$. For any $0 \neq \xi = (\xi_\gamma) \in E(\bigoplus A_\gamma)$, choose $r_0 \in R$ such that $0 \neq \xi r_0 = a_1 + \ldots + a_n \in A_{\gamma(1)} \oplus \ldots \oplus A_{\gamma(n)}$ with all $0 \neq a_i \in A_{\gamma(i)}$, and with the length $n \neq 0$ of ξr_0 minimal. Then $(\xi r_0)^\perp = a_1^\perp = \ldots = a_n^\perp \neq R$. Thus $\xi r_0 R \cong a_1 R \subset A_{\gamma(1)}$. In particular, every nonzero submodule of $E(\bigoplus A_\gamma)$ contains an isomorphic copy of a nonzero submodule of some A_γ.*

1.3. Trace projection argument. *Let $\{A_\gamma \mid \gamma \in \Gamma\}$ be any family of torsion free modules and B any torsion free module. Suppose that $0 \neq V \leqslant E[\mathrm{tr}_{EB} E(\bigoplus_\Gamma A_\gamma)]$. Then there exists a $\gamma \in \Gamma$ and elements $0 \neq a \in A_\gamma$, $0 \neq b \in B \cap V$ such that $aR \cong bR$ and $a^\perp = b^\perp$.*

Proof. By [D; p. 55, Theorem 2.1] there are $0 \neq \xi \in E(\bigoplus_\Gamma A_\gamma)$, $0 \neq v \in B \cap V$ with $\xi^\perp = v^\perp$. Choose $r_0 \in R$ such that $0 \neq \xi r_0 = a_1 + \ldots + a_n$, $0 \neq a_i \in A_{\gamma(i)}$ with $A_{\gamma(1)} \oplus \ldots \oplus A_{\gamma(n)}$ a direct sum, and where n is minimal. Then $(\xi r_0)^\perp = a^\perp$, where $\gamma = \gamma(1)$, $a = a_1 \in A_\gamma$, and $b = vr_0 \in B \cap V$. Then $b^\perp = (vr_0)^\perp = (\xi r_0)^\perp = a^\perp$.

1.4. Lemma. *Suppose that $ZA \subseteq K \leqslant A$ and B are modules and that f: $A \to B$ is a right R-module map with $ZB \subseteq fK$. Then*

(i) $fK \leqslant f\bar{K} \leqslant (fK)^-$.

(ii) *In particular, $\forall L \leqslant A$, $K \leqslant L$ also $fK \leqslant fL$.*

Proof. If $fx \in f\bar{K}$ with $x \in \bar{K}$, then $x^{-1}K \leqslant R$, and $x^{-1}K \subseteq (fx)^{-1}(fK)$. Since $ZB \subseteq fK$, we have $fx \in (fK)^-$. Consequently $f\bar{K} \subseteq (fK)^-$. But $(fK) \leqslant (fK)^-$. Hence $fK \leqslant f\bar{K} \leqslant (fK)^-$.

From [D; p. 56, 2.4], the result below will be needed. The fact that an arbitrary intersection of injective submodules remains injective may lead the reader to conjecture that below, the "E" may be omitted from in front of the next summand. A counterexample (5.4) shows that this is not so.

1.5. *For any torsion free modules A and B, $EB = E(\mathrm{tr}_{EB} A) \oplus \mathrm{rej}_{EB} EB$ is a direct sum of two fully invariant submodules.*

1.6. Remarks. Let $K < M$ be any modules. Form $\hat{M} = \hat{K} \oplus \hat{C}$. Then

$$\hat{K} < \hat{M} \text{ is fully invariant} \Leftrightarrow \mathrm{Hom}_R(\hat{K}, \hat{C}) = 0.$$

In particular, if M is torsion free, then

$$\hat{K} < \hat{M} \text{ is f.i.} \Leftrightarrow \hat{C} < \hat{M} \text{ is f.i.}$$

Note that in case M is torsion free, Lemma 1.4 immediately proves the next lemma, but not in general, for there is no guarantee that $ZM \subseteq fK$ below.

1.7. Lemma. *For any modules $ZM \subseteq K < M$,*

$$K < M \text{ is f.i.} \Rightarrow \bar{K} < M \text{ is also f.i.}$$

Proof. Let $f \in \mathrm{Hom}_R(M, M)$ and $\xi \in \bar{K}$ be arbitrary. Then since $\bar{K} = \{x \in M \mid x^{-1}K \leqslant R\}$ (by [D; p. 53, 1.3(0)]), $\xi^{-1}K \leqslant R$. But then $(f\xi)\xi^{-1}K \subseteq fK \subseteq K$. Hence $\xi^{-1}K \subset (f\xi)^{-1}K \leqslant R$, and also $f\xi \in \bar{K}$. Thus $f\bar{K} \subseteq \bar{K}$.

1.8. Proposition. *Let $K = \bar{K} < M$ be a complement submodule with $ZM \subseteq K$. Then*

(i) $\hat{K} < \hat{M}$ *is f.i.* $\Rightarrow$ $K < M$ *is also fully invariant.*

(ii) $K < \hat{M}$ *is f.i.* $\Rightarrow$ $\hat{K} < \hat{M}$ *is fully invariant.*

Proof. (i) Since $K \leqslant \hat{K} \cap M$, $K = \hat{K} \cap M$. For any $\phi \in \mathrm{Hom}_R(M, M)$, extend ϕ to $\hat{\phi}$: $\hat{M} \to \hat{M}$. By hypotheses, $\hat{\phi}\hat{K} \subseteq \hat{K}$. But then $\phi K \subseteq \hat{\phi}\hat{K} \cap M \subseteq \hat{K} \cap M = K$.

(ii) Let $\psi \in \mathrm{Hom}_R(\hat{M}, \hat{M})$. Write $\hat{M} = \hat{K} \oplus U$ for some $U < \hat{M}$. For $\xi \in \hat{K}$, suppose that $\psi\xi \notin \hat{K}$. Then $\psi\xi = x + y$, $x \in \hat{K}$, $0 \neq y \in U$. Set $L = \xi^{-1}K \cap x^{-1}K \leqslant R$. For any $r_0 \in L$, $\psi\xi r_0 \in K$, $xr_0 \in K$, and hence also $yr_0 \in K$. Thus $L \subseteq y^{-1}K \leqslant R$ shows (by [D; p. 53, 1.3(0)]) that $y \in \hat{K}$, a contradiction.

1.9. Corollary. *If $ZM \subseteq K < M$, then*

$$K < \hat{M} \text{ is f.i.} \Rightarrow \hat{K} < \hat{M} \text{ is f.i.}; \quad K < M \text{ is f.i.} \Rightarrow \bar{K} < M \text{ is f.i.}$$

Note that in the next proposition, $K < M$ need not be a complement necessarily.

1.10. Proposition. *Let $ZM \subseteq K < M$ and assume that $\hat{K} < \hat{M}$ is fully invariant. Let C be any*

(a) *complement submodule $C < M$ such that*

(b) $K \oplus C \leqslant M$.

Then

(i) *$C < M$ is fully invariant.*

(ii) *Unique: If $C_1 < M$ satisfies* (a) *and* (b), *then $C_1 = C$.*

Proof. Write $\hat{M} = \hat{K} \oplus \hat{C}$. By 1.6 it suffices to show that $\mathrm{Hom}_R(\hat{C}, \hat{K}) = 0$. If not, let $0 \neq \phi$: $\hat{C} \to \hat{K}$. Write $\hat{C} = U \oplus \ker\phi$. Then

$$\hat{C} > U \cong \phi U < \hat{K}$$

contradicts the fact that $\hat{K} < \hat{M}$ is fully invariant.

1.11. Lemma. *If $ZM \subset K = \bar{K} < M$, then below* (ii) $\Rightarrow$ (i), (iii) $\Leftrightarrow$ (iv), (iii) $\Rightarrow$ (i), *and* (iv) $\Rightarrow$ (ii):

(i) *$K < M$ is fully invariant.*

(ii) *$K/Z_2M < M/Z_2M$ is fully invariant.*

(iii) *$\hat{K} < \hat{M}$ is fully invariant.*

(iv) *$E(K/Z_2M) < E(M/Z_2M)$ is fully invariant.*

Proof. Conclusions (iii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (ii) were shown in 1.8. It will be shown that (iii) $\Leftrightarrow$ (iv). The proof of (ii) $\Rightarrow$ (i) is similar and is omitted.

Take any $D < M$ such that $Z_2M \oplus D \leqslant M$, and hence $Z_2M \oplus (K \cap D) \leqslant K$. Then since $Z_2M < M$ is a complement,

$$\frac{Z_2M \oplus (K \cap D)}{Z_2M} \leqslant \frac{K}{Z_2M}, \quad E\left(\frac{K}{Z_2M}\right) \cong E(K \cap D);$$

$$\frac{Z_2M \oplus D}{Z_2M} \leqslant \frac{M}{Z_2M}, \quad E\left(\frac{M}{Z_2M}\right) \cong E(D).$$

Since $EZ_2M = Z_2EM < EM$ is fully invariant it follows that:

$$\hat{K} < \hat{M} \text{ is fully invariant} \Leftrightarrow E(K \cap D) < E(D) \text{ is fully invariant}$$
$$\Leftrightarrow E(K/Z_2M) < E(M/Z_2M) \text{ is fully invariant.}$$

1.12. Lemma. *For any modules $L < K < M$ assume that $ZM \subseteq L$ and $\bar{L} \subseteq K$. Then*

(i) $Z(M/L) \subseteq K/L$.

(ii) *$K < M$ is a complement $\Leftrightarrow$ $K/L < M/L$ is a complement.*

Proof. (i) If $m + L \in Z(M/L)$ then $(m+L)^\perp = m^{-1}L \leqslant R$, and $m \in \bar{L} \subseteq K$ by [D; p. 53, 1.3(0)].

(ii) $\Leftarrow$ If not, $K \ll B \leqslant M$. By hypothesis, $K/L <\!\!\not< B/L$. Hence there exists a cyclic submodule of B/L such that $(b_0+L)R \cap K/L = 0$, or $(b_0R+L) \cap K = L$. The latter implies that $b_0^{-1}K = b_0^{-1}L$. But $K \ll B$, and hence $b_0^{-1}K \ll R$. Thus $b_0^{-1}L \ll R$ and $b_0 \in \bar{L} \subseteq K$, a contradiction. Hence $K < M$ is a complement.

(ii) $\Rightarrow$ This proof will only require that $ZM \subseteq K$. By way of contradiction, suppose that $K/L \ll B/L$ for some $K < B$. Then $K <\!\!\not< B$, and hence $K \neq K \oplus b_0 R$ for some $b_0 \in B$. Since $K/L \ll B/L$, $b_0^{-1}K = (b_0+L)^{-1}(K/L) \ll R$. By [D; p. 53, 1.3(0)] from $ZM \subseteq K$ we conclude that $b_0 \in \bar{K} = K$, a contradiction. Thus $K/L < M/L$ is a complement.

1.13. Corollary. *If $ZM \subseteq K < M$, then $K < M$ is a complement $\Leftrightarrow$ $K/Z_2M < M/Z_2M$ is a complement.*

1.14. Corollary. *Suppose that $Z_2M \subset K < M$ and $E(K/Z_2M) < E(M/Z_2M)$ is a fully invariant complement. Let $L, C < M$ be any submodules satisfying the following: $K \oplus C \ll M$ and*

(a) *$L < M$ is a complement with $L \cap K = Z_2M$; and*

(b) *$K+L \ll M$.*

Then

(i) *$K/Z_2M \oplus L/Z_2M \ll M/Z_2M$ and also L/Z_2M is fully invariant.*

(ii) *Unique: If $L_1 < M$ satisfies* (a) *and* (b), *then $L_1 = L$. Consequently,*

(iii) *$L = (Z_2M + C)^-$.*

1.15. Lemma. *Suppose that $\{A_i \mid i \in I\}$ is a family of torsion free modules such that $\mathrm{Hom}_R(\hat{A}_i, \hat{A}_j) = 0$ for all $i \neq j$. Then*

$$E(\bigoplus_{i \in I} A_i) = \prod_{i \in I} \hat{A}_i.$$

Proof. View $E(\bigoplus_I A_i) < \prod_I \hat{A}_i$. It suffices to show that for any $0 \neq \xi = (\xi_i) \in \prod_I \hat{A}_i$, and any $\xi_i \neq 0$, there exists an $r \in R$ with $0 \neq \xi r = \xi_i r \in \hat{A}_i < \prod_I \hat{A}_i$. Since A_i is t.f., we have $\xi_i^\perp <\!\!\not< R$. Take any $0 \neq C < R$ with $\xi_i^\perp \oplus C \leqslant R$. Suppose that $\xi_j C \neq 0$ for some $j \neq i$. Since $\xi_i^\perp \cap C = 0$, $C \cong \xi_i C$, and the map $\xi_i C \to \xi_j C$, $\xi_i c_0 \to \xi_j c_0$, $c_0 \in C$, extends to a nonzero element of $\mathrm{Hom}_R(\hat{A}_i, \hat{A}_j)$, a contradiction. Hence for any $0 \neq r \in C \subseteq \bigcap \{\xi_j^\perp \mid j \in I, j \neq i\}$ we have $0 \neq \xi r = \xi_i r \in \hat{A}_i$.

2. The functor Ξ.

This section reformulates, extends, and generalizes further some of the facts about the lattice $\Xi(R)$ from [D].

2.1. *Types.* On the class of all torsion free right R-modules $A, B, C, \ldots$ define a quasi-order $\prec$ by $A \prec B$ if there is an embedding $A \subset E(\bigoplus\{B \mid J\})$ for some (arbitrary) index set J. Then $A \sim B$ if $A \prec B$ and $B \prec A$ defines an equivalence relation. The equivalence classes $[A] = \{C \mid C \sim A\}$ are called *types.* The class $\Xi(R) = \{[A], [B], [C], \ldots\}$ of all equivalence classes becomes a poset, where $[A] \leqslant [B]$ if and only if $A \prec B$. Whenever the least upper bound or the greatest lower bound exist in $\Xi(R)$, they are denoted by $[A] \vee [B]$ and $[A] \wedge [B]$, and similarly for infinite suprema and infima. Thus so far $\Xi(R)$ is a poset with least element $[(0)] \in \Xi(R)$.

Next some facts from [D] are summarized and notation is established which we later need to use.

2.2. Type lattice. *Let $\Xi(R) = \{[A], [B], \ldots\}$ be the partially ordered class of equivalence classes of torsion free right R-modules $A, B, \ldots$ Set $\tilde{R} = R/Z_2R$. Let $\{[A_\gamma] \mid \gamma \in \Gamma\} \subset \Xi(R)$ be any indexed subset. Then*

(1) *$\Xi(R)$ is a set. $|\Xi(R)| \leqslant |\mathscr{P}(\tilde{R})|$.*

(2) *$\sup\{[A_\gamma] \mid \gamma \in \Gamma\} = \bigvee_{\gamma \in \Gamma}[A_\gamma] = [\bigoplus_{\gamma \in \Gamma} A_\gamma]$; and $\exists \bigwedge_{\gamma \in \Gamma}[A_\gamma] \in \Xi(R)$.*

(3) *$\Xi(R)$ is a complete Boolean lattice (= distributive, with 0 and 1; and complemented): $[A] \vee [B] = [A \oplus B]$, $[A] \wedge [B] = [\mathrm{tr}_A B] = [\mathrm{tr}_B A]$; $0 = [(0)]$, $1 = [\tilde{R}]$; $[A]^C = [\mathrm{rej}_A \tilde{R}]$, $[A] \vee [A]^C = 1$, $[A] \wedge [A]^C = 0$.*

See 3.17 for some very special cases of the next theorem.

2.3. Module Decomposition Theorem. *Let M be any torsion free module and $\alpha \leqslant [M] \in \Xi(R)$. Define $M_\alpha = \sum\{U \mid U \leqslant M, [U] \leqslant \alpha\}$. Then*

(i) *$[M_\alpha] = \alpha$; hence M_α is the unique largest submodule of M of type α; $M_\alpha = \bar{M}_\alpha \leqslant M$; both $M_\alpha < M$ and $EM_\alpha < EM$ are fully invariant.*

Suppose that $\Gamma \subset \Xi(R)$ is any pairwise disjoint subset such that $\bigvee \Gamma = [M]$. Then

(ii) *$EM = E(\bigoplus\{M_\gamma \mid \gamma \in \Gamma\}) = \prod\{EM_\gamma \mid \gamma \in \Gamma\}$.*

Proof. Everything in 2.3 was established in [D; p. 63, 3.20] except the product representation in (ii), which follows from 1.15.

2.4. Definition. Let $\mathbf{A}$ be the category whose objects are all rings R and S with identity, and morphisms ϕ: $R \to S$ are ring homomorphisms satisfying: (a) $\phi 1_R = 1_S$; (b) $\phi R = S$; and (c) $I = \ker\phi < R_R$ is a right complement, i.e. with closed kernels. Dually $\mathbf{B}$ is the category of all complete Boolean lattices L_1 and L_2 whose morphisms ϕ^*: $L_1 \to L_2$ satisfy: (a*) $\phi^* 0 = 0$; (b*) ϕ^* is monic; and (c*) $\phi^* L_1 \subset L_2$ is convex, i.e. with closed images.

2.5. *Functoriality.* For any typical morphism $\phi \in \mathbf{A}$, for simplicity, take ϕ: $R \to S = R/I$ to be the natural projection with $I = \ker\phi \lhd R$. For any right S-module N, N_ϕ denotes the induced R-module, i.e. $NI = 0$. If N is S-torsion free we write $[N]_S \in \Xi(S)$, and define ϕ^*: $\Xi(S) \to \Xi(R)$ by $\phi^*([N]_S) = [N_\phi] \in \Xi(R)$. For a proof that ϕ^* is a well defined lattice homomorphism, see [D; pp. 65–68]. Define $\Xi(\phi) = \phi^*$. Then

(i) Ξ: $\mathbf{A} \to \mathbf{B}$ is a contravariant functor; ϕ^*: $\Xi(S) \to \Xi(R)$ is a zero preserving monic lattice homomorphism which preserves arbitrary suprema and infima.

(ii) $\phi^* \Xi(S) \subset \Xi(R)$ is a convex (and complete) sublattice, and hence $\Xi(R)$ is a lattice direct sum $\Xi(R) = \phi^* \Xi(S) \oplus A$ for a (unique) convex and complete sublattice $A \subset \Xi(R)$.

Among other things, the next theorem is a generalization of [D; p. 69, 4.12]. In the latter the results were proved for α equal to one of four special elements of $\Xi(R)$, while here below for every $\alpha \in \Xi(R)$. The proof proceeds as follows. The previous decomposition 2.3 is applied to $M = R/Z_2R$, and then with the aid of 1.11 and 1.13 the results lifted back up to R. Below (3) follows by use of 1.14. Since the ideal lattice $\mathscr{I}(R)$ defined below is the simplest and in many cases the only way to explicitly compute, describe, or determine $\Xi(R)$ for examples of rings, or for concrete classes of rings, several descriptions or characterizations are given for the elements $R_\alpha \in \mathscr{I}(R) \cong \Xi(R)$.

2.6. Ideal Lattice Theorem. *For $\alpha \in \Xi(R)$ define R_α in $Z_2R \subseteq R_\alpha < R$ by $(R/Z_2R)_\alpha = R_\alpha/Z_2R$. Let $c(\alpha) \in \Xi(R)$ be the complement of α, i.e. $\alpha \vee c(\alpha) = 1$, $\alpha \wedge c(\alpha) = 0$. Define $\mathscr{I}(R) = \{R_\alpha \mid \alpha \in \Xi(R)\}$. Then*

(1) *$\mathscr{I}(R)$ consists exactly of all J with $Z_2R \subseteq J \leqslant R$ such that $J \leqslant R$ is a right complement and $EJ \leqslant ER$ is fully invariant. Hence $J \lhd R$. For $J \in \mathscr{I}(R)$ define J^c to be maximal with respect to the properties*

$$Z_2R \subset J^c \leqslant R, \qquad J \cap J^c = Z_2R.$$

Then $J^c \in \mathscr{I}(R)$ and $\mathscr{I}(R)$ is a Boolean lattice under

$$J_1 \wedge J_2 = J_1 \cap J_2, \qquad J_1 \vee J_2 = (J_1 + J_2)^-;$$

$$J \vee J^c = R = 1, \qquad J \wedge J^c = Z_2R = 0, \qquad J, J_1, J_2 \in \mathscr{I}(R).$$

Then there is an isomorphism of Boolean lattices

(2) *$\Xi(R) \to \mathscr{I}(R)$, $\alpha \to R_\alpha$, where*

(i) *$R_{\alpha\wedge\beta} = R_\alpha \cap R_\beta$, $R_{\alpha\vee\beta} = (R_\alpha + R_\beta)^-$, $R_{c(\alpha)} = R_\alpha^c$, $R_\alpha \cap R_{c(\alpha)} = Z_2R$,*

(ii) *$\alpha = [R_\alpha/Z_2R]$, $c(\alpha) = [R/R_\alpha] = [R_{c(\alpha)}/Z_2R]$.*

(3) *For any $\alpha \in \Xi(R)$ let $C_\alpha \leqslant R$ be any right ideal maximal with respect to*

$$C_\alpha \cap Z_2R = 0, \qquad \alpha = [C_\alpha].$$

Then $R_\alpha = (Z_2R \oplus C_\alpha)^-$, $Z_2R \oplus C_\alpha \oplus C_{c(\alpha)} \leqslant R$.

(4) *$\forall c(\alpha) \in \Xi(R)$, $R_{c(\alpha)} \leqslant R$ is the* u n i q u e *smallest right ideal of R with respect to the following properties:*

(i) *$Z_2R \subseteq R_{c(\alpha)} = \bar{R}_{c(\alpha)} \leqslant R$;* (ii) *$[R/R_{c(\alpha)}] = \alpha$.*

(5) *$R_\alpha \cap R_{c(\alpha)} = Z_2R$, $R_\alpha + R_{c(\alpha)} \lessdot R$.*

The next corollary is obtained by combining the ideal lattice $\mathscr{I}(R)$ of the last theorem with the functoriality of Ξ. Later 2.7(ii) below will be used to show that $\Xi = \Xi_{\mathrm{I}} \oplus \Xi_{\mathrm{II}} \oplus \Xi_{\mathrm{III}}$ is a direct sum of subfunctors corresponding to Types I, II, and III.

2.7. Corollary. *For any $\alpha \in \Xi(R)$, let $c(\alpha)$, R_α, $R_{c(\alpha)} \lhd R$ be as in 2.6, and let $\Xi_\alpha(R)$ be the Boolean lattice $\Xi_\alpha(R) = \{\beta \in \Xi(R) \mid \beta \leqslant \alpha\}$ with largest element α and with relative complementation $\beta \to c(\beta) \wedge \alpha$. Let $\pi_{c(\alpha)}$: $R \to R/R_{c(\alpha)}$ be the natural projection and form the monic lattice homomorphism $\pi^*_{c(\alpha)}$: $\Xi(R/R_{c(\alpha)}) \to \Xi(R)$. Then*

(i) *$\Xi(R/R_{c(\alpha)}) \cong \Xi_\alpha(R) = \text{image}\, \pi^*_{c(\alpha)}$; i.e. the corestriction of $\pi^*_{c(\alpha)}$ to its image is a (complement preserving) isomorphism of Boolean lattices.*

Now suppose that $\Gamma \subset \Xi(R)$ is any pairwise disjoint subset whose supremum is $\bigvee \Gamma = 1 \in \Xi(R)$. Then there is a canonical isomorphism of Boolean lattices

(ii) *$\Xi(R) \cong \prod_{\alpha\in\Gamma} \Xi_\alpha(R)$; if $|\Gamma| < \infty$, then $\Xi(R) = \bigoplus_{\alpha\in\Gamma} \Xi_\alpha(R)$.*

(iii) *$\forall$ t.f. injective $M = M_R$*

$$M = E(\mathrm{tr}_M R_\alpha) \oplus E(\mathrm{tr}_M R_{c(\alpha)});$$

$$[M] \wedge \alpha = E(\mathrm{tr}_M R_\alpha), \qquad [M] \wedge c(\alpha) = [\mathrm{rej}_{E(R_\alpha/Z_2R)} M];$$

$$E(\mathrm{tr}_M R_{c(\alpha)}) = \mathrm{rej}_{E(R_\alpha/Z_2R)} M.$$

Proof. (i) and (ii). The proof of these conclusions can be modeled after [D; p. 69, 4.12], and is omitted.

(iii) From $(R_\alpha/Z_2R) \oplus (R_{c(\alpha)}/Z_2R) \lessdot R/Z_2R$ it follows that

$$M = E[\mathrm{tr}_M R] = E[\mathrm{tr}_M R/Z_2R] = E(\mathrm{tr}_M R_\alpha/Z_2R) \oplus E(\mathrm{tr}_M R_{c(\alpha)}/Z_2R)$$
$$= E(\mathrm{tr}_M R_\alpha) \oplus E(\mathrm{tr}_M R_{c(\alpha)}).$$

But in 1.5, the second complementary summand is unique because of full invariance, and hence we also have

$$M = E(\mathrm{tr}_M R_\alpha) \oplus (\mathrm{rej}_{E(R_\alpha/Z_2R)} M),$$

and so $E(\mathrm{tr}_M R_{c(\alpha)}) = \mathrm{rej}_{E(R_\alpha/Z_2R)} M$. The rest is clear.

3. Classes of modules. This section relates universal saturated classes of modules to certain points of $\Xi(R)$, and shows how such classes induce decompositions of Ξ as a direct sum of subfunctors. Type I, II, and III modules, and continuous, discrete, molecular, and bottomless modules are examples of universal saturated classes. This makes it possible to apply all the results of Section 2 and this section to these seven classes of modules. The definitions of Types I, II, III are reformulated without reference to idempotents so that later they can be applied to not necessarily injective torsion free modules.

3.1. Definition. Let $C = C_R$ and $D = D_R$; C is *continuous* if C contains no uniform submodules. D is *dicrete* if D contains an essential direct sum of uniform submodules. Let $A = A_R$ and $B = B_R$ be t.f. continuous. A is *atomic* if $[A] \in \Xi(R)$ is an atom. More generally, A is *molecular* if every nonzero submodule of A contains an atomic one. At the other end of the continuous module spectrum, B is *bottomless* if B contains no atomic submodules.

3.2. Definition. Let A, B, C, D generically represent t.f. modules. Define $\Xi_D(R) = \{[D] \mid D \text{ is discrete}\}$, $\Xi_C(R) = \{[C] \mid C \text{ is continuous}\}$, $\Xi_{CA}(R) = \{[A] \mid A \text{ is continuous molecular}\}$, and $\Xi_{CB}(R) = \{[B] \mid B \text{ is bottomless}\}$.

3.3. Definition. Let M be an injective torsion free module, and throughout this definition let N represent any arbitrary nonzero direct summand of M, i.e. $M = N \oplus N'$ for some $N' \leqslant M$, $N \neq 0$. Then M is *directly finite* if for any such N, N is not isomorphic to $N \oplus N$ ([GB; p. 16, 3.1(c)]). The module M is *abelian* if any isomorphic direct summands of M are equal ([GB; p. 12, 2.1]). Thus abelian implies directly finite.

The module M is *Type* I if every N contains a nonzero abelian submodule ([GB; p. 30, 5.1]). Secondly, M is of *Type* II if for any N, N is *not* abelian, and if every N contains a nonzero directly finite direct summand ([GB; p. 34, 5.5]). Thirdly, M is *Type* III if for every N, we have $N \cong N \oplus N$ ([GB; p. 38, 5.9]).

Our module M is *purely infinite* if M has no directly finite fully invariant direct summands ([GB; p. 40, p. 41, 6.2]). Clearly Type III implies purely infinite. Next, M is *Type* I_∞ (or *Type* II_∞) if M is Type I (or Type II respectively) and M is purely infinite ([GB; p. 42]). A module M is *Type* I_f (or *Type* II_f) if M is Type I (or Type II respectively) and directly finite.

3.4. Definition. A torsion free module M has any one of the properties in the last definition, if $\hat{M}$ does (directly finite, abelian, I, II, III, purely infinite, I_∞, II_∞, I_f, and II_f). Consequently, by 1.7, if M is purely infinite then $\hat{M}$ contains no directly finite fully invariant submodules.

Our next objective is to show that there is a simple and natural characterization of the elements of $\Xi(R)$ in terms of saturated classes, and to determine which of the above ten classes of modules in the last definition in some sense correspond to elements of $\Xi(R)$.

3.5. Definition. For any nonempty class of modules Δ whatsoever, define a complementary class $c(\Delta)$ by $c(\Delta) = \{W \mid W = W_R, \forall 0 \neq V \leqslant W, V \notin \Delta\}$. Furthermore, if M is a torsion free module (and only in this case) define an intrinsic submodule $M_\Delta \leqslant M$ by $M_\Delta = \sum\{U \mid U \leqslant M, U \in \Delta\}$, and similarly $M_{c(\Delta)} \leqslant M$. A class Δ is said to be *closed under essential extensions* if $V \in \Delta$, $V \lll W \Rightarrow W \in \Delta$; and *closed under essential submodules* if $W \in \Delta$, $V \lll W \Rightarrow \Delta$.

A nonempty class of torsion free modules Δ will be called a *saturated class* if Δ is closed under direct sums, injective hulls, submodules, and isomorphic copies (i.e. $V \in \Delta$, $V \cong W \Rightarrow W \in \Delta$).

3.6. Consequences. *For any class Δ of modules, closed under submodules, and any torsion free module M, the following hold:*

(i) *$c(\Delta)$ is closed under submodules, essential extensions, and in particular, under injective hulls.*

(ii) $\Delta \cap c(\Delta) = \{(0)\}$; $\Delta \subset c(c(\Delta))$.

(iii) *If Δ is closed under isomorphic copies, so is also $c(\Delta)$.*

(iv) $M_\Delta \cap M_{c(\Delta)} = 0 \Rightarrow M_\Delta \oplus M_{c(\Delta)} \lll M$.

Proof. Suppose that $M_\Delta \oplus M_{c(\Delta)} \oplus W \leqslant M$ for some $0 \neq W \leqslant M$. Since $M_{c(\Delta)} \cap W = 0$, $W \notin c(\Delta)$, and hence there exists a $0 \neq V < W$, $V \in \Delta$. But then $0 \neq V \subseteq M_\Delta \cap W = 0$ is a contradiction.

3.7. Lemma. *Suppose that Δ is a saturated class and M a torsion free module. Then*

(i) *Δ is closed under torsion free homomorphic images and arbitrary (not necessarily direct) sums.*

(ii) *$c(\Delta)$ is a saturated class.*

(iii) *$M_\Delta \oplus M_{c(\Delta)} \lll M$; $M_\Delta, M_{c(\Delta)} \leqslant M$ are fully invariant complement submodules with $M_\Delta \in \Delta$, $M_{c(\Delta)} \in c(\Delta)$.*

(iv) $c(c(\Delta)) = \Delta$.

Proof. (i) The proof of (i) is not difficult and is omitted.

(ii) In view of 3.6(i), (iii), it suffices to show that $c(\Delta)$ is closed under direct sums. If $A_\gamma \in c(\Delta)$, $\gamma \in \Gamma$, with $\bigoplus_\Gamma A_\gamma \notin c(\Delta)$, then there exists a $0 \neq V \leqslant \bigoplus_\Gamma A_\gamma$ where $V \in \Delta$. By 1.2, both V and some one single A_γ contain nonzero submodules which are isomorphic, a contradiction. Hence $c(\Delta)$ is saturated.

(iii) By 3.7(i) above, $M_\Delta \in \Delta$, $M_{c(\Delta)} \in c(\Delta)$, and $M_\Delta \cap M_{c(\Delta)} \in \Delta \cap c(\Delta) = \{(0)\}$. It now follows from 3.6(iv), 3.7(i), (ii) that $M_\Delta \oplus M_{c(\Delta)} \lll M$ where both submodules are fully invariant complements.

(iv) Always for any class Δ closed under submodules, $c(c(\Delta)) = \{A \mid A = A_R;$ $\forall 0 \neq B \leqslant A, \exists 0 \neq C \leqslant B, C \in \Delta\}$, and hence $\Delta \subseteq c(c(\Delta))$. By use of Zorn's lemma and the fact that Δ is closed under direct sums, injective hulls, and submodules, it now follows that $c(c(\Delta)) = \Delta$.

3.8. Remark. Except for 3.7(iii), the proofs of 3.6 and 3.7 did not use the torsion free hypothesis in Definition 3.5.

The next theorem says among other things that the saturated classes are precisely the unions of intervals $\bigcup\{x \mid 0 \leqslant x \leqslant y\}$, $y \in \Xi(R)$, of the lattice $\Xi(R)$.

3.9. Theorem. *Over any ring R, suppose that Δ is any (nonempty, well defined in ZFC) class of torsion free right R-modules which is closed under isomorphic copies (3.5). Let $c(\Delta)$, M_Δ, $M_{c(\Delta)}$, and $\Xi(R)$ be as in 3.5 and 2.1. Let A be any (torsion free) module defined by $\bigvee\{[C] \mid C \in \Delta\} = [A] \in \Xi(R)$.*

(1) *Then the following conditions are all equivalent.*

(a) $\Delta = \{C \mid [C] \leqslant [A]\}$.

(b) *Δ is a saturated class.*

(c) *$\forall$ torsion free module M, $M_\Delta \oplus M_{c(\Delta)} \lll M$ with $M_\Delta \in \Delta$, $M_{c(\Delta)} \in c(\Delta)$; and Δ is closed under injective hulls and essential submodules.*

(2) *Conversely, for any $[D] \in \Xi(R)$, $\{C \mid [C] \leqslant [D]\} = \bigcup\{[C] \mid [C] \leqslant [D]\}$ is a saturated class.*

Proof. Conclusions (2) and (1)(a)$\Rightarrow$(b) are obvious.

(1)(b)$\Rightarrow$(a). Since $\Xi(R)$ is a set, there is an indexed set of torsion free modules $C_\gamma \in \Delta$, $\gamma \in \Gamma$, such that $A = E(\bigoplus_\Gamma C_\gamma)$. Since Δ is saturated, we infer that $A \in \Delta$. From the latter it follows that $\Delta = \{C \mid \exists J$ such that $C \subset E(\bigoplus_J A)\}$.

(1)(b)$\Rightarrow$(c). This was shown in 3.7(iii).

(1)(c)$\Rightarrow$(b). First, for $W \in \Delta$ and $0 \neq V < W$, it has to be shown that $V \in \Delta$. By (1)(c), $W_\Delta \oplus W_{c(\Delta)} \lll W$, and from $W \in \Delta$ we have $W = W_\Delta$. Firstly, since $V < W$, also $V_{c(\Delta)} \subset W_{c(\Delta)} = 0$. By hypotheses (1)(c), secondly, $V_\Delta \in \Delta$; thirdly, $V \lll EV = E(V_\Delta) \in \Delta$; and fourthly, $V \in \Delta$.

Next, suppose that $A_\gamma \in \Delta$, $\gamma \in \Gamma$, is an indexed set of modules. Let $M = E(\bigoplus_\Gamma A_\gamma)$, and $M_\Delta \oplus M_{c(\Delta)} \lll M$. If $M_{c(\Delta)} \neq 0$ then by 1.2 we have for some $W \neq 0$ and $V \neq 0$ and some $\gamma \in \Gamma$

$$M_{c(\Delta)} > W \cong V < A_\gamma,$$

a contradiction. Hence $M_\Delta \lll M$. We now invoke the two hypotheses that $M_\Delta \in \Delta$ and that $E(M_\Delta) \in \Delta$ to conclude that $M \in \Delta$. Thus Δ is saturated.

3.10. Main Corollary. *For any ring R, let $\mathscr{S}$ denote the class consisting of all saturated classes of torsion free right R-modules. Furthermore, let Δ be any one of the following seven classes of right R-modules: Type I, II, III, continuous, discrete, continuous molecular, or bottomless (3.4 and 3.1). Then there exists a $J < R$ such that the following hold.*

(1) *$\mathscr{S}$ is a set; $|\mathscr{S}| \leqslant 2^{|\mathscr{P}(R)|}$.*

(2) *$\Xi(R) \to \mathscr{S}$, $x \to \bigcup\{y \mid 0 \leqslant y \leqslant x\}$, is a bijection.*

(3) Δ *is a saturated class; there exists a* $J \leqslant R$ *such that in* (2) $[J/Z_2R] \to \Delta$ *and such that the following hold:*

(4) (i) $Z_2R \subset J$, $J < R_R$ *is a right complement,* $J/Z_2R \in \mathscr{S}$.

(ii) $\Delta = \{C \mid [C] \leqslant [J/Z_2R]\}$.

(iii) *Unique: if* $I < R$ *satisfies* (i) *and* (ii), *then* $I \subset J$.

(iv) $J < R$ *and* $EJ < ER$ *are fully invariant, in particular,* $J \lhd R$.

3.11. Observations. (1) Purely infinite modules need not be closed under submodules, and hence they do not form a saturated class.

(2) By 3.10 and 3.7(i) it follows that if Δ is any saturated class, and $\{A_\gamma \mid \gamma \in \Gamma\} \subset \Delta$ a subset, then $\sum\{A_\gamma \mid \gamma \in \Gamma\} \in \Delta$. In particular, this holds for Types I, II, and III, and generalizes three separate proofs in [GB] for merely direct sums of Type I ([GB; pp. 31–32, 5.2]), Type II ([GB; p. 35, 5.6]), and Type III ([GB; pp. 38–39, 5.10]).

Our next objective is to show that certain saturated classes induce decompositions of the functor Ξ as a direct sum of subfunctors.

3.12. *Suppose that for any ring R with identity there is a saturated class* $\Delta(R)$ *of torsion free right R-modules. Define* $\delta(R) = \bigvee\{[C] \mid C \in \Delta(R)\} \in \Xi(R)$ *so that* $\Delta(R) = \{C \mid [C] \leqslant \delta(R)\}$ *as before in* 3.9. *Let* $\phi\colon R \to S$ *be in the category* **A**, *and thus* $\phi^*\colon \Xi(S) \to \Xi(R)$. *Then the following are equivalent:*

(i) $\forall S$*-torsion free* $N = N_S$, *if* $N \in \Delta(S)$, *then* $N_\phi \in \Delta(R)$.

(ii) $\phi^*\delta(S) \leqslant \delta(R)$.

Proof. (i) $\Rightarrow$ (ii). The proof requires the use of the fact from [D; p. 68, 4.10(i)] that ϕ^* commutes with arbitrary suprema,

$$\phi^*\delta(S) = \bigvee\{\phi^*[N_S]_S \mid N \in \Delta(S)\} \leqslant \delta(R)$$

because $N_\phi \in \Delta(R)$ implies that $\phi^*[N_S]_S = [N_\phi] \leqslant \delta(R)$ for all N.

(ii) $\Rightarrow$ (i). Since ϕ^* is order preserving we have

$$\bigvee_{N \in \Delta(S)} [N_\phi] = \bigvee_{N \in \Delta(S)} \phi^*([N]_S) \leqslant \phi^*\Big(\bigvee_{N \in \Delta(S)} [N]_S\Big) = \phi^*\delta(S).$$

If $\phi^*\delta(S) \leqslant \delta(R)$, then for any N_ϕ, $[N_\phi] \leqslant \delta(R)$ and hence $N_\phi \in \Delta(R)$.

3.13. Definition. If for every ring R with identity $\Delta(R)$ is a saturated class of torsion free right R-modules, and if for all R, S and all $\phi\colon R \to S$ in the category **A** the above two conditions 3.12(i), (ii) hold, then (Δ, δ) or Δ is called a *universal saturated class.* Note that Δ is completely determined by δ. In this case $(c\Delta, c\delta)$ is another universal saturated class, where $(c\Delta)(R) = c(\Delta(R))$ and $(c\delta)(R) = c(\delta(R)) \in \Xi(R)$ by 3.7(ii).

3.14. Theorem. *Let R denote any ring with* $1 \in R$. *Suppose that* (Δ, δ), $(\Delta_1, \delta_1), \ldots, (\Delta_n, \delta_n)$ *are universal saturated classes, and for any* Δ, $(c\Delta, c\delta)$ *is as in* 3.13. *Define* $\Xi_\Delta(R) = \{[C] \mid C \in \Delta(R)\}$, *and let* $\Xi_{\delta(R)}(R)$ *be as in* 2.7. *Let* $Z_2R \subset R_{\delta(R)}$ *be as in* 2.6(2) *with* $\delta(R) = [R_{\delta(R)}/Z_2R]$. *Then*

(1) $\Xi_\Delta(R) = \Xi_{\delta(R)}(R)$; Ξ_Δ *is a subfunctor of* Ξ.

(2) $\Xi = \Xi_\Delta \oplus \Xi_{c\Delta}$ *is a direct sum of subfunctors of* Ξ.

(3) *The following three conditions are equivalent:*

(i) $\Xi = \Xi_{\Delta_1} \oplus \ldots \oplus \Xi_{\Delta_n}$ *is a direct sum of subfunctors.*

(ii) $\forall R$, $\forall$*t.f.* M_R, $M_{\Delta_1(R)} \oplus \ldots \oplus M_{\Delta_n(R)} \leqslant M$.

(iii) $\forall R$, $\delta_1(R) \vee \ldots \vee \delta_n(R) = 1$, $\delta_i(R) \wedge \delta_j(R) = 0$, $i \neq j$.

(4) *Now assume that the above three conditions* (3)(i)–(iii) *hold. Then for any* $1 \leqslant j \leqslant n$,

$$R_{c(\delta_j(R))} = \Big[\sum_{\substack{i=1\\ i\neq j}}^{n} R_{\delta_i(R)}\Big]^- \lhd R; \quad R_{\delta_j(R)} \cap R_{c(\delta_j(R))} = Z_2R, \quad R_{\delta_j(R)} + R_{c(\delta_j(R))} \lll R.$$

There is a canonical natural lattice (equivalently Boolean ring) isomorphism

$$\Xi_{\Delta_j}(R) \cong \Xi(R/R_{c(\delta_j(R))}).$$

Proof. (1) First upon combining 3.9(1)(a) and 3.12 we get that $\Xi_\Delta(R) = \Xi_{\delta(R)}(R)$ is actually a Boolean lattice. If $\varphi\colon R \to S$ is any ring morphism in **A**, then by 3.12(i), $\varphi^*\Xi_\Delta(S) \subseteq \Xi_\Delta(R)$. Thus Ξ_Δ is a subfunctor of Ξ.

(2) In view of (1) above, and by 2.7(ii) (with $\Gamma = \{\delta(R), (c\delta)(R)\}$), we have $\Xi(R) = \Xi_{\delta(R)}(R) \oplus \Xi_{(c\delta)(R)}(R) = \Xi_\Delta(R) \oplus \Xi_{c\Delta}(R)$. By 3.12, 3.7(ii), and 3.13 it follows that $\varphi^*\Xi_{c\Delta}(S) \subseteq \Xi_{c\Delta}(R)$, and thus $\Xi = \Xi_\Delta \oplus \Xi_{c\Delta}$ is a direct sum of subfunctors.

(3)(i) $\Rightarrow$ (iii). By 3.14(1) above, $\Xi_{\delta_1(R)}(R) \oplus \ldots \oplus \Xi_{\delta_n(R)}(R) = \Xi(R)$ is a direct sum. If $\delta_i(R) \wedge \delta_j(R) \neq 0$ for some $i \neq j$, it is easily seen that this sum would not be direct.

(iii) $\Rightarrow$ (ii). By the definitions of $M_{\delta(R)}$ and $M_{\Delta(R)}$ (in 2.3 and 3.5), from 2.7(ii) it follows that $M_{\Delta(R)} = M_{\delta(R)}$ for any (Δ, δ). With $\Gamma = \{\delta_1(R), \ldots, \delta_n(R)\}$ in 2.3, we get $M_{\delta_1(R)} \oplus \ldots \oplus M_{\delta_n(R)} \leqslant M$.

(ii) $\Rightarrow$ (iii). First, by hypothesis each (Δ_i, δ_i) is a universal saturated class, which entails that for any $\varphi\colon R \to S$ in **A**, $\varphi^*\Xi_{\Delta_i}(S) \subset \Xi_{\Delta_i}(R)$. Set $\beta = \delta_1(R) \vee \ldots \vee \delta_n(R) \in \Xi(R)$. Since always $M_{\Delta(R)} = M_{\delta(R)}$, from 2.3 with $\Gamma = \{\delta_1(R), \ldots, \delta_n(R), c(\beta)\}$ we get that for all t.f. M_R, $M_{\delta_1(R)} \oplus \ldots \oplus M_{\delta_n(R)} \oplus M_{c(\beta)} \leqslant M$. By our hypothesis (ii), this implies that $c(\beta) = 0$, and $\delta_1(R) \vee \ldots \vee \delta_n(R) = 1$. If for some $i \neq j$, $\delta_i(R) \wedge \delta_j(R) \neq 0$, take any M with $0 \neq [M] = \delta_i(R) \wedge \delta_j(R)$. Then 2.3 implies that $M_{\delta_i(R)} = M_{\delta_j(R)} = M \neq 0$, which is a contradiction. Thus $\delta_i(R) \wedge \delta_j(R) = 0$.

(4) Set

$$J = \Big[\sum_{\substack{i=1\\ i\neq j}}^{n} R_{\delta_i(R)}\Big]^-.$$

Thus by 2.6(2)(ii),

$$\left.\begin{array}{l} R_{\delta_j(R)}/Z_2R \oplus J/Z_2R \leqslant R/Z_2R \\ {[R_{\delta_j(R)}/Z_2R] \vee [J/Z_2R] = 1} \\ \delta_j(R) = [R_{\delta_j(R)}/Z_2R] \end{array}\right\} \Rightarrow [J/Z_2R] = (c\delta_j)(R).$$

By 2.3(i) applied to $M = R/Z_2R$ we get first $J \subseteq R_{c\delta_j(R)}$. Let $J \oplus C \leqslant R_{c\delta_j(R)}$ for some torsion free $C \leqslant R$. It then follows from the above that $(c\delta_j)(R) \vee [C] = (c\delta_j)(R)$. Thus $[C] = 0$ and hence $C = 0$. By 1.13, $J \leqslant R$ is a right complement. Hence $J = R_{(c\delta_j)(R)}$.

The next corollary generalizes [D; p. 62, 3.15(iv)] where it was shown that we get lattice direct sums if we apply the functors below to any ring R.

3.15. COROLLARY. *$\Xi = \Xi_C \oplus \Xi_D = \Xi_{CA} \oplus \Xi_{CB} \oplus \Xi_D$ is a direct sum of subfunctors of* Ξ (see 3.2).

In view of their importance, it seems useful to summarize what the previous theory says about Types I, II, and III. The next corollary shows among other things that every associative ring with identity R contains certain six unique ideals. It would be interesting to investigate and relate the algebraic properties of R to those six ideals.

3.16. COROLLARY. *Let R be any ring with $1 \in R$. For (Δ, δ) equal to any one of Types* I, II, *or* III, *denote the corresponding $R_{\delta(R)}$ and $R_{(c\delta)(R)}$ given by* 3.14(4) *by R_{I}, R_{II}, R_{III}, $R_{c(\mathrm{III})}, \ldots$ etc. For $\Delta_1, \Delta_2, \Delta_3 =$* I, II, III, *let $\Xi = \Xi_{\mathrm{I}} \oplus \Xi_{\mathrm{II}} \oplus \Xi_{\mathrm{III}}$ denote the decomposition of Ξ as a direct sum of subfunctors. Then $\Xi_{\mathrm{III}}(R) = \{[M] \mid M$ is t.f. Type* III$\}$ *and similarly for* I *and* II. *Then the following hold for any one of* I, II, *or* III, *say* III.

(1) (i) $Z_2 R \subseteq R_{\mathrm{III}}$, *$R_{\mathrm{III}} < R$ is a right complement,* $R_{\mathrm{III}}/Z_2 R \in \mathrm{III}$.

(ii) *$\forall$ t.f. $M_R \in$ III, there exists an embedding $M \subset E(\bigoplus_\Gamma R_{\mathrm{III}}/Z_2 R)$ for some Γ. i.e.* $[M] \leqslant [R_{\mathrm{III}}/Z_2 R]$.

(iii) *Unique: If $L < R$ satisfies* (i) *and* (ii), *then* $L \subseteq R_{\mathrm{III}}$.

(iv) *$R_{\mathrm{III}} < R$ and $E(R_{\mathrm{III}}) < ER$ are fully invariant right R-modules, hence* $R_{\mathrm{III}} \lhd R$.

(2) $R_{c(\mathrm{III})} = (R_{\mathrm{I}} + R_{\mathrm{II}})^{-}$, $R_{\mathrm{III}} \cap R_{c(\mathrm{III})} = Z_2 R$, $R_{\mathrm{III}} + R_{c(\mathrm{III})} \leqslant\!\!\cdot\; R$.

(3) *$R_{c(\mathrm{III})} \leqslant R$ is the unique smallest right ideal with respect to:*

(i) $Z_2 R \subseteq R_{c(\mathrm{III})} = \bar{R}_{c(\mathrm{III})} \leqslant R$; (ii) $[R/R_{c(\mathrm{III})}] = [R_{\mathrm{III}}/Z_2 R]$.

(4) $R_{\mathrm{I}} + R_{\mathrm{II}} + R_{\mathrm{III}} \leqslant\!\!\cdot\; R$; $R_{c(\mathrm{III})} R_{\mathrm{III}} \subseteq Z_2 R$; $R_{\mathrm{III}} R_{c(\mathrm{III})} \subseteq Z_2 R$.

(5) $\Xi_D(R) \subseteq \Xi_{\mathrm{I}}(R)$; $\Xi_{\mathrm{II}}(R) \oplus \Xi_{\mathrm{III}}(R) \subseteq \Xi_C(R)$; *thus $\Xi_{\mathrm{II}}(R) \oplus \Xi_{\mathrm{III}}(R)$ is atomless.*

We combine 2.3, 3.7, and 3.14 to show how the classical direct sum decomposition into Types I, II, and III is a very special case of a general phenomenon.

3.17. MAIN COROLLARY. *Suppose that $(\Delta_1, \delta_1), \ldots, (\Delta_n, \delta_n)$ are any saturated universal classes satisfying the conditions* 3.14(3) *and that M is any t.f. R-module. Then there exist submodules $M_i < M$ such that*

(1) (a) $M_1 \oplus \ldots \oplus M_n \leqslant\!\!\cdot\; M$; (b) *$M_i = \bar{M}_i < M$ are right complements; and* (c) $M_i \in \Delta$.

(2) *Unique: subject to* (1) *the M_i are unique. Furthermore, $M_i = M_{\Delta_i(R)}$ and $M_i < M$ is fully invariant.*

(3) $\hat{M} = \hat{M}_1 \oplus \ldots \oplus \hat{M}_n$; $\hat{M}_i = E(M_{\Delta_i(R)}) = (EM)_{\Delta_i(R)}$ *for* $i = 1, 2, \ldots, n$.

(4) *Applications.* (a) *If (Δ_i, δ_i) are Types* I, II, *and* III, *then*

$$M_{\mathrm{I}} \oplus M_{\mathrm{II}} \oplus M_{\mathrm{III}} \leqslant\!\!\cdot\; M \leqslant\!\!\cdot\; \hat{M} = \hat{M}_{\mathrm{I}} \oplus \hat{M}_{\mathrm{II}} \oplus \hat{M}_{\mathrm{III}}$$

where M_J, $\hat{M}_J$ are Type J for $J =$ I, II, III.

(b) *If (Δ_i, δ_i) are continuous molecular (CA), continuous bottomless (CB), and the discrete (D) modules, then similarly*

$$M_{CA} \oplus M_{CB} \oplus M_D \leqslant\!\!\cdot\; M \leqslant\!\!\cdot\; \hat{M} = \hat{M}_{CA} \oplus \hat{M}_{CB} \oplus \hat{M}_D.$$

4. Characterizations of types. Various internal module-theoretic conditions are found which will tell us whether a module is atomic, continuous molecular, bottomless, or has some of the eight properties listed in Definition 3.4.

4.1. Atomic. *For a t.f. module A, $[A] \in \Xi(R)$ is an atom $\Leftrightarrow \forall x, y \in A \backslash \{0\}$,* $\mathrm{Hom}_R(E(xR), E(yR)) \neq 0$.

Proof. $\Rightarrow$ This is clear.

$\Leftarrow$ Take any index set Γ of cardinality $|\Gamma| > |\hat{A}|$. We will show that for any $0 \neq y \in A$, there exists an embedding $A \subset E(\bigoplus_\Gamma yR)$. Let $\mathscr{S} = \{(K, f), (L, g), \ldots\}$ where $K \leqslant \hat{A}$ and $f\colon K \to E(\bigoplus_\Gamma yR)$ is monic. Define $(K, f) \prec (L, g)$ if $K \subset L$ and g restricts to $g|K = f$. By Zorn's lemma, $\mathscr{S}$ has a maximal element $(K, f) \in \mathscr{S}$. There exists a $\gamma \in \Gamma$ such that $fK \cap E(yR)_\gamma = 0$, for otherwise $|K| = |fK| \geqslant |\Gamma| > |A|$. If $K \neq A$, then $K \oplus E(xR) \leqslant E(\bigoplus_\Gamma yR)$ for some $0 \neq x \in A$. Thus far we did not use the fact that A is torsion free, but now it is needed to deduce from $\mathrm{Hom}_R(E(xR), E(yR)) \neq 0$ that there exists a submodule $0 \neq V \leqslant E(xR)$ and a monomorphism $h\colon V \to E(yR)_\gamma$. But then the monic map $g\colon K \oplus V \to f(K) \oplus E(yR)_\gamma \leqslant E(\bigoplus_\Gamma yR)$, where $g|K = f$ and $g|V = h$, contradicts the maximality of $(K, f) \in \mathscr{S}$. Thus $K = \hat{A}$.

4.2. Abelian. *Let A_R be torsion free continuous. Then A is abelian $\Leftrightarrow \nexists 0 \neq x, 0 \neq y \in A \backslash \{0\}$ such that $xR \cong yR$ but $xR \cap yR = 0$.*

Proof. $\Rightarrow$ If not, then $\hat{A} = E(xR) \oplus E(yR) \oplus A'$ for some $A' \leqslant A$, and $E(xR) \cong E(yR)$ contradicts 3.3.

$\Leftarrow$ If A is not abelian, then $\hat{A} = K_1 \oplus A' = K_2 \oplus A''$ where $0 \neq K_1 \cong K_2$. By [Bu], $K_1 \backslash K_2 \neq \emptyset$ and $K_2 \backslash K_1 \neq \emptyset$. Let $k \in K_1 \backslash K_2$. Then by [D; p. 53, 1.3(0)], $k^{-1} K_2 <\!\!\!\not\,\cdot\; R$. So there exists $0 \neq B \leqslant R$ such that $k^{-1} K_2 \oplus B \leqslant\!\!\cdot\; R$. Then $kB \cap K_2 = 0$. Let $0 \neq b \in B$, and let $f\colon K_1 \to K_2$ be an isomorphism. Then $x = kb \in K_1$, $y = fkb \in K_2$, $xR \cap yR = 0$, $xR \cong (fx)R = yR$, and $x^\perp = y^\perp$, and hence $xR \cong yR$. This is a contradiction, and hence A is abelian.

Note that (ii) below says that $\hat{M}$ is a direct sum of a Type II and Type III module.

4.3. Continuous molecular. *Let M be a torsion free continuous molecular module. Then*

(i) *$\forall 0 \neq V \leqslant M$, $\exists P \oplus Q \leqslant\!\!\cdot\; V$ with $P \cong Q$.*

(ii) *M does not contain any nonzero Type* I *submodules.*

Proof. (i) By Zorn's lemma, there exists a maximal independent family $\{P_i \oplus Q_i\}$ of submodules of V with $P_i \cong Q_i$. Let $[\bigoplus (P_i \oplus Q_i)] \oplus D \leqslant\!\!\cdot\; V$ and set $P = \bigoplus_i P_i \cong \bigoplus_i Q_i = Q$.

If $D \neq 0$, there exists a continuous atomic submodule $0 \neq A \leqslant D$. Take any $0 \neq a$, $0 \neq b \in A$ with $aR \cap bR = 0$. There is an index set Γ such that $bR \subseteq E(\bigoplus_\Gamma aR)$. From this it follows by 1.2 that $(br_0)^\perp = (ar_1)^\perp$ for some $r_0, r_1 \in R$. Thus $ar_1 R \cong br_0 R$ with $P \oplus Q \oplus ar_1 R \oplus br_0 R \leqslant V$ contradicts the maximality of $P \oplus Q$. Hence $D = 0$ and $P \oplus Q \leqslant\!\!\cdot\; V$, $P \cong Q$.

(ii) By 4.3(i) and 4.2, M cannot contain a nonzero abelian submodule, and hence no Type I submodules.

The next criterion is particularly well adapted to showing that a module M is *not* directly finite. In the proof below, B_1 and B_2 are defined nonsymmetrically.

4.4. Directly finite. *A t.f. module is directly finite* $\Leftrightarrow \nexists$ *a triple* $B_1, B_2, C \leqslant M$ *of nonzero submodules of* M *such that* $B_1 \cong B_2 \cong C$, $C \cap (B_1 \oplus B_2) \leqslant B_1 \oplus B_2$, *and* $C \cap (B_1 \oplus B_2) \leqslant C$.

Proof. $\Rightarrow$ If not, then $\hat{B}_1 \cong \hat{B}_2 \cong \hat{C} = \hat{B}_1 \oplus \hat{B}_2$ is a contradiction.

$\Leftarrow$ If not, then $\hat{M} = V \oplus M'$ with $0 \neq V = V_1 \oplus V_2$, $V_1 \cong V_2 \cong V$. Let $g\colon V_1 \to V$ and $f\colon V_1 \to V_2$ be isomorphisms. Set

$$B_1 = M \cap [f^{-1}(V_2 \cap M)] \cap g^{-1}[(V_1 \cap M) \oplus (V_2 \cap M)].$$

Since $V_i \cap M \leqslant V_i$, $(V_1 \cap M) \oplus (V_2 \oplus M) \leqslant V_1 \oplus V_2 = V$. For t.f. modules the inverse image of a large module is large. Hence $B_1 \leqslant V_1$. By [D; p. 55, Lemma 1.8(i)], $gB_1 \leqslant V$, and $fB_1 \leqslant V_2$. Note that $gB_1 \subseteq (V_1 \cap M) \oplus (V_2 \cap M)$ and that $fB_1 \subseteq V_2 \cap M$. Set $C = gB_1 \leqslant V \cap M$ and $B_2 = fB_1$. Then $C = gB_1 \cong B_1 \cong fB_1 = B_2$ satisfy the required two conditions as follows:

$$\left.\begin{aligned} B_1 \oplus B_2 &\leqslant (V_1 \cap M) \oplus (V_2 \cap M) \\ C &\leqslant (V_1 \cap M) \oplus (V_2 \cap M) \\ C \cap (B_1 \oplus B_2) &\leqslant (V_1 \cap M) \oplus (V_2 \cap M) \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} &C \cap (B_1 \oplus B_2) \leqslant C \\ &\text{and} \\ &C \cap (B_1 \oplus B_2) \leqslant B_1 \oplus B_2. \end{aligned}\right.$$

Note that a sometimes useful equivalent way to reformulate the last two conditions is that $(C \cap B_1) \oplus (C \cap B_2) \leqslant C$ and $(C \cap B_1) \oplus (C \cap B_2) \leqslant B_1 \oplus B_2$. This follows by intersecting both sides of $C \cap (B_1 \oplus B_2) \leqslant B_1 \oplus B_2$ with B_i to get $B_i \cap C \leqslant B_i$.

4.5. *Let M be any torsion free module and let $0 \neq W = \bar{W} \leqslant M$ represent any arbitrary complement submodule. Then*

(1) (i) *M is Type* I $\Leftrightarrow \forall W, \exists 0 \neq A \leqslant W$, *$A$ is abelian* (4.2).

(ii) *M is discrete* $\Rightarrow$ *M is Type* I.

(iii) *M is continuous Type* I $\Rightarrow$ *M is bottomless.*

(2) *M is Type* II $\Leftrightarrow \forall W, \exists 0 \neq V \leqslant W$, *$V$ is directly finite* (4.4), *and M contains no nonzero abelian submodules* (4.2).

(3) *M is Type* III $\Leftrightarrow \forall W, \exists B_1 \oplus B_2 \leqslant W, \exists C \leqslant W$ *such that* $B_1 \cong B_2 \cong C$.

Proof. (1)(i), (ii), and (2). These are clear.

(1)(iii) If not there exists a continuous atomic module $0 \neq A < M$. Thus there exist $0 \neq a$, $0 \neq b \in A$ with $aR \cap bR = 0$, and an embedding $A \subset E(\bigoplus_\Gamma aR)$. By 1.2, there are $r_0, r_1 \in R$ with $(ar_0)^\perp = (br_1)^\perp$. Then $x = ar_0$, $y = br_1$ violate 4.1. Thus M is bottomless.

(3) $\Rightarrow$ Let $\mathcal{S} = \{\langle B_1, B_2, C\rangle \mid B_1 \oplus B_2 \leqslant W, C \leqslant W, 0 \neq B_1 \cong B_2 \cong C, (B_1 \oplus B_2) \cap C \leqslant B_1 \oplus B_2, (B_1 \oplus B_2) \cap C \leqslant C\}$. By 4.4, $\mathcal{S} \neq \emptyset$. Partially order $\mathcal{S}$ by $\langle B_1', B_2', C'\rangle \prec \langle B_1, B_2, C\rangle$ if $B_1' \subseteq B_1$, $B_2' \subseteq B_2$, and $C' \subseteq C$. By Zorn's lemma, let $\langle B_1, B_2, C\rangle \in \mathcal{S}$ be maximal. Set $K = [C \cap (B_1 \oplus B_2)]^- \leqslant W$. If $K \neq W$, let $K \oplus D \leqslant W$, $0 \neq D < W$. By hypothesis, there exist $0 \neq B_1' \oplus B_2' \leqslant D$, $C' \leqslant D$ as in 4.5(3) with $B_1' \cong B_2' \cong C'$. Then $\langle B_1 \oplus B_1', B_2 \oplus B_2', C \oplus C'\rangle$ contradicts the maximality of $\langle B_1, B_2, C\rangle$.

5. Examples. The next two examples illustrate how our characterizations of Type I and III modules can be applied in concrete cases.

5.1. Example. *Suppose that R_j, $j \in J$, is a family of torsion free rings and $R = \prod R_j$. Then*

(i) $\forall j$, $R_j \in \mathrm{I} \Rightarrow R_R \in \mathrm{I}$.

(ii) $\forall j$, $R_j \in \mathrm{III} \Rightarrow R_R \in \mathrm{III}$.

Proof. (i) For any $0 \neq wR \leqslant R$, $w = (w_j) \in R$, there is a cyclic R_j-abelian submodule $a_j R_j \subset w_j R_j = wR_j \subset R_j$, where $a_j \neq 0$ whenever $w_j \neq 0$. Then $a = (a_j) \in wR$. Suppose that $0 \neq x$, $0 \neq y \in aR$ with $xR \cap yR = 0$, but $xR \cong yR$. Take any j with $x_j \neq 0$. Since $xRR_j = x_j R \subset a_j R_j$, necessarily $x_j R \cap y_j R = 0$, and also $0 \neq x_j R_j \cong y_j R_j$ are isomorphic as R- or R_j-modules. This contradicts that $a_j R_j$ is an R_j-abelian module. Therefore $0 \neq aR \leqslant wR$ is an abelian submodule of wR. Thus $R_R \in \mathrm{I}$.

(ii) Let $\pi_j\colon R \to R_j$ be the projection. Take $0 \neq W \leqslant R$. For any j, $\pi_j W \neq 0$ is a directly infinite R_j-module. There exist two large R_j-submodules of $\pi_j W$ as in 4.5(3), where $B_{1j} \oplus B_{2j} \subseteq \pi_j W$ and $C_j \subseteq \pi_j W$ are both R_j-large, with $B_{1j} \cong B_{2j} \cong C_j$. If $\pi_j W \neq 0$, also $C_j \neq 0$. Otherwise when $\pi_j W = 0$, also $C_j = 0$. Set $B_1 = \bigoplus B_{1j}$, $B_2 = \bigoplus B_{2j}$, and $C = \bigoplus C_j$. Then $B_1 \cong B_2 \cong C$, with $B_1 \oplus B_2 \leqslant W$ and $C \leqslant W$. By 4.5(3), R is Type III.

A class of continuous bottomless rings is constructed below, and it is also determined when these are Type I or III. It is known ([D; p. 75, 6.3]) that a domain R that is not right Ore is continuous atomic, and it is easy to see ([D; p. 75, 6.4]) that then R_R is Type III. Atomless Boolean rings are examples of torsion free continuous bottomless Type I_f rings ([D; p. 77, 6.9]). Concrete atomless Boolean rings are given in [P; p. 895, Thm. 3.3], [S; pp. 105–107, (A, B)], [H; p. 96, (4)], and [GHKLMS; p. 113, (3)]. By use of the above in the next construction, it is easy to construct very concrete examples of bottomless Type I and Type III rings.

5.2. Construction. *For any family of rings with identity R_j, $j \in J$, form $P = \prod R_j$, $S = \bigoplus R_j$, and set $R = P/S$. Then*

(i) *R_R is torsion free.*

(ii) *R_R is continuous bottomless.*

If now in addition all the R_j, $j \in J$, are countable domains, then

(iii) $|\{j \in J \mid R_j \notin \mathrm{I}\}| < \infty \Leftrightarrow R_R \in \mathrm{I}$.

(iv) $|\{j \in J \mid R_j \notin \mathrm{III}\}| < \infty \Leftrightarrow R_R \in \mathrm{III}$.

Proof. First it will be shown that for any $\bar{a} = a + S$, $\bar{b} \in R$, $a = (a_j) \in P$, and any R-map $f\colon \bar{a}R \to \bar{b}R$ with $\bar{b} = f\bar{a}$, there exists a coset representative $b = (b_j) \in P$ with $\bar{b} = b + S$ such that $\operatorname{supp} b = \{j \in J \mid b_j \neq 0\} \subseteq \operatorname{supp} a$. For any subset $Y \subset J$, χ_Y will denote the characteristic function of Y. First take any $b \in P$ with $\bar{b} = b + S$, and set $Y = \operatorname{supp} b \setminus [\operatorname{supp} a \cap \operatorname{supp} b]$. Then $a\chi_Y = 0$, $\overline{a\chi_Y} = 0$, and hence $\bar{b} = f[\bar{a} - \overline{a\chi_Y}] = [b(1 - \chi_Y)]^-$. Consequently $\bar{b} = b(1 - \chi_Y) + S$ and $\operatorname{supp}[b(1 - \chi_Y)] = (J \setminus Y) \cap \operatorname{supp} b = \operatorname{supp} a \cap \operatorname{supp} b$ as required.

(i) For $0 \neq \bar{a} \in R$, define $a^{-1}0 = \{p \in P \mid ap = 0\}$. Since $\bar{a}^{\perp} = (a^{-1}0+S)/S = [(1-\chi_{\mathrm{supp}\,a})P+S)]/S$, and since $\bar{a}^{\perp} \oplus [(\chi_{\mathrm{supp}\,a}P+S)/S] = R$, we have $\bar{a}^{\perp} \leqslant R$ iff $|\mathrm{supp}\,a| < \infty$. Thus $ZR = 0$.

(ii) If R_R is not continuous bottomless, then there exists a continuous atomic submodule $0 \neq \bar{a}R$ for some $\bar{a} = a+S \in R$. Partition $\mathrm{supp}\,a = Y \cup Z$, $Y \cap Z = \emptyset$, and $|Y| = |Z| = \infty$. Since $0 \neq \overline{a\chi_Y}$, $0 \neq \overline{a\chi_Z} \in \bar{a}R$ and the latter is atomic, it follows that $\overline{a\chi_Y}R \subseteq E(\bigoplus_{\Gamma} \overline{a\chi_Z}R)$ for some Γ, and by 1.2, that $(\overline{a\chi_Y}s)^{\perp} = (\overline{a\chi_Z}r)^{\perp}$ for some $r, s \in R$ with $\overline{a\chi_Z}r \neq 0$. But then $\overline{a\chi_Y}\bar{\chi}_Z = 0$ and $\overline{a\chi_Z}s\bar{\chi}_Z = \overline{a\chi_Z}r \neq 0$ is a contradiction. Hence R_R is continuous bottomless.

(iii) Set $Y = \{j \in J \mid R_j \notin \mathrm{I}\}$. $\Rightarrow$ If $R_R \notin \mathrm{I}$, then there exist $\bar{a} = a+S$, $\bar{b} = b+S \in R$ with $0 \neq \bar{a}R \cong \bar{b}R$ and with $\bar{a}R \cap \bar{b}R = 0$. But in general

$$\bar{a}R \cap \bar{b}R = 0 \Leftrightarrow |\{j \mid a_jR_j \cap b_jR_j \neq 0\}| < \infty.$$

By a change of coset representatives a and b of $\bar{a}$ and $\bar{b}$ we may assume that the latter set is empty. By the observation at the beginning of the proof, we may also assume that $\mathrm{supp}\,a = \mathrm{supp}\,b$. Thus for all $j \in \mathrm{supp}\,a \cap \mathrm{supp}\,b$, $R_j \cong a_jR_j \cong b_jR_j$ and also $a_jR_j \cap b_jR_j = 0$. This implies that $R_j \notin \mathrm{I}$. Hence $\mathrm{supp}\,a \subseteq Y$. This contradicts that $\bar{a} \neq 0$. Thus $R_R \in \mathrm{I}$.

(iii) $\Leftarrow$ Suppose that $|Y| = \infty$. For each $j \in Y$ choose $0 \neq a_jR_j \cong b_jR_j$ with $a_jR \cap b_jR = 0$. Let π_j: $P \to R_j$ be the projection, and define elements $a, b \in P$ by $\mathrm{supp}\,a = \mathrm{supp}\,b = Y$, and with $\pi_j a = a_j$, $\pi_j b = b_j$ for all $j \in J$. Then $aP \cap bP = 0$, $\bar{a}R \cap \bar{b}R = 0$, and $0 \neq \bar{a}R \cong \bar{b}R$ because $\bar{a}^{\perp} = \bar{b}^{\perp} = (1-\bar{\chi}_Y)R$. Thus $R_R \notin \mathrm{I}$, a contradiction. Hence Y is finite.

(iv) $\Rightarrow$ It suffices to show that any cyclic submodule of R_R is directly infinite. Let $0 \neq \bar{c} = c+S \in R$, $c = (c_n) \in P \backslash S$. For each n, there exists

$$\bigoplus_{i=1}^{\infty} z_n^i R_n = \bigoplus_{i=1}^{\infty} z_n^i P \subseteq c_n P = c_n R_n,$$

where the latter is a large extension of right P-modules, and where we view all $z_n^i \in R_n \subset S \subset P$, and also $c_n \in R_n \subset P$. If $n \in \mathrm{supp}\,c$, all $0 \neq z_n^i \in R_n$. When $c_n = 0$, let all $z_n^i = 0$. Set $z^i = (z_n^i)_{n=1,2,\ldots} \in P$. Then $\bigoplus_{i=1}^{\infty} z^i P \subseteq cP$ is a large extension of right P-modules. Moreover, also $\bigoplus_{i=1}^{\infty} \bar{z}^i R \leqslant \bar{c}R$, where $\bar{z}^i = z^i + S \in R$. Thus

$$\Big(\bigoplus_{i=1}^{\infty} \bar{z}^{2i} R\Big) \oplus \Big(\bigoplus_{i=1}^{\infty} \bar{z}^{2i-1} R\Big) \leqslant cR$$

shows that $E(cR) \cong E(cR) \oplus E(cR)$. Hence R is a continuous bottomless module of Type III. $\Leftarrow$ This is similar to (iii) and is omitted.

5.3. The classes of discrete, continuous, continuous molecular, and bottomless torsion free modules are denoted by D, C, CA, and CB respectively; while the classes of all modules of Types I, II, and III are abbreviated by I, II, and III. By [D; p. 75, 6.4], $CA \cap \mathrm{III} \neq \{(0)\}$; while [D; p. 77, 6.9] shows that $CB \cap \mathrm{I} \neq \{(0)\}$. By 4.3, $D \subset \mathrm{I}$ (and hence $D \cap \mathrm{II} = D \cap \mathrm{III} = \{(0)\}$). From 5.2(iv) we get $CB \cap \mathrm{III} \neq \{(0)\}$. In the table below,

"0" means that the intersection of the two classes (the one in the row and the second one in the column) is $\{(0)\}$; and "yes" means that there exist nonzero modules in the intersection of the two classes, and "?" means that so far the answer is not known.

		I	II	III
	D	yes	0	0
C	CA	0	?	yes
C	CB	yes	?	yes

For any torsion free injective modules A and B, the reject $\mathrm{rej}_A B$ is injective. The next counterexample shows that the dual of this for the trace fails$-\mathrm{tr}_B A$ need not be injective.

5.4. COUNTEREXAMPLE. For an infinite-dimensional right vector space V over a division ring F, take R to be the ring $R = \mathrm{End}_F V$, where R acts on the left of V. Then the ideal $H \lhd R$ of finite rank transformations is a simple ring with $H^2 = H \leqslant R$. Since R is a right self-injective regular ring ([G; p. 53, 2.23]), for any $0 \neq x \in H$, xR is injective because $xR = eR$ for some $e^2 = e \in R$. Then the trace of xR in R is $\mathrm{tr}_R xR = RxR = H$ and is not injective.

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