QUESTION. Is it true that $\wp\varphi(K)-\{1/0\}=\emptyset$ for any nontorus noncabled knot K?

Note that the positive answer to this question implies that to the cabling conjecture, which states: if $K(r)$ is a reducible manifold then K is a torus knot or a cabled knot.

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Normal k'-spaces are consistently collectionwise normal

by

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Abstract. Z. Balogh completed F. Tall's "Toronto project" by proving consistently that every normal, locally compact space is collectionwise normal. The natural generalization is to replace "locally compact" with the classical "k-space property". We prove that in any model obtained by adding supercompact many Cohen or random reals, discrete collections of closed sets in such spaces have a "first-stage" separation; if the space also satisfies the stronger k'-space property, then we can obtain an open separation, so the space is collectionwise normal.

1. Introduction. F. Tall's "Toronto project", to prove consistently that every normal, locally compact space is collectionwise normal, was completed by Z. Balogh who proved that this is so in any model obtained by adding supercompact many Cohen or random reals [B]. A history of the project is contained in his paper. It would be nice to improve this result by replacing "locally compact" by "k-space", where a k-space is one in which a set is closed if, and only if, its intersection with every compact set is closed, since the k-space property is a classical topological property and k-spaces have some nice properties; they are precisely the quotient images of locally compact spaces, and hence are closed under quotient maps. In this paper we prove that in any model obtained by adding supercompact many Cohen or random reals, discrete collections of closed sets in such spaces have what we call a "first-stage" separation, and show that if a space additionally satisfies a stronger property, the k'-property, then the first-stage separation enables us to get an open separation of the sets, and hence the space is collectionwise normal. We also show that if the first-stage separation could be made to be discrete, then the process could be continued and we could get the collectionwise normality of the k-spaces.

We also treat the "countably paracompact" analogue of these results, and obtain, as one familiar with the history of the Toronto project would expect, the results that in such spaces locally finite collections of closed sets have a "first-stage expansion" and that if a space is additionally k', this enables us to get an expansion by locally finite open sets.

The large cardinal assumption is used to obtain a reflection principle: if there is a counterexample, it is forced to be a small one. Once we had the new ideas necessary to show that there could be no small counterexample, we first modelled our consistency proof after Balogh's and so are indebted to his work. A. Dow, however, showed the author how Balogh's result can be proven more simply by following the approach laid out in [DTW], and so this is the approach we follow in this paper.

2. The general result for normal spaces. A collection of disjoint subsets of a space is said to be *normalized* if for every partition of the collection into two subcollections, A and B, there exist disjoint open sets about $\bigcup A$ and $\bigcup B$.

THEOREM 1. *Suppose κ is a supercompact cardinal, $\mathbf{P}$ is the poset for adding either κ-many Cohen reals or κ-many random reals, and G is $\mathbf{P}$-generic over the set-theoretic universe V. Then*

$V[G] \models$ "Suppose X is a normal space with the property

(*) *if Z is a collection of subsets of X such that every countable subcollection of Z is discrete, then Z is discrete,*

$\mathcal{Y} = \{Y_a\colon a \in X\}$ is a normalized discrete collection of closed sets in X, and for each $y \in Y_a$ and $d \in X$ there is a collection $\mathcal{U}_{y,d} = \{U(y, x, d)\colon x \in X\}$ consisting of sets with compact closure and having the following properties:

(i) $\bigcap_{x\in X} U(y, x, d) = \{y\}$,

(ii) *for each $x, x' \in X$ there is an $x'' \in X$ with $\overline{U(y, x'', d)} \subset U(y, x, d) \cap U(y, x', d)$ and*

(iii) *for each open set V, if $y \in V$ then there is an $x \in X$ with $U(y, x, d) \subset V$.*

Then for each $y \in \bigcup \mathcal{Y}$ and $d \in X$ there is an $x(y, d) \in X$ such that $\{\bigcup_{y\in Y_a, d\in X} U(y, x(y, d), d)\colon a \in X\}$ is a (not necessarily open) separation of $\mathcal{Y}$".

Such a separation of $\mathcal{Y}$ we call a *first-stage separation.*

Proof of Theorem 1. Suppose κ, $\mathbf{P}$, and G are as in the hypothesis. For the sake of elementary submodel arguments, let $\Phi(X, \mathcal{Y}, (\mathcal{U}_{y,d})_{y\in\bigcup\mathcal{Y}, d\in X})$ be the statement that X, $\mathcal{Y}$, and $(\mathcal{U}_{y,d})_{y\in\bigcup\mathcal{Y}, d\in X}$ are as stated in the theorem. Also for the sake of elementary arguments, we will think of U as a relation, $U \subset X\times X\times X\times X$, and $(y, x, d, z) \in U$ if, and only if, $z \in U(y, x, d)$.

Suppose that the conclusion of the theorem does not hold.

Using standard facts about supercompact cardinals, Cohen real and random real forcing (see [DTW, Section II]), we may choose an ordinal β large enough, a set G^*, and an elementary embedding $j\colon V[G] \prec M[G^*]$ such that

(a) M is a transitive class with $[M]^\beta \subset M$,

(b) $j''V \subset M$; $j(\alpha) = \alpha$ for each $\alpha < \kappa$; $j(\kappa) > \beta$,

(c) G^* is $j(\mathbf{P})$-generic over M,

(d) $(M[G])_\beta = (V[G])_\beta$ and

(e) $\langle X, \mathcal{Y}, (\mathcal{U}_{y,d})_{y\in\bigcup\mathcal{Y}, d\in X}\rangle \in (V[G])_\beta$ and $(V[G])_\beta \models$ "$\mathcal{Y}$ is a discrete collection in (X, τ) with no first-stage separation".

By (d) and (e), $(M[G])_\beta$, and hence $M[G]$, satisfies the statement that $\mathcal{Y}$ is a discrete collection in (X, τ) with no first-stage separation. In fact, $M[G^*] \models$ "$\mathcal{Y}$ is a discrete collection in (X, τ) with no first-stage separation". Dow, Tall and Weiss show in [DTW] that if $\mathcal{Y}$ is a collection of subsets of a space X, $\mathbf{P}$ is a Cohen partial order, and G is $\mathbf{P}$-generic over V, then if in $V[G]$ there are disjoint open sets about the elements of $\mathcal{Y}$, then the same is true in V. Balogh in [B] essentially shows the same result if $\mathbf{P}$ is the poset for adding random reals. These arguments use the fact that both types of partial orders have endowment properties (see Lemma 2), and can be generalized to first-stage separations. Thus if $M[G^*] \models$ "$\mathcal{Y}$ has a first-stage separation", then since $M[G^*]$ is obtained from $M[G]$ by adding either Cohen or random reals, we would have $M[G] \models$ "$\mathcal{Y}$ has a first-stage separation", which is a contradiction.

By elementarity, $M[G^*] \models \Phi\big(j(X), j(\mathcal{Y}), j((\mathcal{U}_{y,d})_{y\in\bigcup\mathcal{Y}, d\in X})\big)$.

In $M[G^*]$, let

$$\mathcal{X} = \Big\{\bigcup_{y\in j''Y_{j^{-1}(a)}} \bigcup_{d\in X} \bigcap_{x\in X} j(U)\big(y, j(x), j(d)\big)\colon a \in j''X\Big\}.$$

We claim that $\mathcal{X}$ is a discrete collection in $j(X)$. It suffices to show that every countable subcollection of $\mathcal{X}$ is discrete. So let $\{a_n\colon n\in\omega\} \subset j''X$ and consider

$$\mathcal{X}' = \Big\{\bigcup_{y\in Y_{j^{-1}(a_n)}} \bigcup_{d\in X} \bigcap_{x\in X} j(U)\big(y, j(x), j(d)\big)\colon n \in \omega\Big\}.$$

Let $h\colon \omega \to X$ be the function defined by $h(n) = j^{-1}(a_n)$. Then $h \in M[G^*] = M[G][G^*/G]$, where G^*/G is $j(\mathbf{P})/\mathbf{P}$-generic over $M[G]$. Since $j(\mathbf{P})/\mathbf{P}$ is ccc, we may let $H \in M[G]$, $H\colon \omega \to (X)$ be such that for each $n \in \omega$, $h(n) \in H(n)$ and $|H(n)| \leqslant \omega$. We have $H \in (M[G])_\beta$, so $H \in V[G]$. So in $V[G]$, consider $\{Y_a\colon a \in \bigcup_n H_n(n)\}$, a countable discrete collection of closed sets in a normal space. It has an open discrete separation, say $\{U_a\colon a \in \bigcup_n H(n)\}$. For each $a \in \bigcup_n H(n)$, $y \in Y_a$, and $d \in X$, let $x(y, d) \in X$ be such that $U(y, x(y, d), d) \subset U_a$. Then

$$\Big\{\bigcup_{y\in Y_a} \bigcup_{d\in X} U\big(y, x(y, d), d\big)\colon a \in \bigcup_n H(n)\Big\}$$

is discrete in X. By elementarity, and using the fact that $\bigcup_n H(n)$ is countable and hence $j(\bigcup_n H(n)) = \bigcup_n j''H(n)$, $M[G^*]$ satisfies that the image of this set,

$$\Big\{\bigcup_{y\in j(Y_a)} \bigcup_{d\in j(X)} j(U)\big(y, j(x)(y, d), d\big)\colon a \in \bigcup_n H(n)\Big\},$$

is discrete in $j(X)$. But

$$\bigcup_{y\in j''Y_{j^{-1}(a_n)}} \bigcup_{d\in X} \bigcap_{x\in X} j(U)\big(y, j(x), j(d)\big) \subset \bigcup_{y\in j(Y_{j^{-1}(a_n)})} \bigcup_{d\in j(X)} j(U)\big(y, j(x)(y, d), d\big),$$

for if in $M[G^*]$, $z \in \bigcap_{x\in X} j(U)\big(y, j(x), j(d')\big)$, for $y \in j''_n Y_{j^{-1}(a)}$ and $d' \in X$, then since $x\big(j^{-1}(y), d'\big) \in X$,

$$z \in j(U)\Big(y, j\big(x(j^{-1}(y), d')\big), j(d')\Big) = j(U)\Big(y, j(x)\big(y, j(d')\big), j(d)\Big)$$

which means

$$z\in \bigcup_{y\in j(Y_{j^{-1}(a_n)})}\ \bigcup_{d\in j(X)} j(U)(y, j(x)(y, d), d).$$

So, $\mathscr{Z}'$ is discrete and thus $\mathscr{Z}$ is discrete.

Since $j(X)$ is normal, $\mathscr{Z}$ is normalized.

We now want to show that there are collections satisfying "filter-base properties" (ii) and (iii) for $\mathscr{Z}$. A convenient equivalent description of $\mathscr{Z}$ is

$$\mathscr{Z}=\Big\{\bigcup_{y\in Y_a}\bigcup_{d\in X}\bigcap_{x\in X} j(U)\big(j(y), j(x), j(d)\big):\ a\in X\Big\}.$$

We now show in $M[G^*]$ that if $a\in X$, $y\in Y_a$, $d\in X$, and $\bigcap_{x\in X} j(U)\big(j(y), j(x), j(d)\big)\subset W$ for some $W\in j(\tau)$, then there is an $x\in X$ with $j(U)\big(j(y), j(x), j(d)\big)\subset W$. Suppose not and let $\mathscr{C}=\{\overline{j(U)\big(j(y), j(x), j(d)\big)}\setminus W:\ x\in X\}$. By elementarity, each $\overline{j(U)\big(j(y), j(x), j(d)\big)}$ is compact, so $\mathscr{C}$ consists of compact sets. Also by using elementarity on property (ii) of the collections $\mathscr{U}_{y,d}$, $\mathscr{C}$ has the finite intersection property, and so $\bigcap\mathscr{C}\neq\emptyset$. But again using property (ii) and elementarity, $\bigcap_{x\in X}\overline{j(U)\big(j(y), j(x), j(d)\big)}\subset W$, which by supposition is empty. We have a contradiction, and so there must be an $x\in X$ with $j(U)\big(j(y), j(x), j(d)\big)\subset W$.

We now show that there is no first-stage separation of $\mathscr{Z}$ in $j(X)$. On the contrary, suppose that $M[G^*]$ satisfies that for every $a\in X$, $y\in Y_a$, and $d\in X$, there is an $x(y, d)\in X$ such that

$$\Big\{\bigcup_{y\in Y_a}\bigcup_{d\in X} j(U)\big(j(y), j(x(y, d)), j(d)\big):\ a\in X\Big\}\ \text{is a separation of}\ \mathscr{Z}.$$

Then we claim that in $M[G^*]$, $\{\bigcup_{y\in Y_a}\bigcup_{d\in X} U(y, x(y, d), d):\ a\in X\}$ separates $\mathscr{Y}$, which is a contradiction. Suppose that $a, b\in X$; $y_a\in Y_a$; $y_b\in Y_b$; $d_a, d_b\in X$; and $z\in U(y_a, x(y_a, d_a), d_a)\cap U(y_b, x(y_b, d_b), d_b)$. Then by fact (d) and absoluteness,

$$V[G]\models z\in U(y_a, x(y_a, d_a), d_a)\cap U(y_b, x(y_b, d_b), d_b).$$

Therefore in $M[G^*]$,

$$j(z)\in j(U)\big(j(y_a), j(x(y_a, d_a)), j(d_a)\big)\cap j(U)\big(j(y_b), j(x(y_b, d_b)), j(d_b)\big),$$

a contradiction. Thus x gives a first-stage separation of $\mathscr{Y}$ in X, but there is no such separation. Hence there is no first-stage separation of $\mathscr{Z}$ in $j(X)$.

To summarize the key information, in $M[G^*]$ we have $j''X\subset j(X)$, $|j''X|<j(\kappa)$; for each $a\in j''X$ there is a $Z_a\subset j''X$ (namely $Z_{j(x)}=j''Y_x$) such that

(1) $\{\bigcup_{y\in Z_a}\bigcup_{d\in j''X}\bigcap_{x\in j''X} j(U)(y, x, d):\ a\in j''X\}$ is discrete in $j(X)$,

(2) for each $a, d\in j''X$, for each $y\in Z_a$, for each $x, x'\in j''X$ there is an $x''\in j''X$ such that $j(U)(y, x'', d)\subset j(U)(y, x, d)\cap j(U)(y, x', d)$,

(3) for each $W\in j(\tau)$, if

$$\bigcup_{y\in Z_a}\bigcup_{d\in j''X}\bigcap_{x\in j'X} j(U)(y, x, d)\subset W,$$

then for each $y\in Z_a$, for each $d\in j''X$ there is an $x(y, d)\in j''X$ such that $j(U)(y, x(y, d), d)\subset W$ and

(4) for each $h: j''X\to 2$ there exist disjoint $W_0, W_1\in j(\tau)$ with

$$\bigcup\Big\{\bigcup_{j\in Z_a}\bigcup_{d\in j''X}\bigcap_{x\in j''X} j(U)(y, x, d):\ a\in j''X\ \text{and}\ h(a)=i\Big\}\subset W_i,$$

but there is no $f:\bigcup_{a\in j''X} Z_a\times j''X\to j''X$ such that $\{\bigcup_{y\in Z_a}\bigcup_{d\in j''X} j(U)(y, f(y, d), d):\ a\in j''X\}$ is pairwise disjoint.

Thus we may say $M[G^*]\models\exists X'\subset j(X)$, $|X'|<j(\kappa)$ such that $\forall a\in X'\,\exists Z_a\subset X'\cap\bigcup j(\mathscr{Y})$ with

(1) $\{\bigcup_{y\in Z_a}\bigcup_{d\in X'}\bigcap_{x\in X'} j(U)(y, x, d):\ a\in X'\}$ is discrete in $j(X)$,

(2) $\forall a, d\in X'\ \forall y\in Z_a\ \forall x, x'\in X'\ \exists x''\in X'$ with

$$j(U)(y, x'', d)\subset j(U)(y, x, d)\cap j(U)(y, x', d),$$

(3) $\forall W\in j(\tau)$, if $\bigcup_{y\in Z_a}\bigcup_{d\in X'}\bigcap_{x\in X'} j(U)(y, x, d)\subset W$, then $\forall y\in Z_a\,\forall d\in X'\,\exists x(y, d)\in X'$ such that $j(U)(y, x(y, d), d)\subset W$,

(4) $\forall h: X'\to 2\ \exists$ disjoint $W_0, W_1\in j(\tau)$ with

$$\bigcup\Big\{\bigcup_{y\in Z_a}\bigcup_{d\in X'}\bigcap_{x\in X'} j(U)(y, x, d):\ a\in X'\ \text{and}\ h(a)=i\Big\}\subset W_i,\quad\text{but}$$

(5) $\nexists f:\bigcup_{a\in X'} Z_a\times X'\to X'$ such that $\{\bigcup_{y\in Z_a}\bigcup_{d\in X'} j(U)(y, f(y, d), d):\ a\in X'\}$ is pairwise disjoint.

By elementarity, $V[G]\models$ "$\exists X'\subset X$, $|X'|<\kappa$, such that $\forall a\in X'\,\exists Z_a\subset X'\cap\bigcup\mathscr{Y}$ with

(1) $\mathscr{Y}'=\{\bigcup_{y\in Z_a}\bigcup_{d\in X'}\bigcap_{x\in X'} U(y, x, d):\ a\in X'\}$ discrete in X,

(2) $\forall a, d\in X'\,\forall y\in Z_a\,\forall x, x'\in X'\,\exists x''\in X'$ with $U(y, x'', d)\subset U(y, x, d)\cap U(y, x', d)$,

(3) $\forall W\in\tau$ if $\bigcup_{y\in Z_a}\bigcup_{d\in X'}\bigcap_{x\in X'} U(y, x, d)\subset W$, then $\forall y\in Z_a\,\forall d\in X'\ \exists x(y, d)\in X'$ with $U(y, x(y, d), d)\subset W$,

(4) $\forall h: X'\to 2\ \exists$ disjoint $W_0, W_1\in\tau$ with

$$\Big\{\bigcup_{y\in Z_a}\bigcup_{d\in X'}\bigcap_{x\in X'} U(y, x, d);\ a\in X'\ \text{and}\ h(a)=i\Big\}\subset W_i,\quad\text{but}$$

(5) $\nexists f:\bigcup_{a\in X'} Z_a\times X'\to X'$ such that $\{\bigcup_{y\in Z_a}\bigcup_{d\in X'} U(y, f(y, d), d):\ a\in X'\}$ is pairwise disjoint".

If we can show that in $V[G]$ there can be no such X', then we will have shown our supposition that the conclusion of the theorem does not hold is false, i.e., that the conclusion holds. We show there is no such X' in a separate lemma.

LEMMA 2. *Suppose that, in V,* $\mathscr{Y}=\{\bigcup_{y\in Y_a}\bigcup_{d\in X}\bigcap_{x\in X} U(y, x, d):\ a\in X\}$ *is a discrete collection of sets in a space X, and for each* $y\in Y_a$ *and* $d\in X$, *for each* $x, x'\in X$, *there is an* $x''\in X$ *with* $U(y, x'', d)\subset U(y, x, d)\cap U(y, x', d)$. *Also suppose that* κ *is an infinite cardinal* $\geqslant|X|$, $\mathbf{P}$ *is the poset for adding either* κ*-many Cohen reals or* κ*-many random reals, and G is* $\mathbf{P}$*-generic over V. Then if* $V[G]\models$ *"for each* $f: |X|\to 2$ *there is* *a* $g\colon \bigcup_{a\in X} Y_a\times X\to X$ *such that if*

$$U_i = \bigcup\{\bigcup_{y\in Y_a}\bigcup_{d\in X} U(y, g(y, d), d)\colon a\in X, f(a) = i\}$$

for $i = 0, 1$ *then* $U_0\cap U_1 = \emptyset$", *then in* V *there is a first-stage separation of* $\mathscr{Y}$.

Proof. Assume the hypothesis. $\bigcup G$ is a function from κ into 2, so without loss of generality we consider it as a function from X into 2. Then by hypothesis we may let $g\colon \bigcup_{a\in X} Y_a\times X\to X$ be such that for each $a\in X$, for each $y\in Y_a$ and $d\in X$, $U(y, g(y, d), d)\subset U_{\cup G(a)}$ and $U_0\cap U_1 = \emptyset$. Let $p\in G$ be such that, in V, $p\Vdash$ "$\bigcup G\colon X\to 2$, and $g\colon \bigcup_{a\in X} Y_a\times X\to X$ is such that for each $a\in X$, each $y\in Y_a$, and each $d\in X$, $U(y, g(y, d), d)\subset U_{\cup G(a)}$, and $U_0\cap U_1 = \emptyset$". By [DTW], $\mathbf{P}$ is endowed, i.e., for each $p\in\mathbf{P}$ and $n\geqslant 2$, there is a family $\mathscr{L}_n$ of finite subsets of $\mathbf{P}$ such that (a) for every maximal antichain A below p, there is an $L\in\mathscr{L}_n$ with $L\subset A$, and (b) whenever $L_1, \ldots, L_n$ are members of $\mathscr{L}_n$ and $p'\leqslant p$ is such that $|p'\backslash p|\leqslant n$ if $\mathbf{P}$ is the Cohen real partial order or $\mu(p') < (1/n)\mu(p)$ if $\mathbf{P}$ is the random real partial order, then there exist $q_i\in L_i$ $(i = 1, \ldots, n)$ such that $\{p', q_1, \ldots, q_n\}$ has a common lower bound. So let $\mathscr{L}\subset\mathscr{P}(\mathbf{P})$ be a 5-dowment below p (i.e., consider $\mathscr{L}_5$). For each $a, d\in X$ and for each $y\in Y_a$, $p\Vdash g(y, d)\in X$, so let $A(y, d)$ be a maximal antichain below p deciding $g(y, d)$. Let $L(y, d)\subset A(y, d)$, with $L(y, d)\in\mathscr{L}$. For each $q\in L(y, d)$, let $x(y, d, q)\in X$ be such that $q\Vdash g(y, d) = x(y, d, q)$. Let $\tilde{g}(y, d)\in X$ be such that $U(y, \tilde{g}(y, d), d)\subset\bigcap_{q\in L(y,d)} U(y, x(y, d, q), d)$, by property (i).

We claim that $\tilde{g}$ gives a separation. The proof that this is so follows the standard arguments laid out in [DTW] and [B]. We outline the procedure. Suppose

$$z\in\bigcup_{y\in Y_a}\bigcup_{d\in X} U(y, \tilde{g}(y, d), d)\cap\bigcup_{y\in Y_{a'}}\bigcup_{d\in X} U(y, \tilde{g}(y, d), d),$$

for $a\neq a'$ in X. Let $y\in Y_a$, $y'\in Y_{a'}$, and $d, d'\in X$ be such that $z\in U(y, \tilde{g}(y, d), d)\cap U(y', \tilde{g}(y', d'), d')$. If $\mathbf{P}$ is the Cohen real poset and $a, a'\notin\operatorname{dom} p$, then let $p' = p\cup\{\langle a, 0\rangle, \langle a', 1\rangle\}$. So there exist $q\in L(y, d)$, $q'\in L(y', d')$ such that $\{p', q, q'\}$ has a common lower bound r. Since $q\Vdash U(y, \tilde{g}(y, d), d)\subset U_{\cup G(a)}$ and $q'\Vdash U(y', \tilde{g}(y', d'), d')\subset U_{\cup G(a')}$, $r\Vdash z\in U_{\cup G(a)}\cap U_{\cup G(a')}$, but since $r\leqslant p'$, $r\Vdash U_{\cup G(a)}\cap U_{\cup G(a')} = \emptyset$. Thus we have a contradiction and $\tilde{g}$ separates Y_a and $Y_{a'}$. The domain of p is finite, so using the fact that Y is normalized, we can get a first-stage separation of $\{Y_a\colon a\in\operatorname{dom} p\}$, and of $\{\bigcup_{a\in\operatorname{dom} p} Y_a, \bigcup_{a\notin\operatorname{dom} p} Y_a\}$, and so we have a first-stage separation of Y. If $\mathbf{P}$ is the random real poset, the argument is similar, but more technical, we refer the reader to [B]. ∎

Continuation of the proof of Theorem 1. Since we wish to get a separation of $\mathscr{Y}'$ in X, and not merely in X', we expand X' as follows. For each $y, y', x, x', d, d'\in X'$, let $z(y, x, d, y', x', d')\in U(y, x, d)\cap U(y', x', d')$, if possible. Let X'' contain $X'\cup\{z(y, x, d, y', x', d')\colon y, x, d, y', x', d'\in X'\}$. By standard factorization properties of $\mathbf{P}$, we may assume that the ground model V satisfies

$$\{\bigcup_{y\in Z_a}\bigcup_{d\in X'}\bigcap_{x\in X'} U(y, x, d)\cap X''\colon a\in X'\} \text{ is discrete in } X'',$$

and that for each $y\in Z_a$ and $d\in X'$, x and $x'\in X'$, there is an $x''\in X'$ with

$$U(y, x'', d)\cap X''\subset U(y, x, d)\cap X''\cap U(y, x', d).$$

$V[G]\models$ "for each $f\colon |X''|\to 2$ there is a $g\colon \bigcup_{a\in X'} Z_a\times X'\to X'$ such that if

$$U_i = \bigcup\{\bigcup_{y\in Z_a}\bigcup_{d\in X'} U(y, g(y, d), d)\colon a\in X', f(a) = i\}\quad \text{for } i = 0, 1,$$

then $U_0\cap U_1 = \emptyset$". So by Lemma 2, in V there is a first-stage separation of

$$\{\bigcup_{y\in Z_a}\bigcup_{d\in X}\bigcap_{x\in X'} U(y, x, d)\cap X''\colon a\in X'\}$$

in X''; since each $z(y, x, d, y', x', d')\in X''$, we thus have a first-stage separation in X, which contradicts statement (5) which said there is no such separation. Thus in $V[G]$ there is no such X', and so we have proven the theorem. ∎

3. Applications of the normal case

THEOREM 3. *Suppose κ is a supercompact cardinal, $\mathbf{P}$ is the poset for adding either κ-many Cohen reals or κ-many random reals, and G is $\mathbf{P}$-generic over the set-theoretic universe V. Then $V[G]\models$ "if X is a normal k-space, and $\mathscr{Y} = \{Y_a\colon a\in X\}$ is a discrete collection of closed sets in X, then $\mathscr{Y}$ has a first-stage separation".*

Proof. Suppose κ, $\mathbf{P}$, and G are as in the statement of the theorem. Suppose $V[G]\models$ "X is a normal k-space and $\mathscr{Y} = \{Y_a\colon a\in X\}$ is a discrete collection of closed sets in X". Let $\{C_d\colon d\in X\}$ list all compact subsets of X (expand X by adding a closed discrete set, if necessary). For each $y, d\in X$ such that $y\in C_d$, let $\mathscr{U}_{y,d} = \{U(y, x, d)\colon x\in X\}$ be a basis for y in C_d. Clearly the collections $\mathscr{U}_{y,d}$ satisfy properties (i), (ii), and (iii) listed in the statement of Theorem 1. We show that X also satisfies property (*). Suppose $\mathscr{Z}$ is a collection of subsets of X such that each countable subset is discrete. We first show that for each $\mathscr{Z}'\subset\mathscr{Z}$, $\bigcup\{\bar{Z}\colon Z\in\mathscr{Z}'\}$ is closed in each compact set C. Suppose $\mathscr{Z}'' = \{Z_n\colon n\in\omega\}$ lists distinct elements of $\mathscr{Z}'$ and $C\cap\bar{Z}_n\neq\emptyset$ for each $n\in\omega$. Since $\mathscr{Z}''$ is discrete, for each $p\in C$ there is an open set containing p that meets at most one element of $\mathscr{Z}''$, and so by the compactness of C there is an open set containing C that meets only finitely many elements of $\mathscr{Z}''$, a contradiction. So $|\{Z\in\mathscr{Z}'\colon C\cap\bar{Z}\neq\emptyset\}| < \omega$ and so $\{\bar{Z}\colon Z\in\mathscr{Z}'\}$ is closed in C. Since C is arbitrary and X is a k-space, $\bigcup\{\bar{Z}\colon Z\in\mathscr{Z}'\}$ is closed in X for each $\mathscr{Z}'\subset\mathscr{Z}$. Now suppose p is a limit point of $\bigcup\mathscr{Z}$, and hence of $\bigcup\{\bar{Z}\colon Z\in\mathscr{Z}\}$. By the above argument there is a $Z\in\mathscr{Z}$ such that $p\in\bar{Z}$. p is not a limit point of $\bigcup(\mathscr{Z}\backslash\{Z\})$, since otherwise there is a $Z'\in\mathscr{Z}\backslash\{Z\}$ with $p\in\overline{Z'}$, which contradicts the assumption that $\{Z, Z'\}$ is discrete. It follows that $\mathscr{Z}$ is discrete. Clearly property (i) holds for the collections $\mathscr{U}_{y,d}$, and since each C_d is compact, it is easily checked that properties (ii) and (iii) hold as well. Thus by Theorem 1, $\mathscr{Y}$ has a first-stage separation.

THEOREM 4. *Suppose κ, $\mathbf{P}$, and G are as in the statement of Theorem 3. Then $V[G]\models$ "if X is a normal k'-space, then X is collectionwise normal".*

DEFINITION. A space X is a *k'-space* provided that whenever $x \in \bar{A}$, there is a compact C such that $x \in \overline{A \cap C}$.

Proof. Suppose κ, $\mathbf{P}$, and G are as in the statement of Theorem 3. Suppose $V[G] \models$ "X is a normal k'-space, and $\mathscr{Y} = \{Y_a\colon a \in X\}$ is a discrete collection of closed sets in X". By Theorem 3, there is a first-stage separation of $\mathscr{Y}$ of the form $\{\bigcup_{y \in Y_a} U(y, x(y, d), d)\colon a \in X\}$, where $U(y, x(y, d), d)$ is open in the compact set C_d and contains y, $\{C_d\colon d \in X\}$ being a listing of all compact subsets of X. We claim that

$$Y_a \subset \operatorname{int}\Big(\bigcup_{y \in Y_a} U(y, x(y, d), d)\Big).$$

Suppose not and let $y \in Y_a \setminus \operatorname{int}\big(\bigcup_{d \in X} U(y, x(y, d), d)\big)$. Then $y \in \overline{X \setminus \bigcup_{d \in X} U(y, x(y, d), d)}$. Since X is a k'-space, there is an $a \in X$ with $y \in \overline{C_a \cap \bigcup_{d \in X} U(y, x(y, d), d)}$, which contradicts the fact that $y \in U(y, x(y, a), a)$. Thus $\{\operatorname{int}\big(\bigcup_{y \in Y_a, d \in X} U(y, x(y, d), d)\colon a \in X\}$ is an open separation of $\mathscr{Y}$. ■

It may also be of interest to see that the Product Measure Extension Axiom (PMEA), which states that the product measure on ${}^{\lambda}2$ can be extended to a $\mathfrak{c}$-additive measure on all the subsets of ${}^{\lambda}2$, for any cardinal λ, and which holds if supercompact-many random reals are added to a model of set theory, can be used directly to show that small normal k-spaces are collectionwise normal.

THEOREM 5. (PMEA) *Suppose X is a normal space, $\{Y_\alpha\colon \alpha < \lambda\}$ is a discrete collection of closed sets in X, and for each $\alpha < \lambda$,*

$$Y_\alpha = \bigcup_{y \in A_\alpha} \bigcup_{d \in \Gamma_\alpha} \bigcap_{\beta < \varrho(y)} U(y, \beta, d),$$

where $\varrho(y) < \mathfrak{c}$ and the collection $\{U(y, \beta, d)\colon \beta < \varrho(y)\}$ satisfies (1) *for each open set $U \supset \bigcap_{\beta < \varrho(y)} U(y, \beta, d)$ there is a $\beta < \varrho(y)$ with $U \supset U(y, \beta, d)$, and* (2) *for each $\beta, \beta' < \varrho(y)$, there is a $\beta'' < \varrho(y)$ such that $U(y, \beta'', d) \subset U(y, \beta', d) \cap U(y, \beta, d)$. Then there is a function $\bar{\beta}\colon \bigcup_{\alpha < \lambda}(A_\alpha \times \Gamma_\alpha) \to \mathrm{ORD}$ such that $\bar{\beta}(y, d) < \varrho(y)$ and $U(y, \bar{\beta}(y, d), d) \cap U(y', \bar{\beta}(y', d'), d') = \emptyset$ for $y \in A_\alpha$, $y' \in A_{\alpha'}$, $d \in \Gamma_\alpha$, $d' \in \Gamma_{\alpha'}$ and $\alpha \neq \alpha'$.*

The proof of Theorem 5 is very similar to P. Nyikos's proof that PMEA implies normal spaces of character $< \mathfrak{c}$ are collectionwise normal [N] and is left to the reader. Note that we do not necessarily get an open separation of $\{Y_\alpha\colon \alpha < \lambda\}$ in Theorem 5. Using a more delicate argument, we can get an open separation in a space determined by small compact sets:

THEOREM 6. *Suppose X is normal and determined by compact sets of size $< \mathfrak{c}$ (or, X is determined by the compact sets C having the property that for each $x \in X$, the character of x in C is $< \mathfrak{c}$). Then X is collectionwise normal.*

Proof. We do the case where X is normal and determined by compact sets of size $< \mathfrak{c}$. Suppose $\{Y_\alpha\colon \alpha < \lambda\}$ is a discrete collection of closed sets. Let μ be a $\mathfrak{c}$-additive measure on ${}^{\lambda}2$. For each $f\colon \lambda \to 2$, let $U_{f,0}$, $U_{f,1}$ be disjoint open sets such that

$\bigcup_{f(\alpha)=i} Y_\alpha \subset U_{f,i}$ $(i = 0, 1)$. For each $\alpha < \lambda$ and $y \in X$, let $A(\alpha, y) = \{f\colon y \in U_{f,f(\alpha)}\}$. Let $U(\alpha) = \{y\colon \mu(A(\alpha, y)) > 7/8\}$. We claim $U(\alpha)$ is open in X. It suffices to show that $C \setminus U(\alpha)$ is closed in C for each compact C of size $< \mathfrak{c}$. Suppose C is such a set, and p is a limit point of $C \setminus U(\alpha)$, but $p \in U(\alpha)$. Let $\{U(p, \beta)\colon \beta < |C|\}$ be a basis for p in C, let $U(p, \beta) = U_\beta \cap C$ for each $\beta < |C|$, where U_β is open in X, and let $p_\beta \in (C \setminus U(\alpha)) \cap U_\beta$ for each $\beta < |C|$. For each $f \in A(\alpha, p)$, $p \in U_{f,f(\alpha)}$, so let $\beta_f < |C|$ be such that $U(p, \beta_f) \subset U_{f,f(\alpha)}$. Since $\mu(A(\alpha, p)) = \mu\big(\bigcup_{\beta < |C|}\{f \in A(\alpha, p)\colon \beta_f = \beta\}\big) > 7/8$, let $F \in [|C|]^{<\omega}$ be such that $\mu\big(\bigcup_{\beta \in F}\{f \in A(\alpha, p)\colon \beta_f = \beta\}\big) > 7/8$, and let $\beta < |C|$ be such that $U(p, \beta) \subset \bigcap_{\beta \in F} U(p, \beta)$. We show that $p_\beta \in U(\alpha)$. For each $g \in \bigcup_{\beta' \in F}\{f \in A(\alpha, p)\colon \beta_f = \beta'\}$, $p_\beta \in U(p, \beta) \subset U(p, \beta_g) \subset U_{g,g(\alpha)}$, so $g \in A(\alpha, p_\beta)$. Therefore $\mu(A(\alpha, p_\beta)) > 7/8$, and so $p_\beta \in U(\alpha)$. This contradicts our choice of p_β, so we must have $p \notin U(\alpha)$, which is what we wanted to show. So the $U(\alpha)$'s are open sets, and $U(\alpha) \supset Y_\alpha$. The fact that the $U(\alpha)$'s separate the Y_α's is again a standard Nyikos-type argument and is left to the reader. ■

4. The general result for countably paracompact spaces

THEOREM 7. *Suppose κ, $\mathbf{P}$, and G are as in the statement of Theorem 1. Suppose $V[G] \models$ "X is a space satisfying the property that if $\mathscr{Z}$ is a collection of subsets of X such that every countable subcollection of $\mathscr{Z}$ is locally finite, then $\mathscr{Z}$ is locally finite; $\mathscr{Y} = \{Y_a\colon a \in X\}$ is a locally finite collection of subsets of X such that every countable subcollection has a locally finite (point finite) open expansion, and for each $y, d \in X$ there is a collection $\mathscr{U}_{y,d} = \{U(y, x, d)\colon x \in X\}$ consisting of sets with compact closure and satisfying properties* (i), (ii), *and* (iii) *listed in Theorem* 1".

Then $V[G] \models$ "there is a function $f\colon \bigcup_{a \in X} Y_a \times X \times \mathbf{Z}^+ \to X$, a sequence $\langle \mathscr{W}_n \rangle_{n \in \mathbf{Z}^+}$, where

$$\mathscr{W}_n = \Big\{\bigcup_{\substack{y \in Y_a \\ d \in X}} U(y, f(y, d, n), d)\colon a \in X\Big\},$$

and a function $g\colon X \times X \to X$ so that for each $x \in X$ there is an $n \in \mathbf{Z}^+$ such that $\bigcup_{d \in X} U(x, g(x, d), d)$ meets only finitely many elements of $\mathscr{W}_n$ (for each $x \in X$ there is an $n \in \mathbf{Z}^+$ such that x is in only finitely many elements of $\mathscr{W}_n$)".

Proof. The argument parallels that of Theorem 1: in $M[G^*]$ the collection $\{\bigcup_{y \in Y_a} \bigcup_{d \in X} \bigcap_{x \in X} j(U)(j(y), j(x), j(d))\colon a \in X\}$ is shown to be locally finite in $j(X)$, but without a "locally-finite expansion sequence"; we use the elementary embedding to reflect this to a small counterexample in $V[G]$. We then need the analogue of Lemma 2 to show there can be no such counterexample: the proof of the analogue is similar to Balogh's analogue of the same lemma for countably paracompact spaces [B]. ■

5. Applications of the countably paracompact case

THEOREM 8. *Suppose κ, $\mathbf{P}$, and G are as in Theorem 1. Then $V[G] \models$ "if X is a countably paracompact k-space and $\mathscr{Y} = \{Y_a\colon a \in X\}$ is a discrete collection of closed sets in X, then $\mathscr{Y}$ has a 'first-stage-locally-finite expansion sequence'."*

Proof. We need only verify that if $\mathscr{X}$ is a collection of subsets of a k-space such that every countable subcollection is locally finite, then $\mathscr{X}$ is locally finite, in order to have the conclusion of Theorem 6 hold. The proof that this is so is similar to the proof presented in Theorem 3 that the statement is true if we replace "locally finite" with "discrete". ■

THEOREM 9. *Suppose κ, $\mathbf{P}$, and G are as in Theorem 1. Then $V[G] \models$ "if X is a countably paracompact k'-space, then every locally finite collection in X has a locally finite expansion by open sets".*

Proof. Similar to the proof of Theorem 4. ■

Again, the Product Measure Extension Axiom (PMEA) can be used directly to show that small countably paracompact k-spaces have the property that locally finite collections of closed sets have locally finite open expansions. In this case our argument is a more delicate version of Burke's argument that countably paracompact spaces of character $< \mathfrak{c}$ have this property [Bu].

6. Final remarks. If we could get that for each $y \in \bigcup \mathscr{Y}$ and $d \in X$ there is an $x(y, d) \in X$ such that $\{\bigcup_{y \in Y_a, d \in X} U(y, x(y, d), d)\colon a \in X\}$ is a discrete collection (not just a separation), then we could repeat the argument using this collection instead of $\mathscr{Y}$ to get a discrete separation of a similar form. After repeating the argument ω times, we would have an open separation of the original collection $\mathscr{Y}$. Thus we would have that it is consistent that normal k-spaces are collectionwise normal. Unfortunately, we have not been able to make the first-stage separation discrete.

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