approximablement par des fonctions \( k \) telles que, pour tout \( K \) dans \( U \), la borne inférieure de \( k(K) \) ne soit pas un point isolé de \( k(K) \). La construction d'une telle fonction \( k \) peut se faire en deux temps, comme celle de \( h \). On approxime d'abord \( id_y \), par une fonction \( k_l \), telle que, pour tout \( K \), la borne inférieure \( q_1(K) \) de \( k_l(K) \) soit \( >0 \). Notant \( L = \{0\} \cup \{1/n \mid n \geq 1\} \), on pose alors
\[
k(K) = k_l(K) \cup \{[q_1(K) - \alpha(K), q_1(K)](L)\},
\]
ou \( \alpha : U \rightarrow \{0, 1\} \) est une fonction continue suffisamment petite.

Compte tenu du corollaire 5.2, le lemme suivant achève de vérifier les conditions du théorème 1.1.

5.6. Lemme. \( \mathcal{H} \) est réunion dénombrable de \( Z \)-ensembles.

Démonstration. Soit \( Z_0 \) l'ensemble des éléments de \( 2^I \) ne contenant qu'un seul point. Pour \( n \geq 1 \), soit \( Z_n \) le sous-ensemble de \( 2^I \) de \( n \)-formes des \( K \) pour lesquels il existe un \( x \) appartenant à \( K \) tel que \( K \setminus \{x\} \) soit non vide et que \( d(x, K \setminus \{x\}) \geq 1/n \). Il est facile de vérifier que \( Z_n \) est fermé dans \( 2^I \). La fonction \( \varphi : 2^I \times I \rightarrow 2^I \) définie par
\[
\varphi(K, t) = \{x \in I \mid d(x, K, t) < t\}
\]
est une déformation instantanée de \( 2^I \) en \( 2^I \setminus \bigcup_n Z_n \). Cela entraîne que les \( Z_n \) sont des \( Z \)-ensembles dans \( 2^I \). Puisque tout compact dénombrable a un point isolé, \( \mathcal{H} \) est contenu dans la réunion des \( Z_n \). Les sommes 2.6 et 5.1 montrent alors que \( \mathcal{H} \) est la réunion des \( Z \)-ensembles \( \mathcal{H} \cap Z_n (n \geq 0) \).

Bibliographie


Received 8 January 1990; in revised form 17 April 1990

Collectionwise Hausdorffness at limit cardinals

by

Nobuyuki Kemoto (Oita)

Abstract. F. D. Tall conjectured:
If \( \kappa \) is a singular strong limit cardinal and \( X \) is a \( \kappa \)-CWH (Collection-Wise Hausdorff) normal or countably paracompact space of character \( < \kappa \), then \( X \) is \( \kappa \)-CWH.

In this paper, we shall show that the conjecture is true if the singular cardinals hypothesis is assumed. Furthermore, we shall study weak \( \kappa \)-CWH-ness, when \( \kappa \) is a certain limit cardinal.

1. Introduction. F. D. Tall conjectured in [T3]:

TALL'S CONJECTURE. If \( \kappa \) is a singular strong limit cardinal and \( X \) is a \( \kappa \)-CWH (Collection-Wise Hausdorff) normal or countably paracompact space of character \( < \kappa \), then \( X \) is \( \kappa \)-CWH.

W. G. Fleissner proved in [F1] that this conjecture is true if the GCH (Generalized Continuum Hypothesis) is assumed. More generally, as in [T2], this conjecture is true if there is a \( \mu < \kappa \) such that \( 2^\lambda = \lambda^+ \) for every \( \mu < \lambda < \kappa \). Whenever of \( \kappa = \omega \) holds, this conjecture is true without other set-theoretical additional axioms or normality or countable paracompactness by the argument of the proof of [F2, Theorem 1 (b)]. Thus we focus on the case of \( \kappa \geq \omega_1 \).

In Section 2, we shall characterize \( \kappa \)-CWH-ness using the sparse-like argument in [F4], and also that the conjecture is true if the SCH (Singular Cardinals Hypothesis) is assumed. In Section 3, we shall study weak \( \kappa \)-CWH-ness (in the sense of [T1]) for various spaces where \( \kappa \) is a certain limit cardinal.

A closed discrete subspace \( Y \) of a space \( X \) is said to be separated if there is a neighborhood \( U_y \) of \( y \) for each \( y \in Y \) such that \( U_y \cap Y \) is disjoint. \( Y \) is \( \kappa \)-separated if every subset of \( Y \) of size \( < \kappa \) is separated. A space \( X \) is \( \kappa \)-CWH (\( \kappa \)-CWH) if every closed discrete subspace of size \( x \) (\( < \kappa \), respectively) is separated. “Closed UnBounded” is abbreviated as \( CUB \). In this paper, no separation axioms are assumed.

1980 Mathematics Subject Classification (1985 Revision): 54D15, 03E50.

Key words and phrases: collectionwise Hausdorff, strong limit cardinal, singular cardinals hypothesis.
2. The CWI-case. In this section, we shall prove that Tall’s conjecture is true assuming SCH. Some of our arguments will be somewhat similar to the arguments in \cite{F4} or \cite{Wa}. Throughout this paper, \( A \) denotes a neighborhood base at \( y \).

**Definition 2.1.** Let \( X \) be a space and let \( Y \) be a subspace of size \( x \) with \( \aleph_1 \geq \omega_1 \). A countable sequences of partitions \( \{ Y_n : \alpha < \aleph_1 \} \) of \( Y \) is said to be nice partitions if for each \( n < \omega \), there is a \( b_n \in \prod_{\alpha < \aleph_1} A_\alpha \) such that

1. \( \|Y_n\| < \omega \) for each \( n < \omega \) and \( \aleph_1 < \aleph_2 \).
2. \( \{ \alpha < \aleph_1 : \alpha < \aleph_1 \} \) contains a cub set in \( \aleph_2 \).

Here a "partition" means a disjoint cover.

**Lemma 2.2.** If \( Y \) is a \( \aleph_1 \)-separated discrete subspace of size \( x \) with \( \omega_1 \leq \aleph_1 \) which has nice partitions, then \( Y \) is separated.

**Proof.** Take such partitions \( \{ Y_n : \alpha < \aleph_1 \} \) and \( b_n \in \prod_{\alpha < \aleph_1} A_\alpha \) as in Definition 2.1. By \( (1) \) of Definition 2.1, take a cub set \( C \) contained in \( \{ \alpha < \aleph_1 : \alpha < \aleph_1 \} \) for each \( n < \omega \). Put \( C = \{ 0 \} \cap \prod_{\alpha < \aleph_1} C_\alpha \). Enumerate \( C \) in increasing order, say \( C = \{ \mu(\gamma) : \gamma < \aleph_1 \} \). For each \( n < \omega \), put \( Y(\gamma, n) = \bigcup \{ \mu(\gamma) : \mu(\gamma) < \aleph_1 \} \). Since \( C \) is cub in \( \aleph_1 \), \( \{ Y(\gamma, n) : \gamma < \aleph_1 \} \) is a partition of \( Y \) for each \( n < \omega \). By induction fix a \( g_n \in \prod_{\alpha < \aleph_1} A_\alpha \) for each \( n < \omega \) such that

1. \( g_{n+1}(\gamma) = g_n(\gamma) \) for each \( n < \omega \) and \( \gamma < \aleph_1 \).
2. \( \{ \mu(\gamma) : \mu(\gamma) \in \prod_{\alpha < \aleph_1} C_\alpha \} \) is disjoint for each \( n < \omega \) and \( \gamma < \aleph_1 \).

The statement (b) is ensured by \( \aleph_1 \)-separatedness, (c) by \( \alpha(\gamma) \in C \) in \( C_\alpha \), and (2) of Definition 2.1. By (a) and (c), the following holds.

\[
\bigcup_{\alpha < \aleph_1} \{ \alpha(\gamma) : \alpha(\gamma) \in \prod_{\alpha < \aleph_1} C_\alpha \} \times g_{n+1}(\gamma) = 0
\]

For each \( \gamma \in \text{Y} \) and \( n < \omega \), put \( \gamma(\gamma, n) = \gamma(\gamma, n+1) = \gamma \). Then by (2) of Definition 2.1, it is easy to show that \( \gamma(\gamma, n) \in \gamma(\gamma, n+1) \gamma) \) for each \( \gamma \in \text{Y} \) and \( n < \omega \). Thus there is a \( \gamma \in \omega_1 \) such for each \( \gamma \in \text{Y} \) there is \( \gamma(\gamma, n) = \gamma(\gamma, n+1) \gamma) \). It suffices to show that \( \{ \gamma(\gamma, n+1) : \gamma \in \text{Y} \} \) is disjoint. To show this, fix \( \gamma, \delta \in \text{Y} \) with \( \gamma \neq \delta \). Then\( n = \min \{ n(\gamma), n(\delta) \} \). Then

\[
\text{Case 1.} \gamma(\gamma, n) = \gamma(\delta, n) = \gamma.
\]

\[
\text{Case 2.} \gamma(\gamma, n) < \gamma(\delta, n) \text{ (the remaining case is similar)}.
\]

**Subcase 1.** \( \gamma(\gamma, n) < \gamma(\delta, n+1) \gamma) = \gamma \). In this case, since \( \gamma = \gamma(\gamma, n+1) \gamma) \), the claim follows from (b), (c) and (a).

**Subcase 2.** \( \gamma(\gamma, n+1) \gamma) < \gamma(\delta, n) \). In this case, \( \gamma(\gamma, n+1) \gamma) < \gamma(\delta, n) \), we have \( n = n(\gamma) \). First assume \( \gamma(\gamma, n+1) \gamma) = \gamma(\gamma, n) \). Then by \( \gamma(\gamma, n) = \gamma(\gamma, n+1) \gamma) \), \( \gamma \) and \( \delta \) are in \( \text{Y}(n+1) \). Thus the claim follows from (b), (c) and (a).

Next assume \( \gamma(\gamma, n+1) \gamma) < \gamma(\delta, n) \). Then \( \gamma \) is in \( \bigcup_{\gamma < \aleph_1} \text{Y}(n+1) \) and \( \delta \) is in \( \text{Y}(n+1) \) by \( \gamma(\gamma, n+1) \gamma) = \gamma \). Thus the claim follows from (d), (e) and (a). This completes the proof.

**Definition 2.3.** Let \( \aleph_1 \) be a limit cardinal. A sequence \( \{ x_\alpha : \alpha < \aleph_1 \} \) of cardinals in \( \aleph_1 \) is said to be normal if

1. \( x_\alpha < x_{\alpha+1} \) for every \( \alpha < \aleph_1 \),
2. \( x_\alpha = \sup_{\beta < \alpha} x_\beta \) for every limit \( \alpha < \aleph_1 \),
3. \( x = \sup_{\alpha < \aleph_1} x_\alpha \).

**Remark.** Note that there always exists a normal sequence in \( \aleph_1 \) if \( \aleph_1 \) is a limit cardinal, and also that there exists a normal sequence \( \{ x_\alpha : \alpha < \aleph_1 \} \) with \( 2^{\aleph_1} \leq \aleph_1 \) for every \( \aleph_1 < \aleph_1 \) whenever \( \aleph_1 \) is a strong limit cardinal.

The proof of the following lemma is routine.

**Lemma 2.4.** Let \( \aleph_1 \) be a limit cardinal with \( \omega_1 \leq \aleph_1 \) and let \( \{ x_\alpha : \alpha < \aleph_1 \} \) and \( \{ x'_\alpha : \alpha < \aleph_1 \} \) be normal sequences in \( \aleph_1 \). Then \( \{ \alpha < \aleph_1 : x_\alpha < x'_\alpha \} \) is cub in \( \aleph_1 \).

**Lemma 2.5.** Let \( Y \) be a closed discrete subspace of size \( x \) with \( \omega_1 \leq \aleph_1 \). Moreover, let \( \{ Y_\alpha : \alpha < \aleph_1 \} \) be a partition of \( Y \), \( \{ x_\alpha : \alpha < \aleph_1 \} \) be a normal sequence in \( \aleph_1 \), \( C \) a cub set in \( \aleph_1 \), such that \( \{ \alpha : \alpha < \aleph_1 \} \subset \{ \alpha < \aleph_1 : \bigcup \{ [b_\alpha] : \alpha < \aleph_1 \} \cap \bigcup Y_\alpha \in C \} \subset \bigcup \{ \alpha(\gamma) : \gamma < \aleph_1 \} \). Then there is a partition \( \{ Y_\alpha : \alpha < \aleph_1 \} \) of \( Y \) such that

1. \( C \subset Y_\alpha \),
2. \( \{ \alpha < \aleph_1 : \bigcup \{ [b_\alpha] : \alpha < \aleph_1 \} \cap \bigcup Y_\alpha \in C \} \subset \bigcup \{ \alpha(\gamma) : \gamma < \aleph_1 \} \).
to have property $P(\mu)$ if for every $m: \gamma \rightarrow \mu$, there is a $b \in \prod_{x \in Y} \mathcal{A}$, such that \(\{m(\gamma); b(\gamma) \cap \gamma(\gamma) \neq 0, \gamma \in Y\}\) is bounded in $\mu$ for each $\gamma \in Y$. The whole space $X$ is also said to have property $P(\mu)$ if every closed discrete subspace $Y$ has property $P(\mu)$ in the above sense. Thus we shall use property $P(\mu)$ in two different ways, but these differences will be clarified by the context.

Remark. Note that normal or countably paracompact spaces have property $P(\omega)$ and $\kappa$-para-Lindelöf spaces (in the sense of [F4]) have property $P(\kappa)$. The notion of the $\mathcal{A}(\mu)$-property is known as a generalization of countable paracompactness to higher cardinals, see [R]. A space has the $\mathcal{A}(\mu)$-property if for every increasing open cover \(\{U_\alpha: \alpha < \mu\}\) (i.e., $U_\alpha \subseteq U_{\alpha+1}$ if $\alpha < \beta$, each $U_\alpha$ is open and $\bigcup_{\alpha \in \kappa} U_\alpha = X$), there is an increasing open cover \(\{V_\alpha: \alpha < \mu\}\) such that $\text{cl} V_\alpha \subseteq U_\alpha$ for each $\alpha < \mu$. Note that "countable paracompactness $\Rightarrow \mathcal{A}(\omega)$-property" holds, see [En]. And note that the argument of the proof of this equivalence shows "$\kappa$-para-Lindelöfness $\Rightarrow \mathcal{A}(\kappa)$-property". Here we remark the relation between $\mathcal{A}(\mu)$-property and property $P(\mu)$.

**Lemma 2.7.** Every space $X$ having the $\mathcal{A}(\mu)$-property has property $P(\mu)$, where $\mu$ is an infinite cardinal.

**Proof.** Let $Y$ be a closed discrete subspace of a space $X$ having the $\mathcal{A}(\mu)$-property. Fix an arbitrary mapping $m: \gamma \rightarrow \mu$. For each $\alpha < \mu$, put $U_\alpha = X \setminus \bigcup_{\beta \leq \alpha} \text{cl} V_\beta$. Then \(\{U_\alpha: \alpha < \mu\}\) is an increasing open cover of $X$. Take an increasing open cover \(\{V_\alpha: \alpha < \mu\}\) such that $\text{cl} V_\alpha \subseteq U_\alpha$ for each $\alpha < \mu$. For each $y \in Y$, let $f(y)$ be the least $\beta < \mu$ such that $y \in V_\beta$. If $y \in V_\mu$, then $f(y) = \mu$. Note that $a < f(y)$ if $y \in m^{-1}(a)$. For each $y \in m^{-1}(a)$, fix $h(y) \in \mathcal{A}$ such that $h(y) \in \text{cl} V_{f(y)}$. Then it is easy to see that \(\{m(\gamma); b(\gamma) \cap \gamma(\gamma) \neq 0, \gamma \in Y\} \subseteq m(\gamma)\) for each $\gamma \in Y$.

**Lemma 2.8.** Let $\mu$ be an infinite cardinal, $Y$ a closed discrete subspace of a space $X$, and $m_0$ an arbitrary mapping $Y \rightarrow \mu$. Assume $Y$ has property $P(\mu)$. Then for each $\alpha < \omega$, there is a $b_\alpha \in \prod_{x \in Y} \mathcal{A}$ and a $m_\alpha: Y \rightarrow \mu$ such that \(\{m_\alpha(\gamma); b_\alpha(\gamma) \cap \gamma(\gamma) \neq 0, \gamma \in Y\} \subseteq m_\alpha(\gamma)\) for each $\gamma \in Y$.

**Proof.** Assume that $m_0$ and $b_0$ have been defined. By property $P(\mu)$, there is a $b_\alpha \in \prod_{x \in Y} \mathcal{A}$ such that \(\{m_\alpha(\gamma); b_\alpha(\gamma) \cap \gamma(\gamma) \neq 0, \gamma \in Y\}\) is bounded in $\mu$ for each $\gamma \in Y$. Fixing $m_\alpha(\gamma)$ in $\mu$ which contains $A_\alpha$ for each $\gamma \in Y$, we are done.

**Lemma 2.9.** Let $\kappa$ be a strong limit cardinal with $\omega_1 \in \mathcal{C}(\kappa)$, let $Y$ be an infinite cardinal less than $\kappa$, and let \(\{x : \kappa < \mathcal{C}(x)\}\) be a normal sequence of cardinals in $\kappa$ such that $2^\kappa < \kappa_\alpha$ for each $\alpha < \kappa$. Assume $Y$ is a closed discrete subspace of size $\kappa$ such that $Y$ has property $P(\mu)$ and has a partition $\{Y_\alpha; \alpha < \kappa\}$ with $|Y_\alpha| < \kappa$ for each $\alpha < \kappa$ and each $y \in Y$ has a neighborhood basis $\mathcal{A}$ with $|\mathcal{A}| < \kappa$. Then there is a $b \in \prod_{x \in Y} \mathcal{A}$ such that \(\{x \in \mathcal{C}(\kappa); b_\alpha(\gamma) \cap \gamma(\gamma) \neq 0, \gamma \in \alpha \in Y\} \subseteq m_\alpha(\gamma)\) for each $\gamma \in Y$. Fixing $m_\alpha(\gamma)$ in $\mu$ which contains $A_\alpha$ for each $\gamma \in Y$, we are done.

**Proof.** For each $x \in \mathcal{C}(\kappa)$, put $Z_x = x \in Y_\alpha$.

**Claim 1.** \(\alpha \in \mathcal{C}(\kappa); \prod_{x \in Y_\alpha} \mathcal{A}(\kappa) \rightarrow \mathcal{C}(\kappa)\) contains a cub set.

**Proof.** Since $|Y_\alpha| < \kappa$ for each $\alpha < \kappa$, we can fix $f(x) \in \mathcal{A}$ such that $|Y_\alpha| < \kappa^{\alpha}$, Then it is easy to see that $C_\alpha = \{x \in \mathcal{C}(\kappa); \forall \beta < \kappa (f(\beta) < \alpha)\}$ is cub. If $\alpha$ is an element of $C_\alpha$,
Thus case 1 happens and using (b), we obtain

\[ J(y, \Delta, b(y, \Delta)) = \{ b(y, \Delta) \cap b(y, \Delta) \neq 0, y \in Z_{\alpha(y)} \} \]

\[ = \{ m_{\alpha}(y) : b(y, \Delta) \cap b(y, \Delta) \neq 0, y \in Z_{\alpha(y)} \} \]

\[ \subset \{ m_{\alpha}(y) : b(y, \Delta) \cap b(y, \Delta) \neq 0, y \in Y \} \]

\[ = m_{\alpha}(y, \Delta) \prec \mu. \]

Therefore subcase 1 of case 1 happens. Then by the definition of \( m_{\alpha, \beta} \)

\[(d) \quad m_{\alpha, \beta}(y, \Delta) = m_{\alpha}(y, \Delta) \prec \mu. \]

By \( b(y, \Delta) \cap b(y, \Delta) \neq 0 \) and by (d), there is a \( y \in Z_{\alpha(y)} \) such that \( b(y, \Delta) \cap b(y, \Delta) \neq 0 \). By \( b(y, \Delta) \cap b(y, \Delta) \neq 0 \) and by (a),

\[(e) \quad m_{\alpha}(y, \Delta) \in m_{\beta}(y). \]

Also by \( b(y, \Delta) \cap b(y, \Delta) \neq 0 \) and \( y \in Z_{\alpha(y)} \),

\[(f) \quad m_{\alpha}(y, \Delta) \in m_{\beta}(y). \]

Then by (d), (e) and (f), \( m_{\alpha, \beta}(y, \Delta) \in m_{\alpha}(y, \Delta) \prec \mu \). But this is a contradiction. This completes the proof.

**Theorem 2.10.** Let \( x \) be a singular strong limit cardinal with \( \omega_1 \leq cf < \kappa \), let \( \mu, \chi \) be infinite cardinals less than \( \kappa \), and let \( Y \) be a closed discrete subspace of size \( \kappa \) such that \( Y \) has property \( P(\mu) \) and each \( y \in Y \) has a neighborhood base \( \mathfrak{U} \) with \( |\mathfrak{U}| \leq \chi \). Assume that there is a normal sequence \( \{ x_n : n < \kappa \} \) of cardinals in \( \kappa \) such that \( \{ x : cf < \kappa \} : 2^n = x_n \} \) contains a cub set in \( \mathfrak{U} \). Then \( Y \) has nice partitions (Thus \( Y \) is separated if \( Y \) is \( \kappa \)-separated by Lemma 2.2).

**Proof.** Fix a 1-1 onto map \( f : Y \to \kappa \). For each \( \alpha < \kappa \), put \( Y_{\alpha} = f^{-1}(\kappa_\alpha) \). Then \( \{ Y_{\alpha} : \alpha < \kappa \} \) is a partition of \( Y \) with \( |Y_{\alpha}| < \kappa \) for each \( \alpha < \kappa \). Assume a partition \( \{ Y_{\alpha} : \alpha < \kappa \} \) of \( Y \) with \( |Y_{\alpha}| < \kappa \) for each \( \alpha < \kappa \) is defined. By Lemma 2.4, we may assume \( 2^n = \kappa_{n+1} \) for each \( \alpha < \kappa \). Applying Lemma 2.9 to \( \{ Y_{\alpha} : \alpha < \kappa \} \), take a \( b_\alpha \) in \( \bigcap_{\beta < \alpha} \mathfrak{U} \), such that \( \{ x : cf(x) \in \{ Y_{\beta} : \beta < \alpha \} \} \rightarrow Y \). Then \( 2^{n+1} = \kappa_{n+2} \) contains a cub set. Since \( \{ x : cf(x) = x_n \} \) contains a cub set, \( \{ x : cf(x) \in \{ Y_{\beta} : \beta < \alpha \} \} \rightarrow Y \) also contains a cub set. Then by Lemma 2.5, there is a partition \( \{ Y_{\alpha+1} : \alpha < \kappa \} \) of \( Y \) such that \( |Y_{\alpha+1}| < \kappa \) for each \( \alpha < \kappa \) and \( \{ x : cf(x) \in \{ Y_{\alpha} : \alpha < \kappa \} \} \rightarrow Y \). Then repeated applications of this process, one can get nice partitions.

**Remark.** If there is a normal sequence \( \{ x_n : n < \kappa \} \) as in Lemma 2.10, then \( 2^\kappa = \kappa^+ \) by [Je, Lemma 8.2]. Next we shall show such a normal sequence exists assuming SCH (Singular Cardinals Hypothesis).

**Lemma 2.11 [SCH].** Let \( x \) be a singular strong limit cardinal with \( \omega_1 \leq cf < \kappa \), and let \( \{ x_n : n < \kappa \} \) be a normal sequence of cardinals in \( \kappa \) such that \( 2^n = \kappa_{n+1} \) for each \( \alpha < \kappa \). Then \( \{ x : cf(x) = x_n \} \) contains a cub set.

**Proof.** Take \( \{ x_n : n < \kappa \} \) and \( \{ Y_n : n < \kappa \} \) as in the proof of Theorem 3.1. For
each $a < \text{cfx}$, take separated $Y'_a \subseteq Y_a$ of size $|Y_a|$. Fix $b \in \prod_{a < \omega} b_a$, such that $\{b(y) : y \in Y'_a\}$ is disjoint and each $b(y)$ is $\mu$-cc.

Claim. \( \bigcup (b(y) : y \in \bigcup_{a < \omega} Y'_a) \cap Y \subseteq \bigcup_{a < \omega} Y'_a \times \mu \) for each $\alpha < \text{cfx}$.

Proof. Assume the claim fails. Then one can take a separated $Z = \bigcup (b(y) : y \in \bigcup_{a < \omega} Y'_a) \cap Y$ of size $|\bigcup_{a < \omega} Y'_a| < \kappa$ by weak $\kappa$-separability. Take a $b' \in \prod_{a < \omega} b_a$, such that $\{b'(y) : y \in Z\}$ is disjoint. Then for every $y \in Z$, there is a $y \in \bigcup_{a < \omega} Y'_a$ such that $b'(y) \neq b(y)$. Then there are $a, \alpha < \omega$ such that $\alpha < \omega$ and $a \neq \alpha$. This contradicts the $\mu$-cc-ness of $b(y)$.

Then the proof of the claim is complete.

Since for each $\alpha < \text{cfx}$ with $\mu < \kappa$, $|\bigcup_{a < \omega} Y'_a| = |\bigcup_{a < \omega} Y_a| = \kappa$, and $|Y'_a| = \kappa_{a+1}$, $|Y_a| - \kappa_{a+1} = |\bigcup_{a < \omega} Y'_a| = \kappa_{a+1}$ by the claim. Thus we can take a separated $Y_a' = Y_a - \bigcup (b(y) : y \in \bigcup_{a < \omega} Y'_a)$ of size $\kappa_{a+1}$ for such $\alpha$. Then it is straightforward to show that $Y = \bigcup \{Y_a' : \alpha < \text{cfx}, \mu < \kappa\}$ is desired. This completes the proof.

To end this paper, we shall study the relation between property $P(\text{cfx})$ and weak $\kappa$-CHW-ness.

**Lemma 3.3.** Let $Y$ be a subspace of a space and $\kappa$ an infinite cardinal. Then $Y$ has property $P(\text{cfx})$ if and only if $Y$ has property $P(\kappa)$.

Proof. Assume $x$ is a singular cardinal (otherwise, this is clear). Fix a normal sequence $\langle x_a : a < \kappa \rangle$ of cardinals in $\kappa$. First assume $Y$ has property $P(\text{cfx})$. We shall prove $Y$ has property $P(\kappa)$. To show this, fix an arbitrary $m : Y \to \kappa$. Define $m' : Y \to \kappa$ by $m(y) = m(y) + a$ if $m(y) = x_a$ for each $y \in Y$. Then by property $P(\text{cfx})$, there is a $b \in \prod_{a < \omega} b_a$ such that $A_m = \{m'(y) : b(y) \vee b(y) \neq 0, y \neq 0\}$ is bounded in $\kappa$. Thus we can pick $a(y) < \alpha$ such that $a(y) \in A_m$ for each $y \in Y$. It is straightforward to show that $\{m'(y) : b(y) \vee b(y) \neq 0, y \neq 0\} = \kappa_{a+1} < \kappa$ for each $y \in Y$.

Next assume $Y$ has property $P(\kappa)$. Fix an arbitrary $m : Y \to \kappa$. Define $m' : Y \to \kappa$ by $m'(y) = x_a$ for each $y \in Y$. Then by property $P(\kappa)$, there is a $b \in \prod_{a < \omega} b_a$ such that $A_m = \{m'(y) : b(y) \vee b(y) \neq 0, y \neq 0\}$ is bounded in $\kappa$ for each $y \in Y$. Thus we can pick $a(y) < \alpha$ such that $a(y) \in A_m$. Then it is straightforward to show that $\{m'(y) : b(y) \vee b(y) \neq 0, y \neq 0\} = \kappa_{a+1} < \kappa$. The proof is complete.

**Theorem 3.4.** Let $x$ be a singular cardinal, and let $Y$ be a weakly $< \kappa$-separated closed discrete subspace of $x$ having property $P(\kappa)$. Then there is a separated $Y' \subseteq Y$ of size $x$.

Proof. Fix a strictly increasing cofinal sequence $\langle x_a : a < \kappa \rangle$ of successor cardinals in $x$ with $\kappa < x_\omega$ for example, this can be done by putting $x_0 = \lambda_1$ for each $a < \kappa$, where $\langle \lambda_a : a < \kappa \rangle$ is a normal sequence of cardinals in $x$ with $\kappa < \lambda_\omega$. Fix a $1$-1 onto map $f : x \to x$. By putting $Y_a = f^{-1}(x_a)$, $Y_a < \kappa$ is a partition of $Y$ with $|Y_a| = x_a$ for each $a < \kappa$. Define $m' : Y \to \kappa$ by $m'(y) = a$ if $y \in Y_a$ for each $y \in Y$. Then by property $P(\kappa)$, there is a $b \in \prod_{a < \omega} b_a$ such that $\{m'(y) : b(y) \vee b(y) \neq 0, y \neq 0\} = \kappa_{a+1}$. Then we put $Y' = \{y \in Y : a(y) < \beta\}$. Since $Y' = \bigcup_{a < \omega} Y'_a$ and the size of $Y'$ is a successor cardinal $> \kappa$, there is a $\beta < \text{cfx}$ such that $|Y'_\beta| = |Y'\beta|$. Then $\text{cfx} < \alpha$.

$C = \{a < \text{cfx} : \forall a < \alpha (\beta(a) < \alpha)\}$ is unbounded in $\text{cfx}$ (in fact, $C$ is cub in $\text{cfx}$ if $\omega_1 < \text{cfx}$). By weak $< \kappa$-separability, choose a separated $Y'_\beta \subseteq Y'_\beta$ of size $|Y'_\beta| = |Y'\beta|$. Put $Y' = \bigcup_{a < \beta} Y'_a$. Take a $b \in \prod_{a < \omega} b_a$ such that $b(y) = b(y)$ for each $y \in Y'$ and $b(y) \neq b(y)$ otherwise. Since $C$ is unbounded in $\text{cfx}$, the size of $Y'$ is $\kappa$. We shall show $\{b(y) : y \in Y'\}$ separates $Y'$. To show this assume $\forall a < \alpha (\beta(a) < \alpha) \Rightarrow \forall a < \alpha (\beta(a) = \alpha)$. This completes the proof.

Finally, we shall show that weak $< \kappa$-separability can be removed from Theorem 3.4 if "singular" is replaced by "regular".

**Theorem 3.5.** Let $x$ be a regular cardinal, and let $Y$ be a closed discrete subspace of size $x$ having property $P(\kappa)$. Then there is a separated $Y' \subseteq Y$ of size $x$.

Proof. Identify $Y$ with $x$. By property $P(\kappa)$, there is a $b \in \prod_{a < \omega} b_a$ such that for each $a < x$, $\langle b(y) : b(y) \neq 0, y \neq 0\rangle$ is $\kappa$. Thus it is easy to show $C = \{a < x : \forall a < \alpha (\beta(a) < \alpha)\}$ is separable and of size $x$.

References


Department of Mathematics
FACULTY OF EDUCATION
Oita University
Daimaru Oita 870-11, Japan

Received 8 February 1990