# Formalizing a non-linear Henkin quantifier

by

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**Abstract.** L. Henkin introduced the quantifier $(Q_{2,2,d_2}x, y; v, w)$ so that the interpretation of the formula $(Q_{2,2,d_2}x, y; v, w)\phi(x, y; v, w)$ is that for every $x$ there is a $v$, and for every $y$ there is a $w$—depending only on $y$—such that $\phi(x, y; v, w)$ [Henkin, 1961, p. 181]. Shortly after its introduction, it was shown by A. Ehrenfeucht that a first-order predicate calculus enriched with the quantifier $Q_{2,2,d_2}$ is not recursively axiomatizable. We give a sound axiomatization for such a calculus and consider the class of models for which it is complete. The axiomatization is in a Natural Deduction style and we prove various normalization results.

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*Key words and phrases*: quantifiers, completeness, normalization.

## I. INTRODUCTION

In the 1959 Warsaw Symposium on Foundations of Mathematics, Professor Leon Henkin introduced a collection of "new" quantifiers, including some whose natural representation involved two-dimensional arrays. The simplest one is:

$$\left.\begin{matrix}(\forall x)(\exists v)\\(\forall y)(\exists w)\end{matrix}\right\}\phi(x, y; v, w).$$

The interpretation of the above formula is that the $v$ depends only on the $x$ and the $w$ depends only on the $y$; in terms of the uniform notation introduced by Henkin for partially ordered quantifiers the above quantifier is $Q_{2,2,d_2}(x, y; v, w)$. One of the questions posed by Henkin was:

"... if we enrich the first-order predicate calculus by taking $Q_{2,2,d_2}$ as an additional primitive notion, can we axiomatize the resulting system so as to obtain all valid formulas as formal theorems?" [Henkin, 1961, p. 181].

A. Ehrenfeucht has shown that the set of valid sentences of such a system is not recursively axiomatizable and thus the logic of the quantifier $Q_{2,2,d_2}$ is classed with such non-axiomatizable system as (full) second order logic. On the other hand, even such strong systems as full second order logic have powerful recursively axiomatizable subsystems. By and large, the subsystems are obtained by enlarging the collection of possible interpretations by introducing "non-standard models". We propose to do a similar thing with the logic of the $Q_{2,2,d_2}$ quantifier.

One of the principal tools in the theory of proofs is the "strong normalization" theorem which loosely speaking states that any derivation is always transformed into a normal (irreducible) one by "removing" its redundant formulas one by one. The normalization theorems for derivations are best considered in Gentzen systems of natural deduction involving trees of formulas. In such systems of natural deduction the inferences are broken down into atomic steps in such a way that each step involves at most one logical constant. Furthermore, the atomic inferences for the logical constants either *introduce* or *eliminate* the logical constant. The forms of the introduction rule(s) represent, according to Gentzen [Prawitz, 1971, p. 247], the "definitions" of the logical constants; at least they give sufficient conditions for introducing a formula with this logical constant as its principal symbol. On the other hand, the eliminations are motivated by the meaning given to the logical constant by the introduction rules.

All the familiar logical systems formalized in a natural deduction style and whose rules of inference for the logical constants follow the above guidelines have been shown to be strongly normalizable, so it is not unreasonable to conjecture that any *logical system*, formalized in a natural deduction form so that to each "logical" constant there correspond introduction and elimination rules involving only the given constant and whose elimination rules are *determined* by the corresponding introduction rules, satisfies the strong normalization property.

The obvious problem with the above conjecture is the meaning of "determined". In any case, before attempting to prove (or disprove) the conjecture it is advisable to increase the supply of logical systems. Adding more sentential connectives—specially in the classical case—gives little additional insight. On the other hand, as Henkin has shown, there are many "unused" quantifiers, in particular $Q_{2,2,d_2}$.



The interpretation of $Q_{2,2,d_2}$ gives us the form of its introduction rule which in turn leads to a plausible elimination rule. Let $\mathscr{H}\mathscr{Q}$ be the system so obtained. We will show that $\mathscr{H}\mathscr{Q}$ satisfies the strong normalization property. In view of Ehrenfeucht's result, $\mathscr{H}\mathscr{Q}$ cannot be a sound and complete axiomatization of $Q_{2,2,d_2}$. Nevertheless, it is a sound axiomatization and it is complete with respect to an enlarged class of models (analogous to the situation with second order logic).

The strong normalization theorem will be proved by assigning ordinals to the derivations in such a way that "removing" redundant formulas from the derivation lowers the ordinal. The assignment depends on the end rule of the derivation and the burden of the assignment is borne by the introduction rules. Since each logical constant has its own interpretation, each introduction rule determines a particular form of the assignment. On the other hand, all the elimination rules are considered at once. To further illustrate the method we briefly consider a variant of $\mathscr{H}\mathscr{Q}$.

## II. THE LANGUAGE

**1. The primitive symbols of $\mathscr{H}\mathscr{Q}$.**

| | |
|---|---|
| $\perp$ | for absurdity, |
| $\supset$ | for the conditional, |
| $\lambda$ | for a (very restricted) function abstractor, |
| $\stackrel{\circ}{=}$ | for equality of individuals and |
| $(\forall^x_y \exists^v_w)$ | for the Henkin quantifier ($Q_{2,2,d_2}\, x, y; v, w$) |

$\neg A$ is defined as $(A \supset \perp)$. Through the use of variables not occurring in the formula, the existential quantifier, $\exists$, and the universal quantifier, $\forall$, can be considered as special cases of the Henkin quantifier $(\forall^x_y \exists^v_w)$.

Since we are going to use a Natural Deduction System in the style of [Prawitz, 1965], we shall use

$a_1, a_2, \ldots$ for the *individual parameters*,
$x_1, x_2, \ldots$ for the *individual variables*,
$\alpha_1, \alpha_2, \ldots$ for the (unary) *function parameters.*

We shall use $a, b, \ldots$ to represent individual parameters. Individual variables and function parameters will be represented by $x, y, \ldots$ and $\alpha, \beta, \ldots$ respectively.

**2. The non-primitive symbols of $\mathscr{H}\mathscr{Q}$.** In order to reduce the number of subscripts we shall restrict ourselves to a language with the following non-primitive symbols:

$P$ a binary relational constant,
$F$ a binary function constant,
$k$ an individual constant.

**3. Formulas of $\mathcal{HQ}$.** The *terms* and *operators* are defined as follows:

(1) every individual constant is a term,
(2) every individual parameter is a term,
(3) if $t_1, t_2$ are terms, then $Ft_1t_2$ is a term,
(4) if $f$ is an operator and $t$ a term, then $ft$ is a term,
(5) every function parameter is an operator,
(6) if $t$ is a term in which the individual variable $x$ does not occur, then $(\lambda x.t^a_x)$, where $t^a_x$ is the expression obtained by replacing all the occurrences of the individual parameter $a$ in $t$ by $x$, is an operator.

We will use the letters $f, g, \ldots$ for operators.

The *formulas* are inductively defined by:

(1) $\perp$ is an (atomic) formula,
(2) if $t_1, t_2$ are terms, then the expressions $t_1 \triangleq t_2$ and $Pt_1t_2$ are (atomic) formulas,
(3) if $A$, $B$ are formulas and $\circ$ a sentential connective, then $(A \circ B)$ is a formula,
(4) if $A$ is a formula in which there are no bound occurrences of the variables $x, y, v, w$, then

$$(\forall^x_y \exists^v_w) A^{a.b.c.d}_{x.y.v.w}$$

is a formula in which all occurrences of $x, y, v, w$ are bound occurrences.

## III. AN AXIOMATIZATION FOR $\mathcal{HQ}$

As already mentioned, the axiomatization [1] is in the style of [Prawitz, 1965]. That is, the *derivations* are going to be certain annotated trees of formulas and they will be constructed using certain "rules of inferences". By and large, we shall take the rules of inferences from [Prawitz, 1961].

In order to simplify some of the reduction rules we introduce the following generalizations of the rule of repetition:

**1. The structural rules of inference.** For each $n$ and each $1 \leqslant i \leqslant n$ we have the structural rule

$$\frac{\begin{array}{c}\Phi_1 \ldots \Phi_n \\ A_1 \ldots A_n\end{array}}{A_i}\ (S^n_i)$$

where $\Phi_1, \ldots, \Phi_n$ are derivations of $A_1, \ldots, A_n$ respectively.

The premise $A_i$ will be called the "selected" premise (or formula) and $\Phi_i$ the "selected subderivation" of the application of the rule; the other formulas and their subderivations will be called the "auxiliary" ones.

The intuitive content of the rule $(S^n_i)$ is that the auxiliary subderivations are discarded. However, any undischarged assumption formulas in the auxiliary subderivations remain as undischarged formulas of the derivation.

[1] A more apt name would be a "codification" since we are going to give a set of "rules" rather than "axioms".



**2. Rules of inference for the sentential connectives.** The following rules are from [Prawitz, 1965].

$$\frac{\begin{array}{c}(A)\\ B\end{array}}{A \supset B}\ (\supset I) \qquad \frac{A \quad A \supset B}{B}\ (\supset E)$$

$$\frac{\perp}{A}\ (\perp_I) \qquad \frac{\begin{array}{c}(\neg A)\\ \perp\end{array}}{A}\ (\perp_C)$$

*Restrictions.* On the $\perp_I$ rule: $A$ is to be an atomic formula different from $\perp$. On the $\perp_C$ rule: $A$ is not to have the form $\neg B$.

**3. Rules of inference for the Henkin quantifier.**

Introduction rule.

$$\frac{A^{a.b.c.d}_{a.b.fa.gb}}{(\forall^x_y \exists^v_w) A^{a.b.c.d}_{x.y.v.w}}\ (\forall^-_-\exists^-_- I)$$

*Restrictions.* (1) The parameters $a$, $b$, the *eigenparameters* of the application of the inference, must not occur in any assumption formula on which $A^{a.b.c.d}_{a.b.fa.gb}$ depends.
(2) The individual parameters $a$, $b$ must not occur in the operators $f$, $g$.

Elimination rule.

$$\frac{(\forall^x_y \exists^v_w) A^{a.b.c.d}_{x.y.v.w} \qquad \begin{array}{c}(A^{a.b.c.d}_{s.t.\alpha s.\beta t})\\ \Pi \\ B\end{array}}{B}\ (\forall^-_-\exists^-_- E)$$

*Restriction.* The function parameters $\alpha$, $\beta$, the *eigenparameters* of the application, must not occur in $B$, $t$, $s$, $(\forall^x_y \exists^v_w) A^{a.b.c.d}_{x.y.v.w}$, nor in any undischarged assumption in $\Pi$ other than $A^{a.b.c.d}_{s.t.\alpha s.\beta t}$.

The premise $(\forall^x_y \exists^v_w) A^{a.b.c.d}_{x.y.v.w}$ is the "major" premise of the application, $B$ is the "minor".

**4. Rules for equality.** In the rules that follow the $a$'s are individual parameters and the $t$'s and $s$ are terms.

$$t \triangleq t \qquad \frac{t_1 \triangleq t_2}{t_2 \triangleq t_1} \qquad \frac{t_1 \triangleq t_2 \quad t_2 \triangleq t_3}{t_1 \triangleq t_3}$$

$$\frac{t_1 \triangleq t_2 \quad A^a_{t_1}}{A^a_{t_2}}$$

$$(\lambda x.t^a_x)s \triangleq t^a_s$$

**5. Notation for derivations.** Observe that we are using "$\Pi$, $\Phi$" for derivations and "$\Sigma$" for finite sequences of derivations.

If we wish to emphasize that the end formula of $\Pi$ is $A$, we will write

$$\begin{array}{c}\Pi\\ A\end{array}$$

On the other hand, the notation

$$\frac{\Sigma}{A}$$

is to represent a derivation with $A$ as the end formula and the premises of the end rule are the end formulas to the derivations in $\Sigma$.

If $\mathbf{a} = a, b, \ldots$ and $\boldsymbol{\alpha} = \alpha, \beta, \ldots$ are finite sequences of individual and function parameters respectively and if $\Pi$ is a derivation, then we may write "$\Pi(\mathbf{a}, \boldsymbol{\alpha})$" to emphasize that the parameters in $\mathbf{a}$ and $\boldsymbol{\alpha}$ occur in either the undischarged assumption formulas or in the end formula of $\Pi$.

A derivation $\Pi$ is said to be a *proper* derivation iff

(1) No parameter is the eigenparameter of more than one application of a rule of inference.

(2) If a parameter occurs as an eigenparameter, then all of its ocurrences are above the application of the rule for which it is the eigenparameter.

By appropriate renaming of the eigenparameters one may ensure that the derivations are indeed proper.

LEMMA. *If $\Pi(\mathbf{a}, \boldsymbol{\alpha})$ is a proper derivation, then for all substitutions $(\mathbf{t}, \mathbf{f})$ of $(\mathbf{a}, \boldsymbol{\alpha})$, it is possible to rename the eigenparameters of $\Pi$ so that $\Pi(\mathbf{t}, \mathbf{f})$ is a proper derivation.*

**6. Selection rules.** The structural rules and $\forall\exists\exists$ eliminations have the property that the conclusion of an application of the rule is the same formula as one of the premises. It is as if one of the premises is *selected* as the conclusion. Consequently, we shall call them *selection rules* of $\mathscr{H}\mathscr{Q}$.

## IV. GROUNDED AND IRREDUCIBLE DERIVATIONS

Loosely speaking, a reducible derivation is one in which there is an occurrence of a formula, called a *redundant formula*, which is both the conclusion of an introduction rule and the major premise of an elimination rule. However, since the selection rules allow the conclusion to be the same as one of the premises, the introduction–elimination pair of rules might be separated by a finite sequence of occurrences of the same formula (called a redundant segment). Thus a more accurate rendition of a reducible derivation is a derivation in which there are redundant segments. An "irreducible" derivation is one which is not reducible and a "grounded" one is a derivation which is either irreducible or no matter in which way the redundant segments are eliminated, it eventually leads to an irreducible derivation.



**1. Threads and segments in a derivation.** A *thread*[2] in a derivation is a sequence of formula occurrences $A_1, \ldots, A_n$ such that

(1) $A_1$ is a top formula,
(2) $A_n$ is the end formula,
(3) for each $i < n$, $A_{i+1}$ is immediately below $A_i$.

A *segment* in a derivation $\Pi$ is a sequence $B_1, \ldots, B_n$ of consecutive (i.e. $B_{i+1}$ is immediately below $B_i$) formula occurrences in a thread of $\Pi$ such that

(1) $B_1$ is not the conclusion of a selection rule,
(2) $B_n$ is not the selected premise of a selection rule,
(3) for each $i < n$, $B_i$ and $B_{i+1}$ are the selected premise and the conclusion of an application of a selection rule.

In a segment $B_1, \ldots, B_n$, all the formulas $B_i$ are different occurrences of a unique formula, called the "formula" of the segment.

A segment $\sigma = A_1, A_2, \ldots, A_n$ is the conclusion for an application of a rule of inference iff $A_1$ is the conclusion of the application. The segment $\sigma$ is a major premise of an application of a rule iff $A_n$ is a major premise of the application.

DEFINITION. A *redundant occurrence of a formula* in a derivation $\Pi$ is an occurrence which is both the conclusion of an introduction rule and the major premise of an elimination rule.

DEFINITION. A *redundant segment* in a derivation $\Pi$ is a segment which is the conclusion of an introduction rule and the major premise of an elimination rule.

**2. Contractions.** Suppose that $A$ is an occurrence of a redundant formula in the derivation $\Pi$. We now proceed to define the derivation $\mathrm{Contr}_A(\Pi)$ obtained by "contracting"[3] the occurrence of the redundant formula $A$ in $\Pi$.

The definition of $\mathrm{Contr}_A(\Pi)$ depends on the rules of inference that made the particular occurrence of $A$ into a redundant formula.

**2.1.** $\supset$ *contraction.* In this case $A = (B \supset C)$ and $\Pi$ is of the form

$$\dfrac{\dfrac{\begin{array}{c}(B)\\ \Phi_1\\ C\end{array}}{(B \supset C)}\,(\supset I) \qquad \begin{array}{c}\Phi_2\\ B\end{array}}{\begin{array}{c}(C)\\ \Phi_3\end{array}}\,(\supset E)$$

[2] This definition, and some of the others that follow were taken from [Troelstra, 1973, p. 281].

[3] "By-passing" would be a more appropriate name.

Then $\mathrm{Contr}_A(\Pi)$ is the derivation

$$\begin{array}{c}\Phi_2\\ (B)\\ \Phi_1\\ \dfrac{\dfrac{C}{C}\,(S_1^1)\qquad \begin{array}{c}\Phi_2\\ B\end{array}}{(C)}\,(S_1^2)\\ \Phi_3\end{array}$$

**2.2.** $\forall^{-}_{-}\exists^{-}_{-}$ *contraction.* In this case $A = (\forall^x_y \exists^v_w) B^{a.b.c.d}_{x.y.v.w}$ and $\Pi$ is of the form

$$\dfrac{\dfrac{\begin{array}{c}\Phi_1(a,b)\\ B^{a.b.c.d}_{a.b.fa.gb}\end{array}}{(\forall^x_y \exists^v_w) B^{a.b.c.d}_{x.y.v.w}}\,(\forall^{-}_{-}\exists^{-}_{-} I)\qquad \begin{array}{c}(B^{a.b.c.d}_{s.t.\alpha s.\beta t})\\ \Phi_2(\alpha,\beta)\\ C\end{array}}{\begin{array}{c}(C)\\ \Phi_3\end{array}}\,(\forall^{-}_{-}\exists^{-}_{-} E)$$

Then $\mathrm{Contr}_A(\Pi)$ is the derivation

$$\dfrac{\dfrac{\begin{array}{c}\Phi_1(a,b)\\ B^{a.b.c.d}_{a.b.fa.gb}\end{array}}{(\forall^x_y \exists^v_w) B^{a.b.c.d}_{x.y.v.w}}\,(\forall^{-}_{-}\exists^{-}_{-} I)\qquad \begin{array}{c}\Phi_1(s,t)\\ (B^{a.b.c.d}_{s.t.fs.gt})\\ \Phi_2(f,g)\\ C\end{array}}{\begin{array}{c}(C)\\ \Phi_3\end{array}}\,(S_2^2)$$

**3. Permutative reductions.** Consider now the situation in which the *redundant* segment $\sigma = C_1, \ldots, C_{n+1}$ ends in the major premise of an elimination rule. The permutative reductions will have the effect of shortening the length of the segment $\sigma$ by one.

**3.1.** $\forall^{-}_{-}\exists^{-}_{-}$ *permutative reduction.* Suppose that $\Pi$ is the following derivation in which the lowermost displayed occurrence of $C$ is a major premise to an elimination rule:

$$\dfrac{\dfrac{\begin{array}{c}\Phi_1\\ (\forall^x_y \exists^v_w) B\end{array}\quad \begin{array}{c}\Phi_2\\ C\end{array}}{C}\,(\forall^{-}_{-}\exists^{-}_{-} E)\qquad \Sigma}{\begin{array}{c}(D)\\ \Phi_3\end{array}}\,(\cdot E)$$

Then $\mathrm{Perm}_\sigma(\Pi)$ is the derivation

$$\dfrac{\begin{array}{c}\Phi_1\\ (\forall^x_y \exists^v_w) B\end{array}\qquad \dfrac{\begin{array}{c}\Phi_2\\ C\end{array}\quad \Sigma}{D}\,(\cdot E)}{\begin{array}{c}(D)\\ \Phi_3\end{array}}\,(\forall^{-}_{-}\exists^{-}_{-} E)$$



The conditions on *proper derivations* guarantee that if $\Pi$ is a proper derivation then $\mathrm{Perm}_\sigma(\Pi)$ is indeed a derivation (which can then be transformed into a proper derivation).

**3.2.** *Structural permutative reduction.* Suppose that $\Pi$ is the following derivation in which the lowermost displayed occurrence of $C$ is a major premise to an elimination rule:

$$\dfrac{\dfrac{\begin{array}{ccc}\Phi_1 & \Phi_2 & \Phi_3\\ A_1 & A_2 & C\end{array}}{C}\,(S_3^3)\qquad \Sigma}{\begin{array}{c}(D)\\ \Phi_4\end{array}}$$

Then $\mathrm{Perm}_\sigma(\Pi)$ is the derivation

$$\dfrac{\begin{array}{c}\Phi_1\\ A_1\end{array}\quad \begin{array}{c}\Phi_2\\ A_2\end{array}\quad \dfrac{\begin{array}{c}\Phi_3\\ C\end{array}\quad \Sigma}{D}}{\begin{array}{c}(D)\\ \Phi_4\end{array}}\,(S_3^3)$$

**4. Reductions.** $\Phi$ is an *immediate reduction* of $\Pi$, and we write $\Phi \prec_1 \Pi$, iff $\Phi$ is obtained from $\Pi$ by either a single contraction or a single permutative reduction. $\prec$ is the transitive closure of $\prec_1$ and $\preceq$ the transitive and reflexive closure.

A derivation $\Pi$ is *irreducible* (also called *normal*) iff there is no derivation $\Phi$ such that $\Phi \prec \Pi$.

$\Pi_1, \Pi_2, \ldots$ is a *reduction sequence* (starting from $\Pi_1$) iff for all $i$, $\Pi_{i+1} \prec_1 \Pi_i$. If the reduction sequence is finite, it is said to *terminate* (in the last derivation of the sequence) iff the last derivation in the sequence is irreducible.

The *reduction tree* of a derivation $\Pi$ is defined so that at the root of the tree we have the derivation $\Pi$ and the tree relation is precisely $\prec_1$. Obviously $\Pi$ is an irreducible derivation iff its reduction tree consists of exactly one node. We define $\Pi$ to be a *grounded derivation* iff its reduction tree is finite; equivalently, if every reduction sequence starting with $\Pi$ terminates.

## V. ORDINAL ASSIGNMENTS TO DERIVATIONS

We now propose to assign ordinals to the derivations in such a way that if $\Phi \prec_1 \Pi$ then the ordinal assigned to $\Phi$ will be strictly smaller than the one assigned to $\Pi$. The burden of the assignment will be borne by the introduction rules since the introduction rules are supposed to "reflect" the "meaning" of the connective (or quantifier)—at least if the system satisfies Gentzen's inversion principle [Prawitz, 1971, p. 247].

We shall denote the ordinal assigned to the derivation $\Pi$ by "$\mu(\Pi)$".

The method will be to first define $\mu$ as a partial function and then show that in fact it is defined for all derivations. We shall express that $\Pi$ is in the domain of $\mu$ by "$\mu(\Pi)\downarrow$".

**1. Informal definition.**

*Atomic derivations.* If $\Pi$ is a derivation consisting of exactly one formula, then $\mu(\Pi)\downarrow$ and $\mu(\Pi) = 1$.

*Derivations whose end rule is an introduction rule.*

END RULE $\supset$ INTRODUCTION. Assume that $\Pi$ is the derivation

$$\frac{\begin{matrix}(A)\\ \Phi_1\\ B\end{matrix}}{A \supset B}$$

Then *if for all derivations $\Phi_2$ of $A$* which are in the domain of $\mu$, the composed derivation $\Phi_2\cdot(A)\cdot\Phi_1$ is also in the domain of $\mu$, then $\mu(\Pi)\downarrow$ with value the successor of the ordinal supremum of all ordinals of the form $\mu(\Phi_2\cdot(A)\cdot\Phi_1)$.

END RULE $\forall\!=\!\exists\!=$ INTRODUCTION. In this case $\Pi$ is of the form

$$\frac{\begin{matrix}\Phi(a;\, b)\\ A^{a.b.c.d}_{a.b.fa.gb}\end{matrix}}{(\forall^x_y\,\exists^v_w)\,A^{a.b.c.d}_{x.y.v.w}}\ (\forall\!=\!\exists\!= I)$$

Then $\mu(\Pi)\downarrow$ provided that $\mu(\Phi(t, s))\downarrow$ for all terms $t$, $s$, in which case $\mu(\Pi)$ is defined as the successor of the supremum of all ordinals of the form $\mu(\Phi(t, s))$.

*Derivations whose end rule is an elimination rule.* Assume that $\Pi$ is of the form

$$\frac{\Phi_1 \quad \Phi_2 \quad \Phi_3}{A}$$

In order for $\mu(\Pi)\downarrow$ it is required that (a) $\Pi$ be a grounded derivation and (b) $\mu(\Phi_1)\downarrow$, $\mu(\Phi_2)\downarrow$, $\mu(\Phi_3)\downarrow$. Futhermore, letting $\kappa = \mu(\Phi_1) \# \mu(\Phi_2) \# \mu(\Phi_3)$:

Case 1. If $\Pi$ is an irreducible derivation, then $\mu(\Pi)$ is defined to be $\kappa$.

Case 2. If $\Pi$ is a reducible derivation, then it is required that for all derivations $\Psi \prec_1 \Pi$, $\mu(\Psi)\downarrow$; and $\mu(\Pi)$ is defined to be the successor of the supremum of the ordinals $\mu(\Psi) \# \kappa$ where $\Psi \prec_1 \Pi$.

*Default derivations.* The only derivations not accounted for are those whose end rules are either the structural rules, intuitionistic absurdity or classical absurdity. In this case, in order for $\mu(\Pi)$ to be defined it is required that the subderivations of all the premises of the end rule of $\Pi$ be in the domain of $\mu$. The value of $\mu(\Pi)$ is to be the natural ordinal sum of the values of $\mu$ on the subderivations of the premises.

**2. A more formal definition.** It might not be obvious that the above conditions could be used for a definition of the function (albeit a partial function) $\mu$. We now give an outline on how to prove that there is a (partial) function $\mu$ having the above properties.

DEFINITION. For derivations $\Phi$ and $\Psi$, $\Phi \ll \Psi$ iff either

(1) $\Phi$ is a (term substitution instance of a) proper subderivation of $\Psi$, or



(2) the end rule of $\Psi$ is $\supset$ Introduction and the end formula of $\Phi$ is either the antecedent or the consequent of the end formula of $\Psi$, or

(3) the end rule of $\Psi$ is an elimination rule, $\Psi$ is a grounded derivation and $\Phi \prec_1 \Psi$.

By considering what would happen if there were an infinite decreasing $\ll$ path we obtain

LEMMA. *$\ll$ is a well-founded relation.*

$\mu$ can then be defined by induction on the well-founded relation $\ll$.

## VI. STRONG NORMALIZATION THEOREM FOR $\mathscr{H}\mathscr{Q}$

The first 4 lemmas are proven by straightforward inductions on the length of the derivation.

LEMMA 1. *If $\mu(\Pi)\downarrow$ and $\Phi$ is a subderivation of $\Pi$ then $\mu(\Phi)\downarrow$ and $\mu(\Phi) \leqslant \mu(\Pi)$.*

LEMMA 2. *If $\mu(\Pi(\mathbf{a}))\downarrow$ where $\mathbf{a}$ is a finite sequence of individual parameters then for all sequences $\mathbf{t}$ (of the same length) of terms $\mu(\Pi(\mathbf{t}))\downarrow$ and $\mu(\Pi(\mathbf{a})) = \mu(\Pi(\mathbf{t}))$.*

LEMMA 3. *If $\mu(\Pi(\boldsymbol{\alpha}))\downarrow$ where $\boldsymbol{\alpha}$ is a finite sequence of function parameters then for all sequences $\mathbf{f}$ (of the same length) of operators $\mu(\Pi(\mathbf{f}))\downarrow$ and $\mu(\Pi(\boldsymbol{\alpha})) = \mu(\Pi(\mathbf{f}))$.*

LEMMA 4. *If $\mu(\Pi)\downarrow$ and $\Phi \prec \Pi$ then $\mu(\Phi)\downarrow$ and $\mu(\Phi) < \mu(\Pi)$. Thus $\Pi$ is a grounded derivation whenever it is in the domain of $\mu$.*

DEFINITION. $\mu^*$ is the restriction of $\mu$ to those derivations $\Pi$ in the domain of $\mu$ which have the property that for all finite sequences of derivations $\Sigma$ in the domain of $\mu$, $\mu(\Sigma\cdot\Pi)\downarrow$.

One of the guiding principles in the definition of $\mu(\Pi)\downarrow$ is to ensure that the following is an immediate consequence of the above lemmas:

LEMMA 5. *If the end rule of $\Pi$ is an introduction rule and the derivations of the premises of the end rule are in the domain of $\mu^*$ then so is $\Pi$.*

The following lemma is even simpler:

LEMMA 6. *If the end rule of $\Pi$ is neither an introduction rule nor an elimination rule and all the derivations of the premises of the end rule are in the domain of $\mu^*$, then so is $\Pi$.*

Eventually we will show that all the derivations are in the domain of $\mu^*$. But first we need the following:

LEMMA 7. *If the end rule of $\Pi$ is an elimination rule and the derivations of the premises of the end rule are in the domain of $\mu^*$ then so is $\Pi$.*

Proof. Assume that $\Pi$ is a derivation of the form

$$\frac{\Phi_1 \;\ldots\; \Phi_i}{B}\ (*\,E)$$

where $\Phi_1$ is the derivation of the *major* premise of the end rule of $\Pi$.

Let $\Sigma$ be a finite sequence of derivations in the domain of $\mu$. Then the lemma is proven by showing that $\mu(\Sigma\cdot\Pi)\downarrow$. Since $\Phi_1, \ldots, \Phi_i$ are derivations in the domain of $\mu^*$, we see that $\Sigma\cdot\Phi_1, \ldots, \Sigma\cdot\Phi_i$ are grounded derivations. Let $k$ be the sum of their reduction trees. We now proceed by induction on $k$.

BASIS STEP: $k = i$. In this case each of $\Sigma\cdot\Phi_1, \ldots, \Sigma\cdot\Phi_i$ is an irreducible derivation. We complete this case by an induction on the length, $l$, of the derivation $\Sigma\cdot\Phi_1$ of the major premise.

Basis step: $l = 1$. In this situation $\Sigma\cdot\Pi$ is an irreducible derivation and hence, in view of the other assumptions, $\mu(\Sigma\cdot\Pi)\downarrow$.

Inductive step: $l > 1$. If $\Sigma\cdot\Pi$ is an irreducible derivation, then we once again find that it is in the domain of $\mu$. Thus assume that $\Sigma\cdot\Pi$ is a reducible derivation. Let $\Psi \prec_1 \Sigma\cdot\Pi$. We will show that $\mu(\Psi)\downarrow$. It will then follow that $\Sigma\cdot\Pi$ is a grounded derivation and hence that $\mu(\Sigma\cdot\Pi)\downarrow$.

Case 1: $\Psi$ *is obtained by a contraction.* Since the derivations of the premises of the end rule are irreducible derivations, this entails that the contraction must have involved the end rule of $\Pi$.

Subcase 1.1: $(\forall^{-}_{-}\exists^{-}_{-})$*-contraction.* For this to be possible, $\Sigma\cdot\Pi$ must be a derivation of the form

$$\dfrac{\dfrac{\begin{array}{c}\Theta_1(a, b)\\ M^{a.b.c.d}_{a.b.fa.gb}\end{array}}{(\forall^x_y\exists^v_w)M^{a.b.c.d}_{x.y.v.w}} \qquad \begin{array}{c}\Sigma, (M^{a.b.c.d}_{s.t.\alpha s.\beta t})\\ \Theta_2(\alpha, \beta)\\ B\end{array}}{B}$$

And hence $\Psi$ is the derivation

$$\dfrac{\dfrac{\begin{array}{c}\Theta_1(a, b)\\ M^{a.b.c.d}_{a.b.fa.gb}\end{array}}{(\forall^x_y\exists^v_w)M^{a.b.c.d}_{x.y.v.w}} \qquad \begin{array}{c}\Theta_1(s, t)\\ \Sigma, (M^{a.b.c.d}_{s.t.fs.gt})\\ \Theta_2(f, g)\\ B\end{array}}{B}$$

where now the end rule of $\Psi$ is the structural rule $(S^2_2)$. Since both derivations of the premises are in the domain of $\mu$ (recall that $\Theta$ was in the domain of $\mu^*$), we see that $\mu(\Psi)\downarrow$.

Subcase 1.2: $(\supset)$*-contraction.* Analogous to the previous subcase.

Case 2: $\Psi$ *is obtained by a permutative reduction.*

Subcase 2.1: $(\forall^{-}_{-}\exists^{-}_{-})$*-permutative reduction.* Assume that $\Sigma\cdot\Pi$ is the derivation

$$\dfrac{\dfrac{\begin{array}{c}\Theta_1\\ (\forall^x_y\exists^v_w)M^{a.b.c.d}_{x.y.v.w}\end{array} \qquad \begin{array}{c}(M^{a.b.c.d}_{s.t.\alpha s.\beta t})\\ \Theta_2(\alpha, \beta)\\ A\end{array}}{A} \qquad \Delta}{B}$$



Consequently, $\Psi$ is the derivation

$$\dfrac{\begin{array}{c}\Theta_1\\ (\forall^x_y\exists^v_w)M^{a.b.c.d}_{x.y.v.w}\end{array} \qquad \dfrac{\begin{array}{c}(M^{a.b.c.d}_{s.t.\alpha s.\beta t})\\ \Theta_2(\alpha, \beta)\\ A\end{array} \quad \Delta}{B}}{B}$$

In this case we make use of the fact that all the displayed elimination rules in $\Psi$ have a shorter derivation of the major premise. Thus $\mu(\Psi)\downarrow$.

Subcase 2.2. *Structural-permutative reduction.* Even simpler than the previous subcase.

INDUCTION STEP: $k > i$. Very similar to the basis step. The only extra possibility is that the derivation $\Psi \prec_1 \Sigma\cdot\Pi$ be obtained by a reduction strictly within the derivation of one of the premises of the end rule of $\Sigma\cdot\Pi$; but then the value of $k$ is reduced and so the inductive hypothesis on $k$ may be applied.

THEOREM. *Every derivation is in the domain of* $\mu^*$.

Proof. By induction on the length of the derivation. It is immediate for derivations consisting of exactly one formula. Lemmas 5 to 7 complete the induction.

COROLLARY. *All derivations of* $\mathscr{H}\mathscr{Q}$ *are grounded.*

## VII. SOUNDNESS AND COMPLETENESS

**1. Structures, assignments and the soundness theorem.** By a *Tarskian structure* for $\mathscr{H}\mathscr{Q}$ we understand a system $\mathfrak{A} = (D^{\mathfrak{A}}, P^{\mathfrak{A}}, F^{\mathfrak{A}}, k^{\mathfrak{A}})$, where $D^{\mathfrak{A}}$ is a non-empty set—the domain of $\mathfrak{A}$, $P^{\mathfrak{A}}$ is a subset of the cartesian product of the domain of $\mathfrak{A}$; $F^{\mathfrak{A}}$ is a binary function on the domain of $\mathfrak{A}$ and $k^{\mathfrak{A}} \in D^{\mathfrak{A}}$.

A *Henkian structure* is a system $\mathfrak{B} = (D^{\mathfrak{B}}, P^{\mathfrak{B}}, F^{\mathfrak{B}}, k^{\mathfrak{B}}, \mathscr{H}^{\mathfrak{B}})$ where $(D^{\mathfrak{B}}, P^{\mathfrak{B}}, F^{\mathfrak{B}}, k^{\mathfrak{B}})$ is a Tarskian structure and $\mathscr{H}^{\mathfrak{B}}$ is a set of functions from (cartesian powers of) $D^{\mathfrak{B}}$ to $D^{\mathfrak{B}}$ which includes $F^{\mathfrak{B}}$, the identity function, the constant functions, all the projection functions and is closed under composition.

A Henkian structure $\mathfrak{B} = (D^{\mathfrak{B}}, P^{\mathfrak{B}}, F^{\mathfrak{B}}, k^{\mathfrak{B}}, \mathscr{H}^{\mathfrak{B}})$, where $\mathscr{H}^{\mathfrak{B}}$ consists of *all* the functions from (cartesian powers of) $D^{\mathfrak{B}}$ to $D^{\mathfrak{B}}$ will be called a *standard structure.*

An *assignment* in a Henkian structure $\mathfrak{B}$ is a mapping $\wp$ of the individual parameters into $D^{\mathfrak{B}}$ and of the function parameters into the set of unary functions of $\mathscr{H}^{\mathfrak{B}}$.

Let $\wp$ be an assignment in a Henkian structure $\mathfrak{B}$. Then $\wp$ can be extended to a function whose domain is the set of all terms and operators such that [4]:

[4] We will use the same name for $\wp$ and its extension.

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(1) $\wp(k) = k^{\mathfrak{B}}$,
(2) $\wp(F) = F^{\mathfrak{B}}$,
(3) if $t$ is a term, then $\wp(t) \in D^{\mathfrak{B}}$,
(4) $\wp(Ft_1 t_2) = F^{\mathfrak{B}}(\wp(t_1), \wp(t_2))$,
(5) if $f$ is an operator, then $\wp(f)$ is an unary function in $\mathscr{H}^{\mathfrak{B}}$,
(6) if $f$ is the operator $(\lambda x.t_x^a)$, then $\wp(f)$ is the function such that $\tau \in D^{\mathfrak{B}} \mapsto \wp_\tau^a(t)$, where $\wp_\tau^a(t)$ is the assignment which is like $\wp$ except that $a$ is mapped to $\tau$,
(7) if $f$ is an operator and $t$ a term, then $\wp(ft) = \wp(f)(\wp(t))$.

Then we may define in the usual way $\models_{\mathfrak{B}} A[\![\wp]\!]$ to mean that the formula $A$ is satisfied in the Henkian structure $\mathfrak{B}$ by the assignment $\wp$. The only case that may need special consideration is when the formula $A$ is of the form $(\forall_y^x \exists_w^v) M_{x.y.v.w}^{a.b.c.d}$. In that case the requirement is that the following two clauses be equivalent:

(I) $\models_{\mathfrak{B}} (\forall_y^x \exists_w^v) M_{x.y.v.w}^{a.b.c.d} [\![\wp]\!]$,

(II) there are unary functions $\theta$ and $\phi$ in $\mathscr{H}^{\mathfrak{B}}$ and function parameters $\alpha$, $\beta$ not occurring in $(\forall_y^x \exists_w^v) M_{x.y.v.w}^{a.b.c.d}$ such that for all elements $\tau$ and $\sigma$ of $D^{\mathfrak{B}}$

$$\models_{\mathfrak{B}} M_{a.b.\alpha a.\beta b}^{a.b.c.d} [\![\wp_{\tau.\sigma.\theta.\phi}^{a.b.\alpha.\beta}]\!].$$

The following routine properties can be verified.

LEMMAS. (1) *If $t$, $s$ are terms and the individual parameter $a$ does not occur in $s$, then* $\wp(t_s^a) = \wp_{\wp(s)}^a(t)$.

(2) *If $t$ is a term and $f$ is an operator and the function parameter $\alpha$ does not occur in $f$, then* $\wp(t_f^\alpha) = \wp_{\wp(f)}^\alpha(t)$.

(3) *If $A$ is a formula and the function parameter $\alpha$ does not occur in the operator $f$ then* $\models_{\mathfrak{B}} A_f^\alpha[\![\wp]\!]$ *iff* $\models_{\mathfrak{B}} A[\![\wp_{\wp(f)}^\alpha]\!]$.

(4) *If the individual parameter $a$ does not occur in the formula $A$ and* $\models_{\mathfrak{B}} A[\![\wp]\!]$, *then for all elements $d$ of* $D^{\mathfrak{B}}$, $\models_{\mathfrak{B}} A[\![\wp_d^a]\!]$.

(5) *If the function parameter $\alpha$ does not occur in the formula $A$ and* $\models_{\mathfrak{B}} A[\![\wp]\!]$, *then for all unary functions $\varepsilon$ of* $\mathscr{H}$, $\models_{\mathfrak{B}} A[\![\wp_\varepsilon^\alpha]\!]$.

The formula $A$ is *valid in the structure* $\mathfrak{B}$, $\models_{\mathfrak{B}} A$, iff $\models_{\mathfrak{B}} A[\![\wp]\!]$ for all $\mathfrak{B}$-assignments $\wp$.

The formula $A$ is *valid*, $\models A$, iff $\models_{\mathfrak{B}} A$ for all Henkian structures $\mathfrak{B}$.

If $\Gamma$ is a set of formulas, then the formula $A$ is a *semantical consequence* of $\Gamma$, $\Gamma \models A$, iff for all Henkian structures $\mathfrak{B}$ and all $\mathfrak{B}$-assignments $\wp$, $\models_{\mathfrak{B}} A[\![\wp]\!]$ whenever the assignment $\wp$ satisfies all the formulas in $\Gamma$.

$\Gamma \vdash A$ iff there is a derivation $\Pi$ of $\mathscr{H}\mathscr{Q}$ whose end formula is $A$ and whose undischarged assumption formulas belong to $\Gamma$.

We now have all the required preliminaries for

SOUNDNESS THEOREM. *If $\Gamma \vdash A$ then $\Gamma \models A$.*

Proof. We prove by induction on the length of the derivation $\Pi$ that

*for all $\mathfrak{B}$-assignments $\wp$, if $\wp$ satisfies all the undischarged assumption formulas of $\Pi$, then $\wp$ satisfies the end formula of $\Pi$.*



The only cases out of the ordinary correspond to the introduction and elimination rules for the Henkin quantifier. Let us thus consider them in some detail.

*Introduction rule.* We may thus assume that the derivation $\Pi$ is of the following form:

$$\cfrac{\begin{matrix}\Phi(a, b)\\ M_{a.b.fa.gb}^{a.b.c.d}\end{matrix}}{(\forall_y^x \exists_w^v) M_{x.y.v.w}^{a.b.c.d}}$$

Let $\wp$ be an assignment that satisfies all the undischarged assumption formulas of $\Pi$. The proof of this case is completed by showing that $\wp$ satisfies the formula $(\forall_y^x \exists_w^v) M_{x.y.v.w}^{a.b.c.d}$. Since $\Phi$ is a derivation of smaller length than $\Pi$ and $\wp$ satisfies all the undischarged assumption formulas of $\Phi$, the induction hypothesis gives us that $\wp$ satisfies the formula $M_{a.b.fa.gb}^{a.b.c.d}$. Since the individual parameters $a$, $b$ do not occur in the undischarged assumption formulas we find that for all elements of the universe, $\tau$ and $\sigma$, $\wp_{\tau.\sigma}^{a.b}$ satisfies the formula $M_{a.b.fa.gb}^{a.b.c.d}$. Let $\theta = \wp(f)$ and $\phi = \wp(g)$. Then, choosing the function parameters $\alpha$ and $\beta$ to be "new" function parameters we conclude that for all $\tau$ and $\sigma$ in $D^{\mathfrak{B}}$,

$$\wp_{\tau.\sigma.\theta.\phi}^{a.b.\alpha.\beta} \quad \text{satisfies the formula} \quad M_{a.b.\alpha a.\beta b}^{a.b.c.d},$$

and hence $\wp$ satisfies the formula $(\forall_y^x \exists_w^v) M_{x.y.v.w}^{a.b.c.d}$.

*Elimination rule.* In this situation we may assume that the form of $\Pi$ is

$$\cfrac{\begin{matrix}\Phi\\ (\forall_y^x \exists_w^v) M_{x.y.v.w}^{a.b.c.d}\end{matrix} \qquad \begin{matrix}(M_{s.t.\alpha s.\beta t}^{a.b.c.d})\\ \Psi(\alpha, \beta)\\ B\end{matrix}}{B}$$

Once again assume that $\wp$ is an assignment that satisfies all the undischarged assumption formulas of $\Pi$. Since $\Phi$ and $\Psi$ are shorter derivations, the induction hypothesis may be applied to them. Since $\Phi$ has no extra undischarged assumption formulas we find that the assignment $\wp$ satisfies the formula $(\forall_y^x \exists_w^v) M_{x.y.v.w}^{a.b.c.d}$. Hence there are unary functions $\theta$ and $\phi$ in $\mathscr{H}^{\mathfrak{B}}$ such that for all elements $m$, $n$ of the universe, the assignment $\wp_{m.n.\phi.\theta}^{a.b.\alpha.\beta}$ satisfies the formula $M_{a.b.\alpha a.\beta b}^{a.b.c.d}$.

Let $\tau = \wp(t)$, $\sigma = \wp(s)$ and $\wp^* = \wp_{\theta.\phi}^{\alpha.\beta}$. Recall that the conditions on the eigen-parameters $\alpha$ and $\beta$ give us that they do not occur in the individual terms $s$ and $t$. Hence $\tau = \wp^*(t)$ and $\sigma = \wp^*(s)$.

The proof of this case is completed in two cases.

Case 1: *$\wp^*$ satisfies $M_{t.s.\alpha t.\beta s}^{a.b.c.d}$.* Since the individual parameters $a$, $b$, $c$ and $d$ are just place holders, they can always be chosen so that they not occur anywhere else in the derivation $\Pi$ and since the function parameters $\alpha$ and $\beta$ are "new" parameters (not occurring in the terms $t$, $s$), the assumption that $\wp$ satisfies all the undischarged assumption formulas of $\Pi$ leads us to $\wp^*$ satisfying all the undischarged assumption formulas of $\Psi$. Thus $\wp^*$ satisfies $B$, and hence $\wp$ satisfies $B$ as required.

Case 2: *$\wp^*$ does not satisfy $M_{t.s.\alpha t.\beta s}^{a.b.c.d}$.* This case is impossible since $\wp^* = \wp_{\tau.\sigma.\theta.\phi}^{a.b.\alpha.\beta}$.

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Since the standard structures are indeed Henkian structures, the above soundness theorem gives us that the calculus $\mathscr{H}\mathscr{Q}$ is also sound with respect to the original interpretation of $(\forall^x_y \exists^v_w)$ of Henkin. More specifically, if $S$ is a provable sentence of $\mathscr{H}\mathscr{Q}$ then $S$ is valid with respect to Henkin's original interpretation.

**2. Completeness with respect to the Henkian structures.** A set $\Gamma$ of formulas is *syntactically consistent* iff $\Gamma \nvdash \bot$.

$\Gamma$ is *semantically consistent* iff there is a Henkian structure $\mathfrak{B}$ and an assignment $\wp$ which satisfies all the formulas in $\Gamma$.

THEOREM (Completeness with respect to Henkian structures). *If $\Gamma$ is a syntactically consistent set of formulas then it is semantically consistent.*

Proof. In order to simplify the exposition and reduce the number of "new" individual parameters to be introduced, we will assume that the universal quantifier $\forall$ is part of the primitive symbols of the language.

Assume that $\Gamma$ is a syntactically consistent set of formulas of $\mathscr{H}\mathscr{Q}$. We shall determine a Henkian structure $\mathfrak{B}$ and an assignment $\wp$ which satisfies all the formulas of $\Gamma$. Since the method of proof is a variation of the one introduced by Henkin for the completeness of the first (and second) order predicate calculus, we shall content ourselves with presenting an outline in a series of steps.

Step 1. Introduce denumerably many new individual and function parameters. Let $\mathscr{H}\mathscr{Q}^+$ be the new calculus.

Step 2. Let $A_1, A_2, \ldots$ be an enumeration of all the formulas of $\mathscr{H}\mathscr{Q}^+$ which start with the universal quantifier $\forall$.

Step 3. Let $H_1, H_2, \ldots$ be an enumeration of all the formulas of $\mathscr{H}\mathscr{Q}^+$ which start with the Henkin quantifier $\forall^-_-\exists^-_-$.

Step 4. Let $c_1, c_2, \ldots$ be an infinite sequence of distinct individual parameters of $\mathscr{H}\mathscr{Q}^+$ such that $c_i$ does not occur in the set of formulas $\Gamma \bigcup \{A_1, \ldots, A_i\}$.

Let $\alpha_1, \alpha_2, \ldots$ and $\beta_1, \beta_2, \ldots$ be infinite sequences of distinct function parameters of $\mathscr{H}\mathscr{Q}^+$ such that $\alpha_i, \beta_i$ do not occur in the set of formulas $\Gamma \cup \{H_1, \ldots, H_i\}$.

Step 5. Let $\Gamma^*$ be the union of $\Gamma$ and all the formulas of the form

$$M_{c_i} \supset \forall x M_x, \qquad (\forall^x_y \exists^v_w) N_{x,y,v,w} \supset \forall x \forall y N_{x,y,\alpha_i x,\beta_i y}$$

where we are assuming that $A_i = \forall x M_x$ and $H_i = (\forall^x_y \exists^v_w) N_{x,y,v,w}$.

Step 6. $\Gamma^*$ is a syntactically consistent set of formulas of $\mathscr{H}\mathscr{Q}^+$.

This step follows because of the way in which the "new" parameters are chosen.

Step 7. Let $\Gamma^+$ be a maximally consistent extension of $\Gamma^*$.

Step 8. Let $T$ be the set of terms of $\mathscr{H}\mathscr{Q}^+$ and let $\simeq$ be the relation on $T$ defined by

$$t \simeq s \quad \text{iff} \quad (t \doteq s) \in \Gamma^+.$$

$\simeq$ is a congruence relation on $T$. Let $t/\simeq$ be the congruence class of $t$ and let $D$ be the set of congruence classes.

Step 9. For each operator $f$ of $\mathscr{H}\mathscr{Q}^+$ we let $f/\simeq$ be the set of pairs of the form $(t/\simeq, ft/\simeq)$ where $t$ is a term in $T$. It can be shown that $f/\simeq$ is a function.



Step 10. The Henkian structure $\mathfrak{B}$ is obtained as follows: The domain of $\mathfrak{B}$ is the set $D$ of $\simeq$ congruence classes,

$$(t/\simeq, s/\simeq) \in P^{\mathfrak{B}} \quad \text{iff} \quad P(t, s) \in \Gamma^+, \qquad F^{\mathfrak{B}}[t/\simeq, s/\simeq] = F[t, s]/\simeq, \qquad k^{\mathfrak{B}} = k/\simeq,$$

$\mathscr{H}^{\mathfrak{B}}$ is the smallest set of functions which includes $F^{\mathfrak{B}}$, the identity, the projection functions, all functions of the form $f/\simeq$ — where $f$ is an operator of $\mathscr{H}\mathscr{Q}^+$ — and is closed under composition.

Step 11. The assignment $\wp$ is defined by $\wp(c) = c/\simeq$, $\wp(\alpha) = \alpha/\simeq$ for each individual parameter $c$ and function parameter $\alpha$ of $\mathscr{H}\mathscr{Q}^+$ respectively.

Step 12. For all formulas $G$ of $\mathscr{H}\mathscr{Q}^+$, $\models_{\mathfrak{B}} G[\![\wp]\!]$ iff $G \in \Gamma^+$.

## VIII. A VARIANT OF $\mathscr{H}\mathscr{Q}$

The method that Ehrenfeucht used to show that first-order logic extended with the Henkin quantifier $\forall^-_-\exists^-_-$ is not recursively axiomatizable was to show that the quantifier "there exists infinitely many elements such that", can be defined in terms of $\forall^-_-\exists^-_-$. The latter was done by showing that $\forall^-_-\exists^-_-$ could be used to define the quantifier "there exists a (1-1) unary function such that" [Henkin, 1961, p. 182]. Consequently, it is possible to define in $\mathscr{H}\mathscr{Q}$ the counting quantifier $\leqslant_x$ where the interpretation of the sentence

$$(A_x \leqslant_x B_x)$$

is that there are at least as many elements that satisfy $B$ as there are those satisfying $A$.

Let $\mathscr{C}\mathscr{Q}$ be the extension of first-order logic obtained by adding the above counting quantifier $\leqslant_*$. Since the quantifier "there exist infinitely many such that" is definable in terms of $\leqslant_*$, it follows that, in its standard interpretation, $\mathscr{C}\mathscr{Q}$ is not recursively axiomatizable. On the other hand, if instead of using an arbitrary function to do the counting, we require that it be one already in the structure, then the methods of the previous section can be used.

Our interest is focused on the $\mathscr{C}\mathscr{Q}$ strong normalization theorem.

We propose the following rules of inference for the counting quantifier $\leqslant_*$:

INTRODUCTION RULE FOR $\leqslant_*$.

$$\frac{\begin{matrix} (fa \doteq fb) & (A_c) \\ a \doteq b & B_{fc} \end{matrix}}{(A_x \leqslant_x B_x)} \; (\leqslant_* I)$$

*Restrictions.* The individual parameters $a$, $b$, $c$, called the *eigenparameters* of the application, are to be distinct and not to occur in undischarged assumption formulas—except for the displayed ones. $f$ is to be a unary operator in which there are no occurrences of the eigenparameters.

ELIMINATION RULE FOR $\leqslant_*$.

$$\frac{A_x \leqslant_x B_x \quad A_t \quad \begin{matrix} (B_{\alpha t}) \\ C \end{matrix}}{C} \; (\leqslant_* E)$$

*Restrictions*. The function parameter $\alpha$, the *eigenparameter* of the application, is not to occur in any undischarged assumption — except the displayed one — nor in any of the premises. The formula $C$ is to be distinct from the formula $B_{\alpha t}$.

$\leqslant_*$ CONTRACTION. The derivation

$$\dfrac{\dfrac{\begin{matrix}(fa \doteq fb) \\ \Phi_1 \\ a \doteq b\end{matrix} \quad \begin{matrix}(A_c)\\ \Phi_2(c) \\ B_{fc}\end{matrix}}{A_x \leqslant_x B_x} \quad \begin{matrix}\Phi_3 \\ A_t\end{matrix} \quad \begin{matrix}(B_{\alpha t}) \\ \Phi_4(\alpha) \\ C\end{matrix}}{C}\;(\leqslant_* E)$$

"contracts" to

$$\dfrac{\dfrac{\begin{matrix}(fa \doteq fb) \\ \Phi_1 \\ a \doteq b\end{matrix} \quad \begin{matrix}(A_c)\\ \Phi_2(c) \\ B_{fc}\end{matrix}}{A_x \leqslant_x B_x} \quad \begin{matrix}\Phi_3 \\ A_t\end{matrix} \quad \begin{matrix}\Phi_3 \\ (A_t) \\ \Phi_2(t) \\ (B_{ft}) \\ \Phi_4(f) \\ C\end{matrix}}{C}\;(S_3^3)$$

ORDINAL ASSIGNMENTS. In order for the derivation

$$\dfrac{\begin{matrix}(fa \doteq fb) \\ \Phi_1 \\ a \doteq b\end{matrix} \quad \begin{matrix}(A_c)\\ \Phi_2(c) \\ B_{fc}\end{matrix}}{(A_x \leqslant_x B_x)}\;(\leqslant_* I)$$

to be in the domain of $\mu$ it is required that both $\Phi_1$ and $\Phi_2(c)$ be in the domain and that for all terms $t$ and all derivations $\Psi$ of $A_t$ which are in the domain of $\mu$, the composite derivation $\Psi \cdot \Phi(t)$ of $B_{ft}$ be also in the domain. The actual ordinal assigned is not too important as long as the analogues of Lemmas 1–4 of Section VI hold.

The case when the end rule corresponds to the elimination rule for $\leqslant_*$ is taken under the uniform condition for elimination rules (see Section VI).

It should be clear that the method used to show that all derivations of $\mathscr{H}\mathscr{Q}$ are grounded can be applied to $\mathscr{C}\mathscr{Q}$ to obtain the analogous result.

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