Nonintegral boundary-slopes exist

by

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Abstract. The existence of nonintegral boundary-slopes for knots in S^3 is proved, which gives a negative answer to a question asked by Hatcher and Thurston.

1. Introduction. Let K in S^3 be a nontrivial knot, let $N(K)$ be a tubular neighborhood of K in S^3 and let $M = S^3 - \operatorname{int} N(K)$ with a preferred meridian-longitude framing pair on ∂M. If $(F, \partial F) \subset (M, \partial M)$ is an orientable, incompressible and boundary-incompressible surface (with ∂F nonempty), then the components of ∂F all have the same slope on ∂M and such a slope is called a *boundary-slope*. Consider $\varphi(K) \subset \boldsymbol{Q} \cup \{1/0\}$, the set of boundary-slopes of K. Questions about $\varphi(K)$ are closely related to understanding the structure of 3-manifolds obtained by Dehn surgery on K. In [9] Hatcher and Thurston completely described $\varphi(K)$ for 2-bridge knots. In particular they found that $\varphi(K) \subset \boldsymbol{Z} \cup \{1/0\}$ for every 2-bridge knot. The following natural question was thus raised in [9].

Question. Is it true that $\varphi(K) \subset \boldsymbol{Z} \cup \{1/0\}$ for every knot K in S^3?

In this paper we give the question a negative answer as stated in the title ([1]). In fact we give the example that for the $(-2, 3, 7)$ pretzel knot there exists a nonintegral boundary-slope. The argument consists of the following two sections. In the next section we prove

Theorem 1. *If K is hyperbolic and non-sufficiently large (i.e. K is a non-torus knot and there is no closed incompressible non-peripheral surface in M, the complement of K) and if K admits two nontrivial cyclic surgeries, then there exists at least one nonintegral boundary-slope for K.*

The set of knots satisfying the conditions given in Theorem 1 is not empty. In Section 3 we explain

Example 1. *The $(-2, 3, 7)$ pretzel knot is a hyperbolic and non-sufficiently large knot which admits 18- and 19-cyclic Dehn surgeries.*

([1]) After this work was done we learned that a negative answer has also been given by Hatcher and Oertel, and by Takahashi (to appear).

The paper is closed in Section 4 where a quick view over recent research results on $\varphi(K)$ is given, companioned with some remarks and open questions.

We refer to [10] and [15] for standard terminology and we work in the smooth category.

Before leaving this section, we give a remark below. The proof of the remark is easy and is thus omitted. Recall that an orientable surface $(F, \partial F) \subset (M, \partial M)$ is *essential* if each component of F is incompressible and not parallel to a subsurface of ∂M (the definition is from [3]).

Remark. An orientable surface $(F, \partial F) \subset (M, \partial M)$ is essential iff F is incompressible and boundary-incompressible. Therefore the definition of boundary-slope defined in the first paragraph (from [9]) agrees with that defined in [3].

2. Proof of Theorem 1. The proof is based on an application of the main results of [3]. Let $K(r)$ denote the manifold obtained by surgery along K with slope r.

By [3] Corollary 1, the two nontrivial cyclic surgery slopes that K admits are successive integers, say, m and $m+1$.

CLAIM. *Neither m nor $m+1$ is a boundary-slope.*

Proof of Claim. Suppose that one of the two slopes, say m, is a boundary-slope. Let $(F, \partial F) \subset (M, \partial M)$ be an orientable essential surface such that ∂F is a nonempty set of boundary curves in ∂M of slope m and such that the number of components of ∂F is minimal subject to these conditions. Note that in any knot complement all orientable essential surfaces except those with 0 boundary-slope are separable surfaces. Now applying [3] Proposition 2.2.1 if F is nonplanar or applying [3] Proposition 2.3.1 if F is planar, we get a contradiction either with the condition that $\pi_1(K(m))$ is cyclic or with the condition that K is non-sufficiently large in both cases. ■

Since K is a hyperbolic knot, the interior of M has a complete hyperbolic metric of finite volume. We can now apply the main results in [3] Chapter 1. It follows that there exists a norm $\|\cdot\|$ on the 2-dimensional real vector space $H_1(\partial M, \mathbf{R})$ such that

(1) $\|\cdot\|$ is positive integer valued for each $(m, l) \in H_1(\partial M, \mathbf{Z}) - \{(0, 0)\} \subset H_1(\partial M, \mathbf{R})$. Note that every slope $r = m/l \in \mathbf{Q} \cup \{1/0\}$ corresponds to the pair of primitive elements $(\pm m, \pm l) \in H_1(\partial M, \mathbf{Z})$.

(2) Define $n = \min\{\|(m, l)\|;\ (m, l) \in H_1(\partial M, \mathbf{Z}) - (0, 0)\}$ and consider the ball B of radius n in $H_1(\partial M, \mathbf{R})$. Then B is a compact, convex, finite sided polygon which is symmetric about the origin (i.e. $-B = B$). Note that $\operatorname{int} B \cap H_1(\partial E, \mathbf{Z}) = (0, 0)$.

(3) For any vertex of B, there is a primitive element $(m, l) \in H_1(\partial M, \mathbf{Z})$ such that (m, l) lies on the semi-line starting at $(0, 0)$ and passing through the vertex and moreover $r = m/l$ is a boundary-slope.

(4) If $r = m/l$ is not a boundary-slope and $K(r)$ has cyclic fundamental group, then $(\pm m, \pm l) \in \partial B$ (of course they are not vertices of B by (3)).

(5) Assume the area of a parallelogram spanned by any pair of generators of $H_1(\partial M, \mathbf{Z})$ is 1. Then $\operatorname{Area} B \leqslant 4$.

Now to get Theorem 1 it suffices to show that there exists a vertex of B which provides a nonintegral boundary-slope in the way described in (3). By the Claim and (4) above, the points $(\pm m, \pm l)$ and $(\pm m \pm 1, \pm 1)$ are all on the boundary of B and none of them are vertices of B. Let E be the closed edge segment of ∂B on which $(m+1, 1)$ lies (as an interior point) and let $v_1 = (s_1, s_2)$ and $v_2 = (t_1, t_2)$ be the two vertices of E. Let L be the line in $H_1(\partial M, \mathbf{R})$ passing through the points $\{(m, 1);\ m \in \mathbf{Z}\}$.

Case 1. E is not parallel to L. Then one of the vertices of E, say $v_1 = (s_1, s_2)$ must lie above the line L in the sense that $s_2 > 1$. Such a vertex certainly yields a nonintegral boundary-slope in the way described in (3).

Case 2. E is parallel to L. Then $m \in E$ (as an interior point) and $v_1 = (s_1, 1)$, $v_2 = (t_1, 1)$. We may assume that $s_1 < m < m+1 < t_1$. We must have $m-1 < s_1 < t_1 < m+2$ since otherwise the area of B would be larger than 4, violating (5). Now both v_1 and v_2 contribute nonintegral boundary-slopes as we required. ■

3. Proof of Example 1. Throughout this section let K denote the $(-2, 3, 7)$ pretzel knot. We understand that Fintushel and Stern have shown (unpublished) the following

LEMMA 2.1. *18 and 19 Dehn surgeries on K yield lens spaces.*

For the sake of the completeness of the paper we give the following verification of their result.

Proof. The idea is to show that 18 and 19 surgeries on K yield manifolds that double branched cover S^3 with branched set in S^3 a 2-bridge link and a 2-bridge knot respectively. The manifolds are therefore lens spaces. Actually we will see that they are $L(18, 5)$ and $L(19, 8)$. We provide below an explicit pictorial illustration.

Note that K is a strongly invertible knot (Fig. 1). The quotient under the involution shown in Fig. 1 is S^3 and hence S^3 double branched covers S^3 with branch set downstairs the unknot as shown in Fig. 4 (the process is shown through Figs. 1–4).

Note that the strong inversion on K can be extended to an involution on each of the manifolds $K(r)$ and the quotient under the corresponding involution is S^3. Moreover, the branched set in S^3 of the corresponding double covering can be obtained

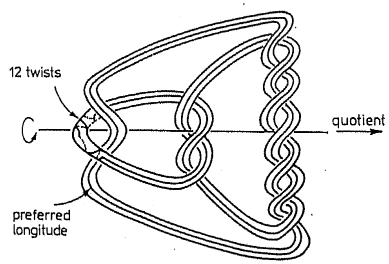

Fig. 1

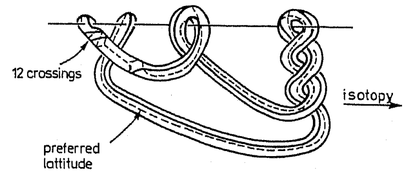

Fig. 2

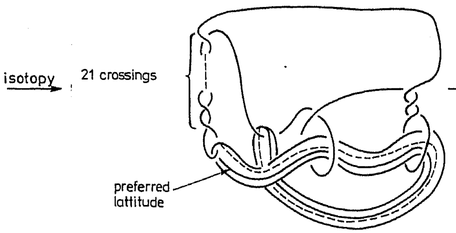

Fig. 3

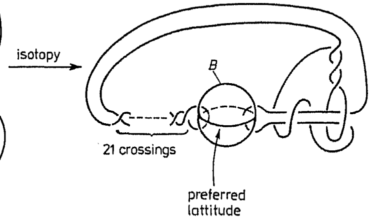

Fig. 4

by removing the trivial 1/0-tangle (ball B shown in Fig. 4) from the unknot and replacing it by the rational r-tangle (bewear that the sign of a rational r-tangle given here is opposite to that given in [2]). In particular, the branch sets in S^3 corresponding to 18 and 19 surgeries are shown in Fig. 5 and Fig. 6 respectively. They turn out to be (by isotopy) the 18/5 2-bridge link and the 19/8 2-bridge knot. Therefore the manifolds upstairs are lens spaces $L(18, 5)$ and $L(19, 8)$. ■

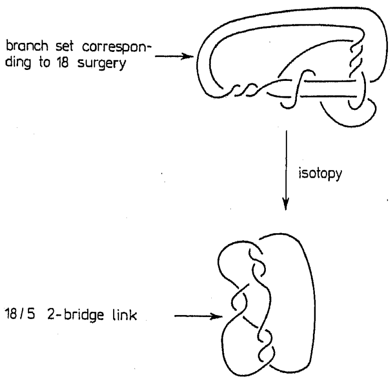

Fig. 5

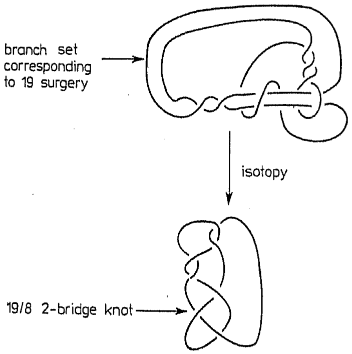

Fig. 6

References for the argument above are [1], [2], [12], [15].

Lemma 2.2. *K is hyperbolic and non-sufficiently large.*

Proof. Note that K is the $K(-1/2, 1/3, 1/7)$ star knot (notation as in [14]) and hence by [14] Corollary 4(a), K is non-sufficiently large. K can not be a torus knot either for, according to Moser's formula [13], there is no nontrivial torus knot which could admit 18 and 19 cyclic surgeries. ■

Example 1 follows from Lemmas 2.1 and 2.2. ■

4. Facts, remarks and questions on $\varphi(K)$. In this section we list several results that are known about the general properties of $\varphi(K)$ for arbitrary knot $K \subset S^3$, make some remarks and raise some open questions.

Theorem 4.1 ([4]). *$|\varphi(K)| \geqslant 2$ for any nontrivial knot K in S^3.*

Theorem 4.1 is sharp as a torus knot $T(p, q)$ has exactly two boundary-slopes, namely $\varphi(T(p, q)) = \{0, pq\}$.

Question. Is it true that for a nontorus knot K in S^3, $|\varphi(K)| > 2$?

Theorem 4.2 ([8]). *$\varphi(K)$ is a finite set for any knot in S^3.*

In spite of Theorem 4.2, there is no upper bound restriction on distance among boundary-slopes in $\varphi(K)$ when K varies over all knot types (the distance between two slopes $r_1 = m_1/l_1$ and $r_2 = m_2/l_2$ is defined to be $|m_1 l_2 - m_2 l_1|$). This is easily seen to be true when K varies in the set of cabled knots of a fixed knot, namely the distance between the boundary-slopes 0 ($0 \in \varphi(K)$ for all knots $K \subset S^3$) and pq (the slope of the cabling annulus) can be arbitrarily large. This is also true when K varies over the set of hyperbolic knots. In fact, Fintushel and Stern have shown [5] that for any even integer $2n$, $|n| > 1$, there is a hyperbolic knot K_{2n} in S^3 such that $18n$ surgery on K_{2n} produces the lens space $L(18n, 6n+1)$. Then the similar argument to that of Theorem 1 will give a boundary-slope m/l of K_{2n} with $|m| > |18n|$.

Take the notations as in the proof of Theorem 1. One of properties of the fundamental domain B is that each vertex of B corresponds to a boundary-slope. Let $m/l \neq 0$ be a boundary-slope of a hyperbolic knot K in S^3 and let $L \subset H_1(\partial M, \mathbf{R})$ be the semi-line which starts from $(0, 0)$ and passes (m, l).

Question. Does L intersect B at a vertex of B?

If the answer is yes, then some interesting information about cyclic surgery and boundary-slopes can be drawn. In particular, Theorem 4.2 follows for hyperbolic knots.

Question. Is Theorem 1 still true if in Theorem 1 the condition "K admits two nontrivial cyclic surgeries" is reduced to "K admits one nontrivial cyclic surgery"?

If the answer is yes, then all K_{2n} ($|n| > 1$) have nonintegral boundary-slopes.

Let $\wp\varphi(K)$ be the set of boundary-slopes of essential planar surfaces in $S^3 - \operatorname{int} N(K)$.

Theorem 4.3 ([6]). *$|\wp\varphi(K)| \leqslant 6$ for any knot K.*

Theorem 4.4 ([7]). *$\wp\varphi(K) \subset \mathbf{Z} \cup \{1/0\}$ for any knot K.*

For a torus knot $K = T(p, q)$ or cabled knot $K = C(p, q)$, $pq \in \wp\varphi(K)$. It is also known that for certain prime knots, e.g., those which have prime tangle decompositions [11], and even for certain hyperbolic knots, e.g. those which have simple tangle decompositions [16], $1/0 \in \wp\varphi(K)$ (the proof is not too hard and is omitted).

QUESTION. Is it true that $\wp\varphi(K)-\{1/0\}=\emptyset$ for any nontorus noncabled knot K?

Note that the positive answer to this question implies that to the cabling conjecture, which states: if $K(r)$ is a reducible manifold then K is a torus knot or a cabled knot.

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Normal k'-spaces are consistently collectionwise normal

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Abstract. Z. Balogh completed F. Tall's "Toronto project" by proving consistently that every normal, locally compact space is collectionwise normal. The natural generalization is to replace "locally compact" with the classical "k-space property". We prove that in any model obtained by adding supercompact many Cohen or random reals, discrete collections of closed sets in such spaces have a "first-stage" separation; if the space also satisfies the stronger k'-space property, then we can obtain an open separation, so the space is collectionwise normal.

1. Introduction. F. Tall's "Toronto project", to prove consistently that every normal, locally compact space is collectionwise normal, was completed by Z. Balogh who proved that this is so in any model obtained by adding supercompact many Cohen or random reals [B]. A history of the project is contained in his paper. It would be nice to improve this result by replacing "locally compact" by "k-space", where a k-space is one in which a set is closed if, and only if, its intersection with every compact set is closed, since the k-space property is a classical topological property and k-spaces have some nice properties; they are precisely the quotient images of locally compact spaces, and hence are closed under quotient maps. In this paper we prove that in any model obtained by adding supercompact many Cohen or random reals, discrete collections of closed sets in such spaces have what we call a "first-stage" separation, and show that if a space additionally satisfies a stronger property, the k'-property, then the first-stage separation enables us to get an open separation of the sets, and hence the space is collectionwise normal. We also show that if the first-stage separation could be made to be discrete, then the process could be continued and we could get the collectionwise normality of the k-spaces.

We also treat the "countably paracompact" analogue of these results, and obtain, as one familiar with the history of the Toronto project would expect, the results that in such spaces locally finite collections of closed sets have a "first-stage expansion" and that if a space is additionally k', this enables us to get an expansion by locally finite open sets.

The large cardinal assumption is used to obtain a reflection principle: if there is a counterexample, it is forced to be a small one. Once we had the new ideas necessary to show that there could be no small counterexample, we first modelled our consistency