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W R O C Ł A W S K A D R U K A R N I A N A U K O W A

Partial confluence of maps onto graphs and inverse limits of single graphs

by

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Abstract. $P(M)$ is the smallest integer such that if X is any continuum, f is any map from X onto M, and K is any subcontinuum of M, then there are $P(M)$ or fewer continua in X the union of whose images under f is K. A formula is given for $P(G)$ when G is a graph. In addition, an affirmative answer is given to a question of Hagopian who asked if an aposyndetic continuum that is the inverse limit of a single graph is locally connected.

1. Introduction. A general problem for a continuum M is to find the smallest integer $P(M)$ such that if X is any continuum, f is any map from X onto M, and K is any subcontinuum of M, then there are $P(M)$ or fewer continua in X the union of whose images under f is K. For example, class$[W]$ is the set of all continua M for which $P(M) = 1$, and if M is a simple closed curve or a simple triod, $P(M) = 2$. The first author has shown [7, Theorem II.2] if M is a continuum that, for some integer n, contains an n-od but no $(n+1)$-od, $n > 1$, then $P(M) \leqslant n(n-1)$. One purpose of this paper is to show that if M is a graph, $P(M) \leqslant \frac{3}{2}n-1$. More precisely, $P(M) = \frac{3}{2}n - \frac{1}{2}t(M) - 1$ where $t(M)$ is the number of points in M of order 1.

A second purpose is to answer a question of Charles Hagopian who asked if an aposyndetic continuum that is the inverse limit of a single graph is locally connected. It is proved here that if X is semi-aposyndetic and is the inverse limit of continua for which there is an integer n such that no factor contains an n-od, then X is a graph.

2. Partial confluence of maps onto graphs. A *continuum* is a compact connected metric space (with metric ϱ). A continuum M is an *n-od*, where n is an integer greater than 1, if M contains a subcontinuum K, called the *core* of the n-od, such that $M \backslash K$ has n components. If M is a continuum, let $n(M)$ be the largest integer (if it exists) such that M contains an $n(M)$-od. A *map* is a continuous function. If f is a map from a continuum X onto a continuum Y, then a subcontinuum K of Y is a *w_f-set* if there is a continuum K' in X such that $f(K') = K$. A subcontinuum J of a subcontinuum K of Y is a *maximal w_f-set* in K if J is a w_f-set, and J is not a proper subcontinuum of a w_f-set which is contained in K. The map f is *n-partially confluent* if every subcontinuum of Y is the union of n or fewer w_f-sets. For the continuum M let $P(M)$ be the largest integer such that there is a map f from a continuum onto M that is not $(P(M)-1)$-partially confluent. Note that $P(M)$ is the smallest integer such that for every map of a continuum onto M, every subcontinuum of M is the union of $P(M)$ or fewer w_f-sets.

A subcontinuum A of a continuum X is a *free arc* in X if A is an arc such that the boundary of A is contained in the set of endpoints of A. A continuum G is a *graph* if it is the union of a finite number of free arcs. For a graph G, let $e(G)$ be the number of edges of G, $t(G)$ the number of points of order one in G, called *terminal points*, and $v(G)$ the number of vertices of G (here, a point of order one is not a vertex). Let $\beta(G)$ be the first Betti number for G. A *spanning tree* for G is an acyclic subcontinuum of G that contains all of the vertices and terminal points of G, and whose edges are edges of G. If H is a spanning tree, $\beta(G)=\beta(G/H)=e(G)-e(H)=e(G)-\big(t(G)+v(G)\big)+1$ [1, Theorem 5, p. 36]. If K is a subcontinuum of G, a component of $G\backslash K$ whose closure is an arc with both endpoints in K or a simple closed curve with one point in K is a *chord* of K. Clearly, every spanning tree has $\beta(G)$ chords. The following lemma is probably well known, but the proof is short, and is included here for completeness.

LEMMA 1. *If G is a graph, then every finite collection of points of order two in G that does not separate G is contained in a collection of $\beta(G)$ points of order two in G that does not separate G, and every collection of $\beta(G)+1$ points of order two in G separates G.*

Proof. Let $\{x_1, \ldots, x_t\}$ be a finite collection of points of order two in G that does not separate G, and such that the addition of any point to this collection yields a collection that does separate G. For each j, let I_j be the interior of the edge of G that contains x_j. Then $H=G\backslash(\bigcup I_j)$ is acyclic, since every point of order two in H separates H. Therefore, H is a spanning tree for G, and $\beta(G)=e(G)-e(H)=t$.

THEOREM 1. *If G is a graph, then $n(G)=2\beta(G)+t(G)$.*

Proof. Let D be an $n(G)$-od in G with core K. It follows that K must contain each vertex of G of order greater than two. For if it did not contain a vertex, then K could be extended by an arc to a continuum K' which contains that vertex, and is the core of an $\big(n(G)+1\big)$-od in G.

Each component of $G\backslash K$ is either a chord of K or an arc, one of whose endpoints is a terminal point of G. Each of the latter type of component contains exactly one component of $D\backslash K$, and each chord of K contains exactly two components of $D\backslash K$. A collection of points consisting of one element from each chord of K does not separate G. So, by Lemma 1, the maximum number of chords of K is $\beta(G)$. Thus $n(G)\leqslant 2\beta(G)+t(G)$.

On the other hand, G must contain a spanning tree (see the proof of Lemma 1), and the spanning tree minus the interiors of the edges of G that contain the terminal points of G is the core of a $\big(2\beta(G)+t(G)\big)$-od. Thus, $n(G)=2\beta(G)+t(G)$.

LEMMA 2. *Let f be a map of a continuum X onto the continuum M. Let K be a subcontinuum of M, and C_1 and C_2 be disjoint nonempty closed subsets of K such that $\mathrm{Bd}(K)=C_1\cup C_2$. Then there exists a connected set A that is either a w_f-set in K, or the union of two w_f-sets in K, such that $A\cap C_1\neq\emptyset\neq A\cap C_2$. Moreover, if no w_f-set in K intersects C_1 and C_2, then there is a component of $M\backslash K$ whose closure intersects C_1 and C_2.*

Proof. For $i=1, 2$, let A_i be the set of all points x in K such that there is a continuum in X whose image contains x, lies in K, and intersects C_i. Note that A_1 and A_2 are nonempty closed sets whose union is K. Let y be an element of $A_1\cap A_2$. Then there exist w_f-sets Y_1 and Y_2 such that $y\in Y_1\cap Y_2$, and $Y_1\cap C_1\neq\emptyset\neq Y_2\cap C_2$, so $A=Y_1\cup Y_2$ is the required set (it is possible that Y_1 or Y_2 might intersect both C_1 and C_2).

Suppose the closure of no component of $M\backslash K$ intersects C_1 and C_2. Then $M\backslash K=Q_1\cup Q_2$, a separation, such that $\mathrm{cl}(Q_1)\cap K=C_1$ and $\mathrm{cl}(Q_2)\cap K=C_2$. Let D be a subcontinuum of X irreducible between $f^{-1}(C_1)$ and $f^{-1}(C_2)$. Then $f(D)$ is a w_f-set in K intersecting C_1 and C_2.

LEMMA 3. *If K is a subgraph of a graph G, and $E_1, \ldots, E_n$ are arcs such that both endpoints of each arc are in K while the rest of the arc is in $G\backslash K$, and no one of the arcs is contained in the union of the others, then there exist points $a_1, \ldots, a_n$ such that for $1\leqslant i\leqslant n$, $a_i\in E_i$ and $\bigcup_{i=1}^{n}\{a_i\}$ does not separate G.*

Proof. Let E_1' be a free open arc lying in E_1 such that $E_1'\cap\bigcup_{i=2}^{n}E_i=\emptyset$, and let a_1 be a point in E_1'. Then $G\backslash\{a_1\}$ is connected. Suppose for $1\leqslant k<n$, points $a_1, \ldots, a_k$ and arcs $E_1', \ldots, E_k'$ have been selected so that $a_i\in E_i'\subset E_i$ for $1\leqslant i\leqslant k$, $E_i'\cap\bigcup_{j\neq i}E_j=\emptyset$ for $1\leqslant i\leqslant k$ and $1\leqslant j\leqslant n$, and $\bigcup_{i=1}^{k}\{a_i\}$ does not separate G. Let E_{k+1}' be a free open arc lying in E_{k+1} such that $E_{k+1}'\cap\bigcup_{i=1}^{k}E_i=\emptyset$, and let $a_{k+1}\in E_{k+1}'$. Since $E_{k+1}\cap\bigcup_{i=1}^{k}E_i'=\emptyset$, $\big(G\backslash\bigcup_{i=1}^{k}\{a_i\}\big)\backslash\{a_{k+1}\}=G\backslash\bigcup_{i=1}^{k+1}\{a_i\}$ is connected. By induction $G\backslash\bigcup_{i=1}^{n}\{a_i\}$ is connected, where each point $a_i\in E_i$ for $1\leqslant i\leqslant n$.

THEOREM 2. *If G is a graph then $P(G)=3\beta(G)+t(G)-1$.*

Proof. Suppose f is a map from a continuum onto G. Let K be a subcontinuum of G such that K is acyclic, each boundary point of K is a point of order two in G, and K does not contain a terminal point of G. In this case, K is irreducible about its boundary B. Since $|B|\leqslant n(G)$, it follows from Theorem 1 that $|B|\leqslant 2\beta(G)+t(G)$. Let b_1 be an element of B and let $\mathscr{C}_1=\{b_1, \ldots, b_{\alpha(1)}\}$ be a maximal collection of points in B that contains b_1 and is contained in the union of a collection $\mathscr{E}_1=\{E_1, \ldots, E_{\alpha(1)-1}\}$ of w_f-sets in K such that $\bigcup\mathscr{E}_1$ is connected. Note that if no w_f-set in K contains b_1 and another point of B, then $\mathscr{E}_1$ may be empty. Also, note that no w_f-set in K contains a point of $\mathscr{C}_1$ and a point of $B\backslash\mathscr{C}_1$.

According to Lemma 2, if $B\backslash\mathscr{C}_1\neq\emptyset$, there is a point $b_{\alpha(1)+1}$ in $B\backslash\mathscr{C}_1$ and w_f-sets E_1' and E_1'' in K such that $E_1'\cap\mathscr{C}_1\neq\emptyset$, $b_{\alpha(1)+1}\in E_1''$, and $E_1'\cap E_1''\neq\emptyset$. Let $\mathscr{C}_2=\{b_{\alpha(1)+1}, \ldots, b_{\alpha(1)+\alpha(2)}\}$ be a maximal collection of points in $B\backslash\mathscr{C}_1$ that contains $b_{\alpha(1)+1}$ and is contained in the union of a collection $\mathscr{E}_2=\{E_{\alpha(1)}, \ldots, E_{\alpha(1)+\alpha(2)-2}\}$ of w_f-sets in K such that $\bigcup\mathscr{E}_2$ is connected.

Suppose $\alpha(i)$, $\mathscr{C}_i$, where $\mathscr{C}_i$ is contained in B, and $\mathscr{E}_i$ have been defined for $1\leqslant i\leqslant k$, let $v=\sum_{i=1}^{k}\alpha(i)$, and suppose $B\backslash\bigcup_{i=1}^{k}\mathscr{C}_i\neq\emptyset$. By Lemma 2, there is a point b_{v+1} in

$B\backslash\bigcup_{i=1}^{k}\mathscr{C}_i$ and w_f-sets E_k' and E_k'' in K such that $E_k'\cap\bigcup_{i=1}^{k}\mathscr{C}_i\neq\emptyset$, $b_{v+1}\in E_k''$, and $E_k'\cap E_k''\neq\emptyset$. Let $\mathscr{C}_{k+1}=\{b_{v+1},\ldots,b_{v+\alpha(k+1)}\}$ be a maximal collection of points in $B\backslash\bigcup_{i=1}^{k}\mathscr{C}_i$ that contains b_{v+1} and is contained in the union of a collection $\mathscr{E}_{k+1}=\{E_{v-k+1},\ldots,E_{v+\alpha(k+1)-(k+1)}\}$ of w_f-sets in K such that $\bigcup\mathscr{E}_{k+1}$ is connected.

By induction, there is an integer q such that $B\backslash\bigcup_{i=1}^{q}\mathscr{C}_i=\emptyset$. Since there is no w_f-set in K that contains a point of $\mathscr{C}_1$ and a point of $\mathscr{C}_j$ where $j\neq 1$, it follows from Lemma 2 that there is a j, $j\neq 1$, and an arc A_1 in $\mathrm{cl}(G\backslash K)$ with one endpoint in $\mathscr{C}_1$ and the other in $\mathscr{C}_j$. There is no w_f-set in K that contains a point of $\mathscr{C}_1\cup\mathscr{C}_j$ and a point of $\mathscr{C}_k$ where $k\notin\{1,j\}$. It follows from Lemma 2 that there is a k, $k\notin\{1,j\}$, and an arc A_2 in $\mathrm{cl}(G\backslash K)$ with one endpoint in $\mathscr{C}_1\cup\mathscr{C}_j$ and the other in $\mathscr{C}_k$. Continuing, we obtain arcs $A_1,\ldots,A_{q-1}$. Then $\bigcup_{i=1}^{q-1}A_i$ contains arcs $A_1',\ldots,A_{q-1}'$ such that for $1\leqslant i\leqslant q-1$, A_i' less its endpoints is a subset of $G\backslash K$, both endpoints of A_i' are in K, and A_i' is not contained in $\bigcup_{j\neq i}A_j'$. Then, by Lemma 3, there exists a set of $q-1$ points that does not separate G. Hence, by Lemma 1, $q-1\leqslant\beta(G)$.

Let $\mathscr{E}=\bigcup_{i=1}^{q}\mathscr{E}_i\cup\left(\bigcup_{i=1}^{q-1}\{E_i',E_i''\}\right)$. Then $\bigcup\mathscr{E}$ is connected and it contains B, so $\bigcup\mathscr{E}=K$. Let m be the number of w_f-sets in $\mathscr{E}$. Then

$$m\leqslant|B|-q+2(q-1)=|B|+(q-1)-1\leqslant 2\beta(G)+t(G)+\beta(G)-1=3\beta(G)+t(G)-1.$$

Suppose K is any subcontinuum of G. Then K is the limit of a sequence $\{K_i\}_{i=1}^{\infty}$ of subcontinua of G such that for each i, K_i is acyclic, the boundary points of K_i have order two, and K_i does not contain a terminal point of G. Each K_i is the union of $n=3\beta(G)+t(G)-1$ or fewer w_f-sets. So for each positive integer i, there exist n w_f-sets $Q_1^i,\ldots,Q_n^i$ such that $K_i=\bigcup_{j=1}^{n}Q_j^i$. Choosing subsequences if necessary, assume that $\{Q_j^i\}_{i=1}^{\infty}$ converges to a continuum Q_j for $1\leqslant j\leqslant n$. Clearly Q_j is a w_f-set for $1\leqslant j\leqslant n$, and $\bigcup_{j=1}^{n}Q_j$ is contained in K. To see that $\bigcup_{j=1}^{n}Q_j=K$, let y be an element of K. For every positive integer i, there exists y_i in K_i such that $\lim y_i=y$. For every positive integer i, there exists an integer $\alpha(i)$, $1\leqslant\alpha(i)\leqslant n$, such that $y_i\in Q_{\alpha(i)}^i$. There is an integer α, $1\leqslant\alpha\leqslant n$, such that $\alpha(i)=\alpha$ for infinitely many i's. Without loss of generality assume that $\alpha(i)=\alpha$ for all the i's. Then $y_i\in Q_\alpha^i$ for all i's, and $y=\lim y_i\in\lim Q_\alpha^i=Q_\alpha$ which is contained in $\bigcup_{j=1}^{n}Q_j$. Hence K is the union of $n=3\beta(G)+t(G)-1$ w_f-sets.

Let K be an acyclic subcontinuum of G such that K contains all of the vertices of G, the boundary points of K have order two, and K does not contain any of the terminal points of G. We will produce a map f from a continuum onto G such that K is not the union of fewer than $3\beta(G)+t(G)-1$ w_f-sets.

By an *end arc* of K is meant an arc in K which contains a terminal point of K and is contained in a free arc of K. If $\beta(G)\neq 0$, there are $\beta(G)$ pairs of end arcs, $\{[a_i,b_i],[a_i',b_i']\}$, $1\leqslant i\leqslant\beta(G)$, where b_i and b_i' are terminal points of K, and there is an arc $[b_i,b_i']$ in the closure of $G\backslash K$. If $t(G)\neq 0$, then there are an additional $t(G)$ end arcs of K, $\{[a_i,b_i]\}$, $\beta(G)+1\leqslant i\leqslant\beta(G)+t(G)$, where b_i is a terminal point of K and there is an arc $[b_i,b_i']$ in the closure of $G\backslash K$ from b_i to b_i', where b_i' is a terminal point of G.

Let x be a point in the interior of $[b_1,b_1']$. For each i from 2 to $\beta(G)$ let A_i be an arc in $K\cup[b_1,x]$ which is irreducible from x to a point c_i in (a_i,b_i) and which contains the arc $[b_1,x]$. Note that A_i does not intersect $(x,b_1']$ or $(a_i',b_i']$. Let A_1 be an arc

containing b_1 from x to a point c_1 in (a_1',b_1'). Also, for $2\leqslant i\leqslant\beta(G)$, let A_i' be an arc in $K\cup[b_1,x]\cup[b_i,b_i']$ which is irreducible from c_i to x, and which does not intersect (a_i,c_i) or $(x,b_1']$. Note that A_i' must contain $[a_i',b_i']$. Let A_1' be an arc in $K\cup[b_1',x]$ which is irreducible from c_1 to x and does not contain b_1.

If $t(G)\neq 0$, for $\beta(G)+1\leqslant i\leqslant\beta(G)+t(G)$ and $i\neq 1$, let A_i be an arc in $K\cup[b_1,x]\cup[b_i,b_i']$ which is irreducible from b_i' to x and which does not intersect $(x,b_1']$.

Let F be a simple fan which consists of $2\beta(G)+t(G)$ arcs with the common endpoint x'. Define the map f from F onto G as follows. For $1\leqslant i\leqslant\beta(G)+t(G)$, map one leg of F one-to-one onto A_i sending x' to x. For $1\leqslant i\leqslant\beta(G)$, map one leg of F one-to-one onto A_i' sending x' to x. Note that A_1' and each A_i, $1\leqslant i\leqslant\beta(G)+t(G)$, contains exactly one w_f-set which is maximal in K, and each A_i', $2\leqslant i\leqslant\beta(G)$, contains exactly two w_f-sets which are maximal in K. Observe also that these $[1+\beta(G)+t(G)]+2[\beta(G)-1]=3\beta(G)+t(G)-1$ w_f-sets are all necessary in order for their union to be K. Therefore, K is not the union of fewer than $3\beta(G)+t(G)-1$ w_f-sets. So $P(G)=3\beta(G)+t(G)-1$.

Since $\beta(G)=e(G)-\big(t(G)+v(G)\big)+1$, $P(G)=3\big(e(G)-v(G)\big)-2t(G)+2$, which is a formula that makes $P(G)$ trivial to compute. Also, from Theorem 1 it follows that $\beta(G)=\frac{1}{2}\big(n(G)-t(G)\big)$, so $P(G)=\frac{3}{2}n(G)-\frac{1}{2}t(G)-1$. If $t(G)=0$, $P(G)=\frac{3}{2}n(G)-1$, and, in general, $P(G)\leqslant\frac{3}{2}n(G)-1$, which suggests the following question.

Question 1. Is there a continuum X such that $P(X)>\frac{3}{2}n(X)-1$?

The next theorem will allow us to consider $P(X)$ for a larger collection of continua.

Theorem 3. *Suppose n is a positive integer, and the continuum $X=\varprojlim(X_\alpha,f_\alpha)$, where each X_α is a continuum such that $P(X_\alpha)\leqslant n$. Then $P(X)\leqslant n$.*

Proof. For each positive integer i there is a map g_i from X onto some X_α such that $\mathrm{diam}\big(g_i^{-1}(g_i(x))\big)<1/i$ for each x in X [4, Lemma 1.162, p. 167]. Let f be a map from the continuum M onto X, and let L be a subcontinuum of X. Since g_if is n-partially confluent, for each positive integer i there is a collection $\{K_1^i,\ldots,K_n^i\}$ of continua in M such that $\bigcup_{j=1}^{n}g_if(K_j^i)=g_i(L)$. Let $L_j^i=f(K_j^i)$ for each j, $1\leqslant j\leqslant n$. Choosing subsequences if necessary, assume that for each j, $1\leqslant j\leqslant n$, the sequence $\{L_j^i\}_{i=1}^{\infty}$ converges to a continuum L_j in X, and the sequence $\{K_j^i\}_{i=1}^{\infty}$ converges to a continuum K_j in M. It follows that $f(K_j)=L_j$ for each j, $1\leqslant j\leqslant n$.

If x is a point in L there is a map α from the positive integers into the integers from 1 to n, and a sequence of points $\{k_{\alpha(i)}^i\}_{i=1}^{\infty}$ such that $k_{\alpha(i)}^i\in K_{\alpha(i)}^i$ and $g_if(k_{\alpha(i)}^i)=g_i(x)$ for each positive integer i. There is a j', $1\leqslant j'\leqslant n$, such that $\alpha(i)=j'$ for infinitely many i's. Choosing subsequences if necessary, assume $\alpha(i)=j'$ for each positive integer i. Then $\{k_{j'}^i\}_{i=1}^{\infty}$ converges to a point $k_{j'}$ in $K_{j'}$, and $f(k_{j'})=x$ since $\mathrm{diam}\big(g_i^{-1}(g_i(x))\big)<1/i$ for each positive integer i. So $x\in L_{j'}$. We have shown that $L=\bigcup_{j=1}^{n}L_j$, and, since each L_j is a w_f-set, L is the union of n w_f-sets.

If X is the inverse limit of a single graph, define $P^*(X)$ to be the minimum of $\{P(G) \mid G$ is a graph and X is the inverse limit of $G\}$. According to Theorem 3, $P(X) \leqslant P^*(X)$. For example, if M is the Ingram continuum [2, p. 100] then M is the inverse limit of a simple triod, M is not the inverse limit of an arc [2, Theorem 3, p. 106], and M is in class$[W]$ [3, Theorem 1, p. 190]. So $P^*(M) = 2$ and $P(M) = 1$.

3. Inverse limits of a single graph. The purpose of this section is to answer a question of Charles Hagopian who asked if an aposyndetic continuum that is the inverse limit of a single graph is locally connected. This question is related to the more general problem of when a one-dimensional aposyndetic continuum is locally connected. For example, it is not known if a one-dimensional unicoherent and mutually aposyndetic continuum is locally connected [see the Houston Problem Book, problem 48]. Since every one-dimensional continuum is the inverse limit of graphs, it is natural to view those continua that are the inverse limits of a single graph as an important subclass of one-dimensional continua.

A map $f: X \to Y$ is an *ε-map* if ε is a positive number such that $f^{-1}(y)$ has diameter less than ε for each y in Y. A space X is *semi-aposyndetic* if for each pair of points in X there is a continuum in X that contains one of the points in its interior and does not contain the other point.

THEOREM 4. *If a continuum X contains an n-od, then there is a positive number ε such that if f is an ε-map from X onto a continuum Y, then Y contains an n-od.*

Proof. Suppose C is an n-od with core K in X. Let $\{L_1, \ldots, L_n\}$ be the components of $C \setminus K$. For each i, $1 \leqslant i \leqslant n$, let x_i be an element of L_i. Let $\partial_1 = \min\{\varrho(x_i, K)\}$, and let $\partial_2 = \min_{j \neq i}\{\varrho(x_i, \mathrm{cl}(L_j))\}$. Let $\varepsilon = (\min\{\partial_1, \partial_2\})/2$.

Suppose f is an ε-map from X onto Y. For each i, the set $f(L_i \cup K) \setminus (\bigcup_{j \neq i} f(L_j \cup K))$ is not empty, since it contains $f(x_i)$. Therefore, the continua in the collection $\{f(L_i \cup K)\}_{i=1}^n$ have a point in common and no one of them is contained in the union of the others. The union of this collection contains an n-od [6, Theorem 1].

If a continuum $X = \varprojlim(X_\alpha, f_\alpha)$, then for each positive number ε there is an ε-map into some X_α [4, Lemma 1.162, p. 167]. So, if there is a positive integer n such that each X_α does not contain an n-od, then X does not contain an n-od.

THEOREM 5. *A continuum is a graph if and only if it is semi-aposyndetic and does not contain an infinite-od.*

Proof. Every hereditarily locally connected continuum that does not contain an infinite-od is a graph [5, Theorem III.1, p. 568]. Suppose the continuum X is semi-aposyndetic and not hereditarily locally connected. Then there is a sequence $\{K_i\}_{i=1}^\infty$ of disjoint continua in X that converges to a nondegenerate continuum K in X.

Let x and y be different points in K. Without loss of generality, it can be assumed that there is a continuum J in X that contains x in its interior and does not contain y. There is an integer N such that if $n \geqslant N$, $K_n \cap J \neq \emptyset$, and $K_n \setminus J \neq \emptyset$. Then $J \cup K \cup (\bigcup_{n>N} K_n)$ is an infinite-od.

The next two theorems follow immediately from Theorems 4 and 5.

THEOREM 6. *If $X = \varprojlim(X_\alpha, f_\alpha)$ and each X_α is a continuum, X is semi-aposyndetic, and if there is a positive integer n such that each X_α does not contain an n-od, and each X_α is a continuum, then X is a graph.*

Since, for a positive integer n, there are only finitely many graphs that do not contain an n-od, if each X_α in the statement of Theorem 6 is a graph that does not contain an n-od, then X is the inverse limit of a single graph. Clearly, if G is a graph, there is an integer n such that G does not contain an n-od.

THEOREM 7. *If X is the inverse limit of a single graph G, and X is semi-aposyndetic, then X is a graph.*

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