approximée arbitrairement par des fonctions k telles que, pour tout K dans U, la borne inférieure de $k(K)$ ne soit pas un point isolé de $k(K)$. La construction d'une telle fonction k peut se faire en deux temps, comme celle de h. On approxime d'abord id_U par une fonction k_1 telle que, pour tout K, la borne inférieure $q_1(K)$ de $k_1(K)$ soit > 0. Notant $L = \{0\} \cup \{1/n|\ n \geqslant 1\}$, on pose alors

$$k(K) = k_1(K) \cup [l(q_1(K) - \alpha(K), q_1(K))(L)],$$

où α: $U \to]0, 1]$ est une fonction continue suffisamment petite.

Compte tenu du corollaire 5.2, le lemme suivant achève de vérifier les conditions du théorème 1.1.

5.6. LEMME. *$\mathscr{H}$ est réunion dénombrable de Z-ensembles.*

Démonstration. Soit Z_0 l'ensemble des éléments de 2^I ne contenant qu'un seul point. Pour $n \geqslant 1$, soit Z_n le sous-ensemble de 2^I formé des K pour lesquels il existe un x appartenant à K tel que $K \backslash \{x\}$ soit non vide et que $d(x, K\backslash\{x\}) \geqslant 1/n$. Il est facile de vérifier que Z_n est fermé dans 2^I. La fonction φ: $2^I \times I \to 2^I$ définie par

$$\varphi(K, t) = \{x \in I|\ d(x, K) \leqslant t\}$$

est une déformation instantanée de 2^I en $2^I \backslash \bigcup_{n=0}^{\infty} Z_n$. Ceci entraîne que les Z_n sont des Z-ensembles dans 2^I. Puisque tout compact dénombrable a un point isolé, $\mathscr{H}$ est contenu dans la réunion des Z_n. Les lemmes 2.6 et 5.1 montrent alors que $\mathscr{H}$ est la réunion des Z-ensembles $\mathscr{H} \cap Z_n$ $(n \geqslant 0)$.

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22 rue Jouvenet, 75016 Paris, France

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Collectionwise Hausdorffness at limit cardinals

by

Nobuyuki Kemoto (Oita)

Abstract. F. D. Tall conjectured:

If $\varkappa$ is a singular strong limit cardinal and X is a $<\varkappa$-CWH (CollectionWise Hausdorff) normal or countably paracompact space of character $<\varkappa$, then X is $\varkappa$-CWH.

In this paper, we shall show that the conjecture is true if the singular cardinals hypothesis is assumed. Furthermore, we shall study weak $\varkappa$-CWH-ness, when $\varkappa$ is a certain limit cardinal.

1. Introduction. F. D. Tall conjectured in [T3]:

TALL'S CONJECTURE. *If $\varkappa$ is a singular strong limit cardinal and X is a $<\varkappa$-CWH (CollectionWise Hausdorff) normal or countably paracompact space of character $<\varkappa$, then X is $\varkappa$-CWH.*

W. G. Fleissner proved in [F1] that this conjecture is true if the GCH (Generalized Continuum Hypothesis) is assumed. More generally, as in [T2], this conjecture is true if there is a $\mu < \varkappa$ such that $2^\lambda = \lambda^+$ for *every* $\mu \leqslant \lambda < \varkappa$. Whenever $\mathrm{cf}\,\varkappa = \omega$ holds, this conjecture is true without other set-theoretical additional axioms or normality or countable paracompactness by the argument of the proof of [F2, Theorem 1 (b)]. Thus we focus on the case $\mathrm{cf}\,\varkappa \geqslant \omega_1$.

In Section 2, we shall characterize "$<\varkappa$-CWH $\to \varkappa$-CWH" using the sparse-like argument in [F4], and also show that the conjecture is true if the SCH (Singular Cardinals Hypothesis) is assumed. In Section 3, we shall study weak $\varkappa$-CWH-ness (in the sense of [T1]) for various spaces where $\varkappa$ is a certain limit cardinal.

A closed discrete subspace Y of a space X is said to be *separated* if there is a neighborhood U_y of y for each $y \in Y$ such that $\{U_y: y \in Y\}$ is disjoint. Y is *$<\varkappa$-separated* if every subset of Y of size $<\varkappa$ is separated. A space X is $\varkappa$-CWH ($<\varkappa$-CWH) if every closed discrete subspace of size $\varkappa$ ($<\varkappa$, respectively) is separated. "Closed UnBounded" is abbreviated as *cub*. In this paper, no separation axioms are assumed.

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2. The CWH-case. In this section, we shall prove that Tall's conjecture is true assuming SCH. Some of our arguments will be somewhat similar to the arguments in [F4] or [Wa]. Throughout this paper $\mathscr{B}_y$ denotes a neighborhood base at y.

DEFINITION 2.1. Let X be a space and let Y be a subspace of size $\varkappa$ with $\mathrm{cf}\varkappa \geqslant \omega_1$. A countable sequences of partitions $\{Y_{n\alpha}: \alpha < \mathrm{cf}\varkappa\}$ $(n < \omega)$ of Y is said to be *nice partitions* if for each $n < \omega$, there is a $b_n \in \prod_{y\in Y} \mathscr{B}_y$ such that

(1) $|Y_{n\alpha}| < \varkappa$ for each $n < \omega$ and $\alpha < \mathrm{cf}\varkappa$,

(2) $\{\alpha < \mathrm{cf}\varkappa: \mathrm{cl}(\bigcup\{b_n(y): y \in \bigcup_{\beta<\alpha} Y_{n\beta}\}) \subset \bigcup_{\beta<\alpha} Y_{n+1,\beta}\}$ contains a cub set in $\mathrm{cf}\varkappa$.

Here a "partition" means a disjoint cover.

LEMMA 2.2. *If Y is a $<\varkappa$-separated discrete subspace of size $\varkappa$ with $\omega_1 \leqslant \mathrm{cf}\varkappa$ which has nice partitions, then Y is separated.*

Proof. Take such partitions $\{Y_{n\alpha}: \alpha < \mathrm{cf}\varkappa\}$ and $b_n \in \prod_{y\in Y} \mathscr{B}_y$ $(n < \omega)$ as in Definition 2.1. By (2) of Definition 2.1, take a cub set C_n contained in $\{\alpha < \mathrm{cf}\varkappa: \mathrm{cl}(\bigcup\{b_n(y): y \in \bigcup_{\beta<\alpha} Y_{n\beta}\}) \subset \bigcup_{\beta<\alpha} Y_{n+1,\beta}\}$ for each $n < \omega$. Put $C = \{0\} \cup \bigcap_{n<\omega} C_n$. Enumerate C in increasing order, say $C = \{\alpha(\gamma): \gamma < \mathrm{cf}\varkappa\}$. For each $n < \omega$ and $\gamma < \mathrm{cf}\varkappa$, put $Y(n, \gamma) = \bigcup\{Y_{n\beta}: \alpha(\gamma) \leqslant \beta < \alpha(\gamma+1)\}$. Since C is cub in $\mathrm{cf}\varkappa$, $\{Y(n, \gamma): \gamma < \mathrm{cf}\varkappa\}$ is a partition of Y for each $n < \omega$. By induction fix a $b'_n \in \prod_{y\in Y} \mathscr{B}_y$ for each $n < \omega$ such that

(a) $b'_{n+1}(y) \subset b'_n(y) \subset b_n(y)$ for each $n < \omega$ and $y \in Y$,

(b) $\{b'_n(y): y \in Y(n, \gamma)\}$ is disjoint for each $n < \omega$ and $\gamma < \mathrm{cf}\varkappa$.

(c) $(\bigcup\{b_n(z): z \in \bigcup_{\beta<\alpha(\gamma)} Y_{n\beta}\}) \cap b'_{n+1}(y) = 0$ for each $y \in Y(n+1, \gamma)$ and $\gamma < \mathrm{cf}\varkappa$.

The statement (b) is ensured by $<\varkappa$-separatedness, (c) by $\alpha(\gamma) \in C \subset C_n$ and (2) of Definition 2.1. By (a) and (c), the following holds:

(d) $\bigcup\{b'_{n+1}(z): z \in \bigcup_{\beta<\alpha(\gamma)} Y_{n\beta}\} \cap b'_{n+1}(y) = 0$ for each $y \in Y(n+1, \gamma)$ and $\gamma < \mathrm{cf}\varkappa$.

For each $y \in Y$ and $n < \omega$, put $\gamma(n, y) = \gamma$ such that $y \in Y(n, \gamma)$. Then by (2) of Definition 2.1, it is easy to show that $\gamma(n, y) \geqslant \gamma(n+1, y)$ for each $y \in Y$ and $n < \omega$. Thus there is an $n(y) < \omega$ for each $y \in Y$ such that $\gamma(n(y), y) = \gamma(n, y)$ for each $n \geqslant n(y)$. It suffices to show that $\{b'_{n(y)+2}(y): y \in Y\}$ is disjoint. To show this, fix y, y' in Y with $y \neq y'$. Put $n = \min\{n(y), n(y')\}$. Then

(e) $n \leqslant n+1 \leqslant n+2 \leqslant n(y)+2,\ n(y')+2$.

We shall show

$$b'_{n(y)+2}(y) \cap b'_{n(y')+2}(y') = 0.$$

Case 1. $\gamma(n, y) = \gamma(n, y') = \gamma$. In this case, since y and y' are in $Y(n, \gamma)$, the claim follows from (b), (e) and (a).

Case 2. $\gamma(n, y) < \gamma(n, y')$ (the remaining case is similar).

Subcase 1. $\gamma(n, y) < \gamma(n+1, y') = \gamma$. In this case, since $y \in \bigcup_{\beta<\alpha(\gamma)} Y_{n\beta}$ and $y' \in Y(n+1, \gamma)$, the claim follows from (d), (e) and (a).

Subcase 2. $\gamma(n+1, y') \leqslant \gamma(n, y)$. In this case, since $\gamma(n+1, y') < \gamma(n, y')$, we have $n = n(y)$.

First assume $\gamma(n+1, y') = \gamma(n, y) = \gamma$. Then by $\gamma(n, y) = \gamma(n+1, y) = \gamma$, y and y' are in $Y(n+1, \gamma)$. Thus the claim follows from (b), (e) and (a).

Next assume $\gamma(n+1, y') < \gamma(n, y) = \gamma$. Then y' is in $\bigcup_{\beta<\alpha(\gamma)} Y_{n+1,\beta}$ and y is in $Y(n+2, \gamma)$ by $\gamma(n+2, y) = \gamma$. Thus the claim follows from (d), (e) and (a). This completes the proof.

DEFINITION 2.3. Let $\varkappa$ be a limit cardinal. A sequence $\{\varkappa_\alpha: \alpha < \mathrm{cf}\varkappa\}$ of cardinals in $\varkappa$ is said to be *normal* if

(1) $\varkappa_\alpha < \varkappa_{\alpha+1}$ for every $\alpha < \mathrm{cf}\varkappa$,

(2) $\varkappa_\alpha = \sup_{\beta<\alpha} \varkappa_\beta$ for every limit $\alpha < \mathrm{cf}\varkappa$,

(3) $\varkappa = \sup_{\alpha<\mathrm{cf}\varkappa} \varkappa_\alpha$.

Remark. Note that there always exists a normal sequence in $\varkappa$ if $\varkappa$ is a limit cardinal, and also that there exists a normal sequence $\{\varkappa_\alpha: \alpha < \mathrm{cf}\varkappa\}$ with $2^{\varkappa_\alpha} \leqslant \varkappa_{\alpha+1}$ for every $\alpha < \mathrm{cf}\varkappa$ whenever $\varkappa$ is a strong limit cardinal.

The proof of the following lemma is routine.

LEMMA 2.4. *Let $\varkappa$ be a limit cardinal with $\omega_1 \leqslant \mathrm{cf}\varkappa$ and let $\{\varkappa_\alpha: \alpha < \mathrm{cf}\varkappa\}$ and $\{\varkappa'_\alpha: \alpha < \mathrm{cf}\varkappa\}$ be normal sequences in $\varkappa$. Then $\{\alpha < \mathrm{cf}\varkappa: \varkappa_\alpha = \varkappa'_\alpha\}$ is cub in $\mathrm{cf}\varkappa$.*

LEMMA 2.5. *Let Y be a closed discrete subspace of size $\varkappa$ with $\omega_1 \leqslant \mathrm{cf}\varkappa < \varkappa$. Moreover, let $\{Y_\alpha: \alpha < \mathrm{cf}\varkappa\}$ be a partition of Y, $\{\varkappa_\alpha: \alpha < \mathrm{cf}\varkappa\}$ a normal sequence in $\varkappa$ and $b \in \prod_{y\in Y} \mathscr{B}_y$ such that $\{\alpha < \mathrm{cf}\varkappa: |\mathrm{cl}(\bigcup\{b(y): y \in \bigcup_{\beta<\alpha} Y_\beta\}) \cap Y| \leqslant \varkappa_\alpha\}$ contains a cub set in $\mathrm{cf}\varkappa$. Then there is a partition $\{Y'_\alpha: \alpha < \mathrm{cf}\varkappa\}$ of Y such that*

(1) *$|Y'_\alpha| < \varkappa$ for every $\alpha < \mathrm{cf}\varkappa$,*

(2) *$\{\alpha < \mathrm{cf}\varkappa: \mathrm{cl}(\bigcup\{b(y): y \in \bigcup_{\beta<\alpha} Y_\beta\}) \cap Y \subset \bigcup_{\beta<\alpha} Y'_\beta\}$ contains a cub set in $\mathrm{cf}\varkappa$.*

Proof. For each $\alpha < \mathrm{cf}\varkappa$, put $S_\alpha = \mathrm{cl}(\bigcup\{b(y): y \in \bigcup_{\beta<\alpha} Y_\beta\}) \cap Y$. Take a cub set C in $\mathrm{cf}\varkappa$ such that $|S_\alpha| \leqslant \varkappa_\alpha$ for each $\alpha \in C$. Enumerate C in increasing order, say $C = \{\alpha(\gamma): \gamma < \mathrm{cf}\varkappa\}$. For each limit ordinal $\gamma < \mathrm{cf}\varkappa$, put $T^\gamma = S_{\alpha(\gamma)} - \bigcup_{\gamma'<\gamma} S_{\alpha(\gamma')}$. Since $|T^\gamma| \leqslant \varkappa_{\alpha(\gamma)}$ and $\varkappa_{\alpha(\gamma)} = \sup_{\gamma'<\gamma} \varkappa_{\alpha(\gamma')}$ by the cub-ness of C, T^γ can be partitioned into $\{T^\gamma_{\gamma'}: \gamma' < \gamma\}$ with $|T^\gamma_{\gamma'}| \leqslant \varkappa_{\alpha(\gamma')}$ for each $\gamma' < \gamma$, where γ is a limit ordinal $< \mathrm{cf}\varkappa$. Since $\{S_{\alpha(\gamma)}: \gamma < \mathrm{cf}\varkappa\}$ is increasing with respect to $\subset$ and its union is Y, it is easy to show:

(*) $\{S_{\alpha(\gamma+1)} - S_{\alpha(\gamma)}: \gamma < \mathrm{cf}\varkappa\} \cup \{T^\gamma: \gamma < \mathrm{cf}\varkappa$ and γ is limit$\}$ is a partition of Y.

For each $\gamma < \mathrm{cf}\varkappa$, put

$$Y'_{\alpha(\gamma)} = (S_{\alpha(\gamma+1)} - S_{\alpha(\gamma)}) \cup \bigcup\{T^{\gamma'}_\gamma: \gamma < \gamma' < \mathrm{cf}\varkappa \text{ and } \gamma' \text{ is limit}\}.$$

For each α in $\mathrm{cf}\varkappa - C$, put $Y'_\alpha = 0$. Then it is easy to show that $\{Y'_\alpha: \alpha < \mathrm{cf}\varkappa\}$ is a partition of Y by (*) and that $|Y'_\alpha| < \varkappa$ for each $\alpha < \mathrm{cf}\varkappa$ by the singularity of $\varkappa$. To show (2), it suffices to show that $S_{\alpha(\gamma)} \subset \bigcup_{\alpha<\alpha(\gamma)} Y'_\alpha$ for each limit $\gamma < \mathrm{cf}\varkappa$, since $\{\alpha(\gamma): \gamma < \mathrm{cf}\varkappa$ and γ is limit$\}$ is a cub set contained in C. To show this, let y be an element of $S_{\alpha(\gamma)}$ for a limit ordinal $\gamma < \mathrm{cf}\varkappa$. Then there are two cases.

Case 1. $y \in S_{\alpha(\gamma'+1)} - S_{\alpha(\gamma')}$ *for some* $\gamma' < \gamma$. In this case, y is an element of $Y_{\alpha(\gamma')}$ and $\alpha(\gamma') < \alpha(\gamma)$.

Case 2. *Otherwise.* In this case, there is a limit ordinal $\gamma' \leqslant \gamma$ such that $y \in T^{\gamma'}$. Thus there is a $\gamma'' < \gamma'$ such that $y \in T^{\gamma'}_{\gamma''}$. Then $y \in Y_{\alpha(\gamma'')}$ and $\alpha(\gamma'') < \alpha(\gamma') \leqslant \alpha(\gamma)$. This completes the proof.

DEFINITION 2.6. Let μ be an infinite cardinal and Y a subspace of a space X. Y is said to have *property* $P(\mu)$ if for every $m: Y \to \mu$, there is a $b \in \prod_{y \in Y} \mathscr{B}_y$ such that $\{m(y'): b(y) \cap b(y') \neq 0,\ y' \in Y\}$ is bounded in μ for each $y \in Y$. The whole space X is also said to have property $P(\mu)$ if every closed discrete subspace Y has property $P(\mu)$ in the above sense. Thus we shall use property $P(\mu)$ in two different ways, but these differences will be clarified by the context.

Remark. Note that normal or countably paracompact spaces have property $P(\omega)$ and $\aleph_1$-paraLindelöf spaces (in the sense of [F4]) have property $P(\omega_1)$. The notion of the $\mathscr{B}(\mu)$-property is known as a generalization of countable paracompactness to higher cardinals, see [Ru]. A space has the *$\mathscr{B}(\mu)$-property* if for every increasing open cover $\{U_\alpha: \alpha < \mu\}$ (i.e., $U_\beta \subset U_\alpha$ if $\beta < \alpha$, each U_α is open and $\bigcup_{\alpha<\mu} U_\alpha = X$), there is an increasing open cover $\{V_\alpha: \alpha < \mu\}$ such that $\operatorname{cl} V_\alpha \subset U_\alpha$ for each $\alpha < \mu$. Note that "countable paracompactness $\leftrightarrow$ $\mathscr{B}(\omega)$-property" holds, see [En]. And note that the argument of the proof of this equivalence shows "$\aleph_1$-paraLindelöfness $\leftrightarrow$ $\mathscr{B}(\omega_1)$-property". Here we remark the relation between $\mathscr{B}(\mu)$-property and property $P(\mu)$.

Lemma 2.7. *Every space X having the $\mathscr{B}(\mu)$-property has property $P(\mu)$, where μ is an infinite cardinal.*

Proof. Let Y be a closed discrete subspace of a space X having the $\mathscr{B}(\mu)$-property. Fix an arbitrary $m: Y \to \mu$. For each $\alpha < \mu$, put $U_\alpha = X - \bigcup\{m^{-1}(\beta): \alpha \leqslant \beta\}$. Then $\{U_\alpha: \alpha < \mu\}$ is an increasing open cover of X. Take an increasing open cover $\{V_\alpha: \alpha < \mu\}$ such that $\operatorname{cl} V_\alpha \subset U_\alpha$ for each $\alpha < \mu$. For each y in Y, let $\beta(y)$ be the least $\beta < \mu$ such that $y \in V_\beta$. Note that $\alpha < \beta(y)$ if $y \in m^{-1}(\alpha)$. For each $y \in m^{-1}(\alpha)$, fix $b(y) \in \mathscr{B}_y$ such that $b(y) \subset V_{\beta(y)} - \operatorname{cl} V_\alpha$. Then it is easy to show that $\{m(y'): b(y) \cap b(y') \neq 0,\ y' \in Y\} \subset \beta(y)$ for each $y \in Y$.

Lemma 2.8. *Let μ be an infinite cardinal, Y a closed discrete subspace of a space X, and m_0 an arbitrary map $Y \to \mu$. Assume Y has property $P(\mu)$. Then for each $n < \omega$, there are a $b_n \in \prod_{y \in Y} \mathscr{B}_y$ and a $m_n: Y \to \mu$ such that*

$$\{m_n(y'): b_n(y) \cap b_n(y') \neq 0,\ y' \in Y\} \subset m_{n+1}(y) \quad \textit{for each } y \in Y.$$

Proof. Assume that m_n and b_{n-1} have been defined. By property $P(\mu)$, there is a $b_n \in \prod_{y \in Y} \mathscr{B}_y$ such that $A_y = \{m_n(y'): b_n(y) \cap b_n(y') \neq 0,\ y' \in Y\}$ is bounded in μ for each $y \in Y$. Fixing $m_{n+1}(y)$ in μ which contains A_y for each $y \in Y$, we are done.

Lemma 2.9. *Let $\varkappa$ be a strong limit cardinal with $\omega_1 \leqslant \operatorname{cf}\varkappa$, let μ, χ be infinite cardinals less than $\varkappa$, and let $\{\varkappa_\alpha: \alpha < \operatorname{cf}\varkappa\}$ be a normal sequence of cardinals in $\varkappa$ such that $2^{\varkappa_\alpha} \leqslant \varkappa_{\alpha+1}$ for each $\alpha < \operatorname{cf}\varkappa$. Assume Y is a closed discrete subspace of size $\varkappa$ such that Y has property $P(\mu)$ and has a partition $\{Y_\alpha: \alpha < \operatorname{cf}\varkappa\}$ with $|Y_\alpha| < \varkappa$ for each $\alpha < \operatorname{cf}\varkappa$ and each $y \in Y$ has a neighborhood base $\mathscr{B}_y$ with $|\mathscr{B}_y| \leqslant \chi$. Then there is a $b \in \prod_{y \in Y} \mathscr{B}_y$ such that $\{\alpha < \operatorname{cf}\varkappa: |\operatorname{cl}(\bigcup\{b(y): y \in \bigcup_{\beta<\alpha} Y_\beta\}) \cap Y| < 2^{\varkappa_\alpha}\}$ contains a cub set in $\operatorname{cf}\varkappa$.*

Proof. For each $\alpha < \operatorname{cf}\varkappa$, put $Z_\alpha = \bigcup_{\beta<\alpha} Y_\beta$.

Claim 1. $\{\alpha < \operatorname{cf}\varkappa: |\prod_{y \in Z_\alpha}(\mathscr{B}_y \times \mu)| \leqslant 2^{\varkappa_\alpha}\}$ *contains a cub set.*

Proof. Since $|Y_\alpha| < \varkappa$ for each $\alpha < \operatorname{cf}\varkappa$, fix $f(\alpha) < \operatorname{cf}\varkappa$ such that $|Y_\alpha| \leqslant \varkappa_{f(\alpha)}$. Then it is easy to show that $C_0 = \{\alpha < \operatorname{cf}\varkappa: \forall \beta < \alpha\ (f(\beta) < \alpha)\}$ is cub. If α is an element of C_0, then $|Z_\alpha| = |\bigcup_{\beta<\alpha} Y_\beta| \leqslant |\alpha| \times \varkappa_\alpha = \varkappa_\alpha$ by $|Y_\beta| \leqslant \varkappa_{f(\beta)} < \varkappa_\alpha$ for each $\beta < \alpha$. Put $C = C_0 \cap \{\alpha < \operatorname{cf}\varkappa: |\chi \times \mu| \leqslant \varkappa_\alpha\}$. Note that C is cub in $\operatorname{cf}\varkappa$. If α is in C, then $|\prod_{y \in Z_\alpha}(\mathscr{B}_y \times \mu)| \leqslant 2^{\varkappa_\alpha}$ holds. Thus the proof of this claim is complete.

Next let C be the cub set in the above claim. For $\alpha \in C$, enumerate $\prod_{y \in Z_\alpha}(\mathscr{B}_y \times \mu)$ by $\{u_\alpha^\delta: 1 \leqslant \delta < 2^{\varkappa_\alpha}\}$. By putting $u_\alpha^\delta(y) = \langle b_\alpha^\delta(y), h_\alpha^\delta(y)\rangle$ for each $y \in Z_\alpha$, we mean b_α^δ is an element of $\prod_{y \in Z_\alpha} \mathscr{B}_y$ and $h_\alpha^\delta: Z_\alpha \to \mu$. Enumerate C in increasing order, say $C = \{\alpha(\gamma): \gamma < \operatorname{cf}\varkappa\}$. By induction on $\gamma < \operatorname{cf}\varkappa$ and $\delta < 2^{\varkappa_{\alpha(\gamma)}}$, we shall construct a map $m_{\gamma\delta}$ with $\operatorname{dom}(m_{\gamma\delta}) \subset Y$ and $\operatorname{ran}(m_{\gamma\delta}) \subset \mu$ such that

(1) $m_{\gamma\delta'} \subset m_{\gamma\delta}$ for each $\gamma < \operatorname{cf}\varkappa$ and $\delta' < \delta < 2^{\varkappa_{\alpha(\gamma)}}$,

(2) $m_{\gamma 0} = \bigcup\{m_{\gamma'\delta'}: \gamma' < \gamma,\ \delta' < 2^{\varkappa_{\alpha(\gamma')}}\}$ for each $\gamma < \operatorname{cf}\varkappa$,

(3) $|m_{\gamma\delta}| \leqslant |\varkappa_{\alpha(\gamma)} + \delta|$ for each $\gamma < \operatorname{cf}\varkappa$ and $\delta < 2^{\varkappa_{\alpha(\gamma)}}$.

Here dom (ran) means domain (range, respectively) of a map. To construct such partial functions, first $m_{00} = 0$. Note that the $m_{\gamma 0}$ defined by (2) also satisfies (3) by easy cardinal arithmetics with the inductive assumption. It remains to define $m_{\gamma\delta}$ assuming that $m_{\gamma\delta'}$ has been defined for all $\delta' < \delta$, where $\gamma < \operatorname{cf}\varkappa$ and $0 < \delta < 2^{\varkappa_{\alpha(\gamma)}}$.

Case 1. $(\operatorname{cl}(\bigcup\{b_{\alpha(\gamma)}^\delta(y): y \in Z_{\alpha(\gamma)}\}) \cap (Y - Z_{\alpha(\gamma)})) - \operatorname{dom}(\bigcup_{\delta'<\delta} m_{\gamma\delta'}) \neq 0$. In this case, pick a point $y(\gamma, \delta)$ in this set, and define

$$J(\gamma, \delta, B) = \{h_{\alpha(\gamma)}^\delta(y): B \cap b_{\alpha(\gamma)}^\delta(y) \neq 0,\ y \in Z_{\alpha(\gamma)}\}$$

for each $B \in \mathscr{B}_{y(\gamma,\delta)}$.

Subcase 1. *There is a $B \in \mathscr{B}_{y(\gamma,\delta)}$ such that $J(\gamma, \delta, B)$ is bounded in μ.* In this case, take a $B(\gamma, \delta) \in \mathscr{B}_{y(\gamma,\delta)}$ such that $J(\gamma, \delta, B(\gamma, \delta))$ is bounded in μ. Furthermore, pick a $\alpha(\gamma, \delta) < \mu$ with $\sup J(\gamma, \delta, B(\gamma, \delta)) < \alpha(\gamma, \delta)$. Put

$$m_{\gamma\delta} = \bigcup_{\delta'<\delta} m_{\gamma\delta'} \cup \{\langle y(\gamma, \delta), \alpha(\gamma, \delta)\rangle\}.$$

Subcase 2. *Otherwise.* Put $m_{\gamma\delta} = \bigcup_{\delta'<\delta} m_{\gamma\delta'}$.

Case 2. *Otherwise.* Put $m_{\gamma\delta} = \bigcup_{\delta'<\delta} m_{\gamma\delta'}$.

Then in all cases, such a $m_{\gamma\delta}$ satisfies (3) by easy cardinal arithmetics, and also (1).

Let $m: Y \to \mu$ be a global function extending all $m_{\gamma\delta}$'s. By putting $m_0 = m$, one can take by Lemma 2.8, b_0, $b_1 \in \prod_{y \in Y} \mathscr{B}_y$, and $m_1, m_2: Y \to \mu$ such that $\{m_n(y'): b_n(y) \cap b_n(y') \neq 0,\ y' \in Y\} \subset m_{n+1}(y)$ for each $y \in Y$ and $n = 0, 1$. Take a $b \in \prod_{y \in Y} \mathscr{B}_y$ with $b(y) \subset b_0(y) \cap b_1(y)$ for each $y \in Y$. Then the following hold.

(a) $\{m_0(y'): b(y) \cap b(y') \neq 0,\ y' \in Y\} \subset m_1(y)$ for each $y \in Y$.

(b) $\{m_1(y'): b(y) \cap b(y') \neq 0,\ y' \in Y\} \subset m_2(y)$ for each $y \in Y$.

We shall show this b is the desired one. It suffices to show the next claim.

Claim 2. $|\operatorname{cl}(\bigcup\{b(y): y \in Z_{\alpha(\gamma)}\}) \cap Y| < 2^{\varkappa_{\alpha(\gamma)}}$ *for each* $\gamma < \operatorname{cf}\varkappa$.

Proof. Assume indirectly that $|\operatorname{cl}(\bigcup\{b(y): y \in Z_{\alpha(\gamma)}\}) \cap Y| \geqslant 2^{\varkappa_{\alpha(\gamma)}}$ for some $\gamma < \operatorname{cf}\varkappa$. Then there is a non-zero $\delta < 2^{\varkappa_{\alpha(\gamma)}}$ with $b|Z_{\alpha(\gamma)} = b_{\alpha(\gamma)}^\delta$ and $m_1|Z_{\alpha(\gamma)} = h_{\alpha(\gamma)}^\delta$. Here $b|Z$ denotes the restriction of b to Z. Since $|\bigcup_{\delta'<\delta} m_{\gamma\delta'}| \leqslant |m_{\gamma\delta}| \leqslant |\varkappa_{\alpha(\gamma)} + \delta| < 2^{\varkappa_{\alpha(\gamma)}}$, we have

(c) $(\operatorname{cl}(\bigcup\{b(y): y \in Z_{\alpha(\gamma)}\}) \cap (Y - Z_{\alpha(\gamma)})) - \operatorname{dom}(\bigcup_{\delta'<\delta} m_{\gamma\delta'}) \neq 0$.

Thus case 1 happens and using (b), we obtain

$$\begin{aligned} J(\gamma, \delta, b(y(\gamma, \delta)) &= \{h^{\delta}_{\alpha(\gamma)}(y)\colon\ b(y(\gamma, \delta)) \cap b^{\delta}_{\alpha(\gamma)}(y) \neq 0,\ y \in Z_{\alpha(\gamma)}\} \\ &= \{m_1(y)\colon\ b(y(\gamma, \delta)) \cap b(y) \neq 0,\ y \in Z_{\alpha(\gamma)}\} \\ &\subset \{m_1(y)\colon\ b(y(\gamma, \delta)) \cap b(y) \neq 0,\ y \in Y\} \\ &\subset m_2(y(\gamma, \delta)) < \mu. \end{aligned}$$

Therefore subcase 1 of case 1 happens. Then by the definition of $m_{\gamma\delta}$

(d) $$m_0(y(\gamma, \delta)) = m_{\gamma\delta}(y(\gamma, \delta)) > \sup J(\gamma, \delta, B(\gamma, \delta)).$$

By $b(y(\gamma, \delta))$, $B(\gamma, \delta) \in \mathscr{B}_{y(\gamma,\delta)}$ and by (c), there is a $y \in Z_{\alpha(\gamma)}$ such that $b(y(\gamma, \delta)) \cap B(\gamma, \delta) \cap b(y) \neq 0$. By $b(y(\gamma, \delta)) \cap b(y) \neq 0$ and by (a),

(e) $$m_0(y(\gamma, \delta)) \in m_1(y).$$

Also by $B(\gamma, \delta) \cap b^{\delta}_{\alpha(\gamma)}(y) = B(\gamma, \delta) \cap b(y) \neq 0$ and $y \in Z_{\alpha(\gamma)}$,

(f) $$m_1(y) = h^{\delta}_{\alpha(\gamma)}(y) \in J(\gamma, \delta, B(\gamma, \delta)).$$

Then by (d), (e) and (f), $m_0(y(\gamma, \delta)) \in m_1(y) \in J(\gamma, \delta, B(\gamma, \delta))$ and $\sup J(\gamma, \delta, B(\gamma, \delta)) < m_0(y(\gamma, \delta))$. But this is a contradiction. This completes the proof.

THEOREM 2.10. *Let $\varkappa$ be a singular strong limit cardinal with $\omega_1 \leqslant \mathrm{cf}\varkappa$, let μ, χ be infinite cardinals less than $\varkappa$, and let Y be a closed discrete subspace of size $\varkappa$ such that Y has property $\mathrm{P}(\mu)$ and each $y \in Y$ has a neighborhood base $\mathscr{B}_y$ with $|\mathscr{B}_y| \leqslant \chi$. Assume that there is a normal sequence $\{\varkappa_\alpha\colon \alpha < \mathrm{cf}\varkappa\}$ of cardinals in $\varkappa$ such that $\{\alpha < \mathrm{cf}\varkappa\colon 2^{\varkappa_\alpha} = \varkappa_\alpha^+\}$ contains a cub set in $\mathrm{cf}\varkappa$. Then Y has nice partitions (thus Y is separated if Y is $< \varkappa$-separated by Lemma 2.2).*

Proof. Fix a 1-1 onto map $f\colon Y \to \varkappa$. For each $\alpha < \mathrm{cf}\varkappa$, put $Y_{0\alpha} = f^{-1}(\varkappa_\alpha - \sup_{\beta<\alpha}\varkappa_\beta)$. Then $\{Y_{0\alpha}\colon \alpha < \mathrm{cf}\varkappa\}$ is a partition of Y with $|Y_{0\alpha}| < \varkappa$ for each $\alpha < \mathrm{cf}\varkappa$. Assume a partition $\{Y_{n\alpha}\colon \alpha < \mathrm{cf}\varkappa\}$ of Y with $|Y_{n\alpha}| < \varkappa$ for each $\alpha < \mathrm{cf}\varkappa$ is defined. By Lemma 2.4, we may assume $2^{\varkappa_\alpha} \leqslant \varkappa_{\alpha+1}$ for each $\alpha < \mathrm{cf}\varkappa$. Applying Lemma 2.9 to $\{Y_{n\alpha}\colon \alpha < \mathrm{cf}\varkappa\}$, take a $b_n \in \prod_{y\in Y}\mathscr{B}_y$ such that $\{\alpha < \mathrm{cf}\varkappa\colon |\mathrm{cl}(\bigcup\{b_n(y)\colon y \in \bigcup_{\beta<\alpha} Y_{n\beta}\}) \cap Y| < 2^{\varkappa_\alpha}\}$ contains a cub set. Since $\{\alpha < \mathrm{cf}\varkappa\colon 2^{\varkappa_\alpha} = \varkappa_\alpha^+\}$ contains a cub set, $\{\alpha < \mathrm{cf}\varkappa\colon |\mathrm{cl}(\bigcup\{b_n(y)\colon y \in \bigcup_{\beta<\alpha} Y_{n\alpha}\}) \cap Y| \leqslant \varkappa_\alpha\}$ also contains a cub set. Then by Lemma 2.5, there is a partition $\{Y_{n+1,\alpha}\colon \alpha < \mathrm{cf}\varkappa\}$ of Y such that $|Y_{n+1,\alpha}| < \varkappa$ for each $\alpha < \mathrm{cf}\varkappa$ and $\{\alpha < \mathrm{cf}\varkappa\colon \mathrm{cl}(\bigcup\{b_n(y)\colon y \in \bigcup_{\beta<\alpha} Y_{n\beta}\}) \cap Y \subset \bigcup_{\beta<\alpha} Y_{n+1,\beta}\}$ contains a cub set. Then by repeated applications of this process, one can get nice partitions.

Remark. If there is a normal sequence $\{\varkappa_\alpha\colon \alpha < \mathrm{cf}\varkappa\}$ as in Lemma 2.10, then $2^\varkappa = \varkappa^+$ by [Je, Lemma 8.2]. Next we shall show such a normal sequence exists assuming SCH (Singular Cardinals Hypothesis).

LEMMA 2.11 [SCH]. *Let $\varkappa$ be a singular strong limit cardinal with $\omega_1 \leqslant \mathrm{cf}\varkappa$, and let $\{\varkappa_\alpha\colon \alpha < \mathrm{cf}\varkappa\}$ be a normal sequence of cardinals in $\varkappa$ such that $2^{\varkappa_\alpha} \leqslant \varkappa_{\alpha+1}$ for each $\alpha < \mathrm{cf}\varkappa$. Then $\{\alpha < \mathrm{cf}\varkappa\colon 2^{\varkappa_\alpha} = \varkappa_\alpha^+\}$ contains a cub set.*

Proof. Put $C = \{\alpha < \mathrm{cf}\varkappa\colon \mathrm{cf}\varkappa < \varkappa_\alpha,\ \alpha \text{ is limit}\}$. Note that C is a cub set in $\mathrm{cf}\varkappa$. If $\alpha \in C$, then $\varkappa_\alpha$ is a singular cardinal because of $\mathrm{cf}\varkappa_\alpha \leqslant \mathrm{cf}\alpha < \mathrm{cf}\varkappa < \varkappa_\alpha$. Since $2^{<\varkappa_\alpha} = \varkappa_\alpha$ and $\{2^{\varkappa_\beta}\colon \beta < \alpha\}$ is a strictly increasing sequence of cardinals in $\varkappa_\alpha$, $2^{\varkappa_\alpha} = (2^{<\varkappa_\alpha})^+ = \varkappa_\alpha^+$ by [Je, Lemma 8.1]. The proof is complete.

Remark. Lemma 2.11 also holds for an arbitrary normal sequence in $\varkappa$ by Lemma 2.4.

By Theorem 2.10 and Lemma 2.11, we can conclude;

COROLLARY 2.12 [SCH]. *Let $\varkappa$ be a singular strong limit cardinal with $\omega_1 \leqslant \mathrm{cf}\varkappa$, and let X be a $< \varkappa$-CWH space of character $< \varkappa$. If X is normal or has $\mathscr{B}(\mu)$-property for some $\mu < \varkappa$, then X is $\varkappa$-CWH.*

3. The weak CWH case. A closed discrete subspace of a space is said to be *weakly $< \varkappa$-separated* if for every $A \subset Y$ of size $< \varkappa$, there is a separated $A' \subset A$ with $|A'| = |A|$. A space X is *weakly $\varkappa$-CWH* (*weakly $< \varkappa$-CWH*) if for every closed discrete subspace Y of size $\varkappa$ ($< \varkappa$, respectively), there is a separated $Y' \subset Y$ of size $|Y|$. It is known that if $\varkappa$ is a strong limit cardinal with $\omega_1 \leqslant \mathrm{cf}\varkappa$ and X is a weakly $< \varkappa$-CWH normal or countably paracompact space of character $< \varkappa$, then X is weakly $\varkappa$-CWH, see [T1, Theorems 11 and 13]. First we shall generalize this result to spaces having property $\mathrm{P}(\mu)$ for some $\mu < \varkappa$ using the results in Section 2.

THEOREM 3.1. *Let $\varkappa$ be a strong limit cardinal with $\omega_1 \leqslant \mathrm{cf}\varkappa$, let μ, χ be infinite cardinals less than $\varkappa$, and let Y be a closed discrete weakly $< \varkappa$-separated subspace of size $\varkappa$ having property $\mathrm{P}(\mu)$ such that each $y \in Y$ has a neighborhood base $\mathscr{B}_y$ with $|\mathscr{B}_y| \leqslant \chi$. Then there is a separated $Y' \subset Y$ of size $\varkappa$.*

Proof. Let $\{\varkappa_\alpha\colon \alpha < \mathrm{cf}\varkappa\}$ be a normal sequence of cardinals in $\varkappa$ with $\varkappa_0 = 0$. Fix a 1-1 onto map $f\colon Y \to \varkappa$. Put $Y_\alpha = f^{-1}(\varkappa_{\alpha+1} - \varkappa_\alpha)$ for each $\alpha < \mathrm{cf}\varkappa$. Then $\{Y_\alpha\colon \alpha < \mathrm{cf}\varkappa\}$ is a partition of Y. Then by Lemma 2.9, there are a $b \in \prod_{y\in Y}\mathscr{B}_y$ and a cub set C in $\mathrm{cf}\varkappa$ such that $C \subset \{\alpha < \mathrm{cf}\varkappa\colon |\mathrm{cl}(\bigcup\{b(y)\colon y \in \bigcup_{\beta<\alpha} Y_\beta\}) \cap Y| < 2^{\varkappa_\alpha}\}$. Since $|Y_\alpha| = \varkappa_{\alpha+1} \geqslant 2^{\varkappa_\alpha}$, by weak $< \varkappa$-separatedness, take a separated set $Y'_\alpha \subset Y_\alpha - \mathrm{cl}(\bigcup\{b(y)\colon y \in \bigcup_{\beta<\alpha} Y_\beta\})$ of size $\varkappa_{\alpha+1}$ for each $\alpha \in C$. Then it is easy to show that $Y' = \bigcup_{\alpha\in C} Y'_\alpha$ is the desired one.

Remark. The author showed that Theorem 3.1 also holds for every strong limit cardinal with $\mathrm{cf}\varkappa = \omega$ using a similar argument. But the proof is much simpler, so we omit the proof.

Next we shall study the weak CWH-ness for locally μ-cc spaces. A space is μ-cc if there are at most μ disjoint non-empty open sets. A space is *locally* μ-cc if every point has a μ-cc neighborhood. It is known from [F3, Theorem 3] that if X is a locally μ-cc, $< \varkappa$-CWH space, then X is $\varkappa$-CWH whenever $\varkappa$ is a singular cardinal and $\mu < \varkappa$ (note that normality or countable paracompactness or strong limitness or the character restriction are not needed, cf. 2.12).

THEOREM 3.2. *Let $\varkappa$ be a limit cardinal and μ an infinite cardinal with $\mu < \varkappa$. If Y is a weakly $< \varkappa$-separated closed discrete subspace of size $\varkappa$ such that each $y \in Y$ has a μ-cc neighborhood, then there is a separated $Y' \subset Y$ of size $\varkappa$.*

Proof. Take $\{\varkappa_\alpha\colon \alpha < \mathrm{cf}\varkappa\}$ and $\{Y_\alpha\colon \alpha < \mathrm{cf}\varkappa\}$ as in the proof of Theorem 3.1. For each $\alpha < \mathrm{cf}\varkappa$, take separated $Y''_\alpha \subset Y_\alpha$ of size $|Y_\alpha|$. Fix $b \in \prod_{y \in Y} \mathscr{B}_y$ such that $\{b(y)\colon y \in Y''_\alpha\}$ is disjoint and each $b(y)$ is μ-cc.

CLAIM. $|\mathrm{cl}(\bigcup\{b(y)\colon y \in \bigcup_{\beta<\alpha} Y''_\beta\}) \cap Y| \leqslant |\bigcup_{\beta<\alpha} Y''_\beta| \times \mu$ *for each* $\alpha < \mathrm{cf}\varkappa$.

Proof. Assume the claim fails. Then one can take a separated $Z \subset \mathrm{cl}(\bigcup\{b(y)\colon y \in \bigcup_{\beta<\alpha} Y''_\beta\}) \cap Y$ of size $(|\bigcup_{\beta<\alpha} Y''_\beta| \times \mu)^+$ by weak $<\varkappa$-separatedness. Take a $b' \in \prod_{y \in Y} \mathscr{B}_y$ such that $\{b'(z)\colon z \in Z\}$ is disjoint. Then for every $z \in Z$, there is a $y(z) \in \bigcup_{\beta<\alpha} Y''_\beta$ such that $b'(z) \cap b(y(z)) \neq 0$. Then there are a $Z' \subset Z$ of size $|Z|$ and a $y \in \bigcup_{\beta<\alpha} Y''_\beta$ such that $y(z) = y$ for every $z \in Z'$. This contradicts the μ-cc-ness of $b(y)$. Thus the proof of the claim is complete.

Since for each $\alpha < \mathrm{cf}\varkappa$ with $\mu < \varkappa_\alpha$, $|\bigcup_{\beta<\alpha} Y''_\beta| = |\bigcup_{\beta<\alpha} Y_\beta| = \varkappa_\alpha$ and $|Y''_\alpha| = |Y_\alpha| = \varkappa_{\alpha+1}$, $|Y''_\alpha - \mathrm{cl}(\bigcup\{b(y)\colon y \in \bigcup_{\beta<\alpha} Y''_\beta\})| = \varkappa_{\alpha+1}$ by the claim. Thus we can take a separated $Y'_\alpha \subset Y''_\alpha - \mathrm{cl}(\bigcup\{b(y)\colon y \in \bigcup_{\beta<\alpha} Y''_\beta\})$ of size $\varkappa_{\alpha+1}$ for such α's. Then it is straightforward to show that $Y' = \bigcup\{Y'_\alpha\colon \alpha < \mathrm{cf}\varkappa, \mu < \varkappa_\alpha\}$ is the desired one. This completes the proof.

To end this paper, we shall study the relation between property $\mathrm{P}(\varkappa)$ and weak $\varkappa$-CWH-ness.

LEMMA 3.3. *Let Y be a subspace of a space and $\varkappa$ an infinite cardinal. Then Y has property* $\mathrm{P}(\mathrm{cf}\varkappa)$ *if and only if Y has property* $\mathrm{P}(\varkappa)$.

Proof. Assume $\varkappa$ is a singular cardinal (otherwise, this is clear). Fix a normal sequence $\{\varkappa_\alpha\colon \alpha < \mathrm{cf}\varkappa\}$ of cardinals in $\varkappa$. First assume Y has property $\mathrm{P}(\mathrm{cf}\varkappa)$. We shall show Y has property $\mathrm{P}(\varkappa)$. To show this, fix an arbitrary $m\colon Y \to \varkappa$. Define $m'\colon Y \to \mathrm{cf}\varkappa$ by $m'(y) = \alpha$ if $m(y) \in \varkappa_\alpha\text{-}\sup_{\beta<\alpha}\varkappa_\beta$ for each $y \in Y$. Then by property $\mathrm{P}(\mathrm{cf}\varkappa)$ there is a $b \in \prod_{y \in Y} \mathscr{B}_y$ such that $A_y = \{m'(y')\colon b(y) \cap b(y') \neq 0, y' \in Y\}$ is bounded in $\mathrm{cf}\varkappa$. Thus we can pick $\alpha(y) < \mathrm{cf}\varkappa$ such that $A_y \subset \alpha(y)$ for each $y \in Y$. Then it is straightforward to show that $\{m(y')\colon b(y) \cap b(y') \neq 0, y' \in Y\} \subset \varkappa_{\alpha(y)} < \varkappa$ for each $y \in Y$.

Next assume Y has property $\mathrm{P}(\varkappa)$. Fix an arbitrary $m\colon Y \to \mathrm{cf}\varkappa$. Define $m'\colon Y \to \varkappa$ by $m'(y) = \varkappa_{m(y)}$ for each $y \in Y$. Then by property $\mathrm{P}(\varkappa)$, there is a $b \in \prod_{y \in Y} \mathscr{B}_y$ such that $A_y = \{m'(y')\colon b(y) \cap b(y') \neq 0, y' \in Y\}$ is bounded in $\varkappa$ for each $y \in Y$, thus we can pick $\alpha(y) < \mathrm{cf}\varkappa$ such that $A_y \subset \varkappa_{\alpha(y)}$. Then it is straightforward to show that $\{m'(y)\colon b(y) \cap b(y') \neq 0, y' \in Y\} \subset \alpha(y)$. The proof is complete.

THEOREM 3.4. *Let $\varkappa$ be a singular cardinal, and let Y be a weakly $<\varkappa$-separated closed discrete subspace of size $\varkappa$ having property* $\mathrm{P}(\varkappa)$. *Then there is a separated $Y' \subset Y$ of size $\varkappa$.*

Proof. Fix a strictly increasing cofinal sequence $\{\varkappa_\alpha\colon \alpha < \mathrm{cf}\varkappa\}$ of successor cardinals in $\varkappa$ with $\mathrm{cf}\varkappa < \varkappa_0$ (for example, this can be done by putting $\varkappa_\alpha = \lambda_\alpha^+$ for each $\alpha < \mathrm{cf}\varkappa$, where $\{\lambda_\alpha\colon \alpha < \mathrm{cf}\varkappa\}$ is a normal sequence of cardinals in $\varkappa$ with $\mathrm{cf}\varkappa < \lambda_0$). Fix a 1-1 onto map $f\colon Y \to \varkappa$. By putting $Y_\alpha = f^{-1}(\varkappa_\alpha\text{-}\sup_{\beta<\alpha}\varkappa_\beta)$, $\{Y_\alpha\colon \alpha < \mathrm{cf}\varkappa\}$ is a partition of Y with $|Y_\alpha| = \varkappa_\alpha$ for each $\alpha < \mathrm{cf}\varkappa$. Define $m\colon Y \to \mathrm{cf}\varkappa$ by $m(y) = \alpha$ if $y \in Y_\alpha$ for each $y \in Y$. Then by property $\mathrm{P}(\varkappa)$ (equivalently, property $\mathrm{P}(\mathrm{cf}\varkappa)$), there is a $b \in \prod_{y \in Y} \mathscr{B}_y$ such that for each $y \in Y$, $\{m(y')\colon b(y) \cap b(y') \neq 0, y' \in Y\} \subset \alpha(y)$ for some $\alpha(y) < \mathrm{cf}\varkappa$. For each $\alpha, \beta < \mathrm{cf}\varkappa$, put $Y_\alpha^\beta = \{y \in Y_\alpha\colon \alpha(y) \leqslant \beta\}$. Since $Y_\alpha = \bigcup_{\beta<\mathrm{cf}\varkappa} Y_\alpha^\beta$ and the size of Y_α is a successor cardinal $>\mathrm{cf}\varkappa$, there is a $\beta(\alpha) < \mathrm{cf}\varkappa$ such that $|Y_\alpha^{\beta(\alpha)}| = |Y_\alpha|$ for each $\alpha < \mathrm{cf}\varkappa$. Then $C = \{\alpha < \mathrm{cf}\varkappa\colon \forall \alpha' < \alpha\ (\beta(\alpha') < \alpha)\}$ is unbounded in $\mathrm{cf}\varkappa$ (in fact, C is cub in $\mathrm{cf}\varkappa$ if $\omega_1 \leqslant \mathrm{cf}\varkappa$). By weak $<\varkappa$-separatedness, choose a separated $Y'_\alpha \subset Y_\alpha^{\beta(\alpha)}$ of size $|Y_\alpha^{\beta(\alpha)}|$ $(=|Y_\alpha|)$. Put $Y' = \bigcup_{\alpha \in C} Y'_\alpha$. Take a $b' \in \prod_{y \in Y'} \mathscr{B}_y$ such that $b'(y) \subset b(y)$ for each $y \in Y'$ and $\{b'(y)\colon y \in Y'_\alpha\}$ is disjoint. Since C is unbounded in $\mathrm{cf}\varkappa$, the size of Y' is $\varkappa$. We shall show $\{b'(y)\colon y \in Y'\}$ separates Y'. To show this assume that $y' \in Y'_{\alpha'}$, $y \in Y'_\alpha$ and $\alpha, \alpha' \in C$ with $\alpha' < \alpha$. Since $y' \in Y'_{\alpha'} \subset Y_{\alpha'}^{\beta(\alpha')}$, we have $\alpha(y') \leqslant \beta(\alpha') < \alpha = m(y)$. Thus $b(y') \cap b(y) = 0$ by $m(y) \notin \alpha(y')$. Therefore $b'(y') \cap b'(y) = 0$. This completes the proof.

Finally, we shall show that weak $<\varkappa$-separatedness can be removed from Theorem 3.4 if "singular" is replaced by "regular".

THEOREM 3.5. *Let $\varkappa$ be a regular cardinal, and let Y be a closed discrete subspace of size $\varkappa$ having property* $\mathrm{P}(\varkappa)$. *Then there is a separated $Y' \subset Y$ of size $\varkappa$.*

Proof. Identify Y with $\varkappa$. By property $\mathrm{P}(\varkappa)$, there is a $b \in \prod_{\alpha \in \varkappa} \mathscr{B}_\alpha$ such that for each $\alpha \in \varkappa$, $\{\alpha' \in \varkappa\colon b(\alpha) \cap b(\alpha') \neq 0\} \subset \beta(\alpha)$ for some $\beta(\alpha) < \varkappa$. Then it is easy to show $C = \{\alpha \in \varkappa\colon \forall \alpha' < \alpha\ (\beta(\alpha') < \alpha)\}$ is separated and of size $\varkappa$.

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DEPARTMENT OF MATHEMATICS
FACULTY OF EDUCATION
Oita University
Dannoharu Oita 870-11, Japan

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