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DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DI SIENA
Via del Capitano 15
53100 Siena
Italy

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The Hahn–Banach theorem implies the existence of a non-Lebesgue measurable set

by

Matthew Foreman (Columbus, Ohio) and **Friedrich Wehrung** (Caen)

Abstract. In this paper we show that the Axioms of Zermelo–Fraenkel set theory together with the Hahn–Banach theorem imply the existence of a non-Lebesgue measurable set. Our construction does not make any use of the Axiom of Choice.

§ 0. Introduction. Few methods are known to construct non-Lebesgue measurable sets of reals: most standard ones start from a well-ordering of $\boldsymbol{R}$, or from the existence of a nontrivial ultrafilter over ω, and thus need the axiom of choice AC or at least the Boolean Prime Ideal theorem (BPI, see [5]). In this paper we present a new way for proving the existence of nonmeasurable sets using a convenient operation of a discrete group on the Euclidean sphere. The only choice assumption used in this construction is the Hahn–Banach theorem, a weaker hypothesis than BPI (see [9]). Our construction proves that the Hahn–Banach theorem implies the existence of a nonmeasurable set of reals. This answers questions in [9], [10]. (Since we do not even use the countable axiom of choice, we cannot assume the countable additivity of Lebesgue measure, e.g. the real numbers could be a countable union of countable sets.)

In fact we prove (under the Hahn–Banach theorem) that there is no finitely additive, rotátion invariant extension of Lebesgue measure to $\mathscr{P}(\boldsymbol{R}^3)$. Recall that the Hahn–Banach theorem implies the existence of a finitely additive, isometry invariant extension of Lebesgue measure to $\mathscr{P}(\boldsymbol{R}^2)$ (see [14]).

We use standard set-theoretical notation and terminology. For example, if X is any set, $\mathscr{P}(X)$ is the power set of X. If $A \subseteq X$ and $f\colon X \to Y$ is a map, then $f[A]$ is the image of A under f. ω is the set of all natural numbers.

We assume ZF throughout this paper; no choice assumption (even countable) is made.

§ 1. Definitions. First, let us give one of the many equivalent statements of the Hahn–Banach theorem. We use the version [11]:

THE HAHN–BANACH THEOREM. *Let E be a vector space over the reals, let S be a subspace of E, and f be a linear functional on S. Let p be a map $E \to \boldsymbol{R}$ such that whenever $x, y \in E$ and $\lambda \geqslant 0$, we have $p(\lambda x) = \lambda p(x)$ and $p(x+y) \leqslant p(x)+p(y)$ and for all*

$x \in S$ $f(x) \leqslant p(x)$. *Then there is a linear functional* $\bar{f}$ *on* E, *extending* f, *such that* $(\forall x \in E)(\bar{f}(x) \leqslant p(x))$.

DEFINITION. If B is a Boolean algebra, a *finitely additive probability measure* on B (from now on a *measure*) is a map μ: $B \to [0, 1]$ such that $\mu(1_B) = 1$ and $\mu(x \vee y) = \mu(x) + \mu(y)$ whenever $x \wedge y = 0$.

It is known that ZF+Hahn–Banach implies that every Boolean algebra has a measure (actually in ZF without choice, this last statement is equivalent to the Hahn–Banach theorem, see [7, 15]). It also yields the following statement for collections of Boolean algebras:

PROPOSITION 1 (ZF+Hahn–Banach). *Let* $\langle B_i$: $i \in I\rangle$ *be a sequence of Boolean algebras (with* I *not necessarily well-orderable). Then there exists* $\langle \mu_i$: $i \in I\rangle$ *such that for each* $i \in I$, μ_i *is a measure on* B_i.

Proof. Let $(B, e_i)_{i \in I}$ be the direct sum of $(B_i)_{i \in I}$ in the category of Boolean algebras: so, for every $i \in I$, e_i is an homomorphism $B_i \to B$ (elements of B are formal Boolean combinations of elements of the B_i with no other relations than those from the B_i; one can prove that e_i is one-to-one). By the Hahn–Banach theorem there is a measure μ on B. Put $\mu_i = \mu \circ e_i$. ∎

DEFINITION. A *universally measured space* is an ordered pair (Ω, μ) where Ω is a set and μ is a measure on the Boolean algebra $\mathscr{P}(\Omega)$. A group G is said to *act by measure preserving transformations* on (Ω, μ) when G acts on Ω and $\mu(gA) = \mu(A)$ for all $g \in G$ and $A \in \mathscr{P}(\Omega)$.

We are going to be mainly concerned about the following measure existence statement:

DEFINITION. Let a group G act on a set Ω. $IM(\Omega, G)$ is the statement "there is a G-invariant measure on $\mathscr{P}(\Omega)$".

In the case of a group acting on itself, we get the following classical definition.

DEFINITION. A group G is *amenable* when there is a measure μ on $\mathscr{P}(G)$ such that $\mu(Ag) = \mu(A)$ for all $g \in G$, $A \in \mathscr{P}(G)$.

Assuming the Hahn–Banach theorem many groups are amenable, including finite groups, solvable groups and their extensions. The best known nonamenable group is the free group on two generators.

PROPOSITION 2 (classical, [14]). *The free group on two generators*, F_2, *is not amenable.*

For all integers $n \geqslant 1$, denote by O_n the isometry group of S^{n-1} (with Euclidean norm), $SO_n = \{u \in O_n$: $\det(u) = +1\}$, where $S^n = \{x \in \boldsymbol{R}^{n+1}$: $\|x\| = 1\}$ is the n-dimensional Euclidean sphere. One can prove in ZFC that $IM(S^n, SO_{n+1})$ does not hold for $n \geqslant 2$, and thus SO_{n+1} is not amenable (see [14]). On the other hand, in [10] and [13], the authors construct models of ZF+DC in which $IM(S^n, O_{n+1})$ holds for every $n \geqslant 1$ (in [13], the measure is just normalized Lebesgue measure).

A group G acts on a set Ω *freely* when for all $g \in G$, $x \in \Omega$, $gx = x$ implies $g = 1$.

§ 2. The main results.

We start with a classical result.

PROPOSITION 3. *Assume* $IM(S^2, SO_3)$. *Then there is a free measure preserving action of* F_2 *on some universally measured space* (Ω, μ).

Proof. Consider a subgroup of SO_3 isomorphic to F_2 [14] and D the subset of S^2 consisting of the fixed points of elements of $F_2 \setminus \{1\}$. D is countable since each orbit is effectively countable and it is easy to distinguish fixed points of elements of F_2 acting on S^2. Hence D is the image of a function with domain $\{0, 1\} \times F_2 \times F_2$. (Recall we do not know that a countable union of countable sets is countable.) Let μ be the witness to $IM(S^2, SO_3)$. Since F_2 acts freely on $S^2 \setminus D$, we will be done if we can show $\mu(D) = 0$.

In [14] it is shown that every SO_3-invariant finitely additive measure on S^2 gives each countable set measure zero. We paraphrase the proof given there and check that it works without AC.

It clearly suffices to find a rotation g such that for all $k \in \omega \setminus \{0\}$, $g^k D \cap D = \emptyset$, since then $\{g^k D$: $k \in \omega\}$ is an infinite collection of pairwise disjoint subsets of S^2 of the same μ-measure. Let $\langle a_n$: $n \in \omega\rangle$ be an enumeration of D. Let l be a line through the origin missing D. Let $A_n = \{g \in SO(3)$: g is a rotation about l and for some $i \neq j \in \omega$, $g^n a_i = a_j\}$. Then A_n is countable in a canonical way, since each $g \in A_n$ is determined by a_i and a_j. Hence $\bigcup A_n$ is countable. Choose a rotation g about l such that $g \notin \bigcup A_n$ and g has infinite order. Then for all $n \geqslant 1$, $g^n D \cap D = \emptyset$. ∎

Another example is with $IM({}^{\omega}2, G)$ where ${}^{\omega}2$ is the Cantor space with its canonical metric and G its group of isometries (see [12]).

Our main theorem is:

THEOREM 4 (ZF+Hahn–Banach). *Let a group* G *act freely and measure preserving on a universally measured space* (Ω, μ). *Then* G *is amenable.*

Proof. (Note the similarity to [6].) Denote by Ω/G the set of orbits of Ω modulo G. By Proposition 1, there is a sequence $\langle \mu_{[x]}$: $[x] \in \Omega/G\rangle$ such that for each $[x] \in \Omega/G$, $\mu_{[x]}$ is a measure on $\mathscr{P}([x])$. For each $A \subseteq G$, let a: $\Omega \to [0, 1]$ be the following function: $a(x) = \mu_{[x]}(Ax)$; define λ: $\mathscr{P}(G) \to [0, 1]$ by $\lambda(A) = \int a(x) d\mu(x)$. Note that $x \mapsto a(x)$ is a measurable function since (Ω, μ) is a universally measured space; the integration here is essentially Lebesgue integration, and it does not appeal to any choice (no limit theorems are needed).

We claim that λ is a measure on $\mathscr{P}(G)$, invariant under right translation.

Note that λ takes values in $[0, 1]$ and $\lambda(G) = 1$. If A, B are two disjoint subsets of G and a, b, c are the functions corresponding to A, B, $A \cup B$ respectively, then $(\forall x \in \Omega)(c(x) = a(x) + b(x))$. Hence $\lambda(A \cup B) = \lambda(A) + \lambda(B)$.

Finally, if $B = Ag$ for some $g \in G$ and a, b are the functions corresponding to A and B then, for all $x \in \Omega$,

$$b(x) = \mu_{[x]}(Bx) = \mu_{[x]}(Agx) = \mu_{[x]}(A(gx)) = \mu_{[gx]}(A(gx)) = a(gx).$$

Hence $\lambda(B) = \int b(x) d\mu(x) = \int a(gx) d\mu(x) = \int a(x) d\mu(x) = \lambda(A)$ since g is μ-measure preserving. ∎

COROLLARY 1. ZF+*Hahn–Banach implies not* $IM(S^2, SO_3)$. *Thus, there is a non-Lebesgue measurable subset of* S^2.

Proof. Propositions 2, 3 and Theorem 4. ∎

Note that in the last part of the statement above, S^2 could be replaced by many other spaces, like $\mathbf{R}^n$, $n \geqslant 1$. (See §3 for details.)

COROLLARY 2. *If H is generic for the partial ordering adding ω_1 random reals to a model V of* ZFC *and $V(\mathbf{R})$ is the smallest model of set theory containing V and reals of $V[H]$, then $V(\mathbf{R})$ does not satisfy the Hahn–Banach theorem.*

Proof. $V(\mathbf{R})$ is the model considered by D. Pincus and R. Solovay in [10]. It satisfies $IM(S^n, SO_{n+1})$ for all $n \geqslant 1$, and thus $IM(S^2, SO_3)$; we conclude by Corollary 1. ■

Another way to see Corollary 1 is the following:

COROLLARY 3. *If F_2 acts freely on $\Omega = S^2 \setminus D$ (D as in the proof of Proposition 3) by rotations, and if $\langle \mu_{[x]}\colon [x] \in \Omega/F_2 \rangle$ is any assignment of finitely additive probability measures $\mu_{[x]}$ on $\mathscr{P}([x])$, then there are $A \subseteq F_2$ and $\alpha \in [0, 1]$ such that $\{x\colon \mu_{[x]}(Ax) < \alpha\}$ is not Lebesgue measurable. Further, the set A can be isolated explicitly* (see [14]). ■

§3. Appendix. Lebesgue measure without countable choice. Ordinarily, the theory of Lebesgue measure is developed with use of AC_ω. The use of AC_ω allows one to use arbitrary Borel sets. In this section we explore how to use "coded" Borel sets to eliminate the necessity of AC_ω in many applications. For example, we would still like the existence of nonmeasurable set to be independent of the reference space (here, S^2). The aim of this section is to show how to adapt the proofs of the "classical" theory (with AC_ω) to the study of Lebesgue measure in a totally choiceless context. The ideas here date from [13].

In order to get as many measurable sets as possible, the classical outer measure construction (see [4]) seems convenient enough. This construction, which we will sketch in $\mathbf{R}$, works as well in $\mathbf{R}^n$ or in much more abstract spaces.

Define the outer measure of $A \subseteq \mathbf{R}$ by the greatest lower bound of all sums $\sum_{n\in\omega} \operatorname{length}(I_n)$ where I_n are intervals, and $A \subseteq \bigcup_{n\in\omega} I_n$; call it $\mu^*(A)$. Say that A is Lebesgue measurable when for all $X \subseteq \mathbf{R}$, $\mu^*(X) = \mu^*(X \cap A) + \mu^*(X \setminus A)$. Set $\mathscr{M} = \{A \subseteq \mathbf{R};\ A \text{ is Lebesgue measurable}\}$, $\mu = \mu^* | \mathscr{M}$. It is still possible to prove that $\mathscr{M}$ is a Boolean subalgebra of $\mathscr{P}(\mathbf{R})$ and that μ is a finitely additive function $\mathscr{M} \to [0, \infty]$, and that $\mathscr{M}$ contains all open sets. But one cannot prove any more that $\mathscr{M}$ is a σ-algebra (since $\mathbf{R}$ can be a countable union of countable sets, see [5]). So, instead of considering Borel subsets of $\mathbf{R}$, consider those which have a *code*, as e.g. in [12]; a Borel code is essentially a real, encoding the "construction" of some Borel set. Similarly, say that $(A_n)_{n\in\omega}$ is a *coded sequence* of Borel sets when there is a sequence $(c_n)_{n\in\omega}$ such that for every n, c_n is a code for A_n. And then, we can prove the following properties of $(\mu, \mathscr{M})$:

(a) *$\mathscr{M}$ is a Boolean subalgebra of $\mathscr{P}(\mathbf{R})$, containing all coded Borel subsets of $\mathbf{R}$.*

(b) *μ is a finitely additive map $\mathscr{M} \to [0, \infty]$, and whenever $(A_n)_{n\in\omega}$ is a disjoint coded sequence of Borel sets, we have*

$$\mu\Big(\bigcup_{n\in\omega} A_n\Big) = \sum_{n\in\omega} \mu(A_n).$$

(c) *A subset $A \subseteq \mathbf{R}$ is in $\mathscr{M}$ iff for all $\varepsilon > 0$ and all coded Borel B with $\mu(B) < \infty$, there are coded Borel F and U such that $F \subseteq A \cap B \subseteq U$ and $\mu(U \setminus F) < \varepsilon$.*

(Actually, it is enough to check when B is the bounded interval, and U can be chosen as an open set, F as a closed set.)

(d) *μ is σ-finite: there is a coded sequence $(A_n)_{n\in\omega}$ of Borel sets such that $\mathbf{R} = \bigcup_{n\in\omega} A_n$ and $(\forall n \in \omega)(\mu(A_n) < \infty)$.* (Take $A_n = [-n, n]$.)

The precautions needed by elimination of AC_ω in the classical proof of (a) and (d) above (see [4]) make the proof somewhat more lengthy, but without real difficulties. Note that in (c), the assumption $\mu(B) < \infty$ does not seem to be removable without countable choice.

Let us call the μ above the Lebesgue measure on $\mathbf{R}$; a similar construction yields Lebesgue measure on $\mathbf{R}^n$, for all $n \geqslant 1$.

More generally, let us set the following definition:

DEFINITION. A *coded Borel space* is an ordered pair $(\Omega, \mathscr{B})$ where Ω is a coded Borel subset of the Hilbert cube ${}^\omega[0, 1]$ and $\mathscr{B}$ is the algebra of coded Borel subsets of Ω.

We can naturally extend this definition by taking all isomorphic images; this way, all usual spaces of analysis, like $\mathbf{R}^n$, S^n, or ${}^\omega 2$, together with their coded Borel subsets, become coded Borel spaces. Anyway, even without using countable choice, it turns out that the following is true:

PROPOSITION 5. *Let $(\Omega, \mathscr{B})$ be an uncountable coded Borel space. Then there is a coded Borel isomorphism from $(\Omega, \mathscr{B})$ onto $(I, \mathscr{B}_I)$, where $I = [0, 1]$ and $\mathscr{B}_I$ is the algebra of coded Borel subsets of I.* ■

Here, a *coded Borel isomorphism* $(\Omega, \mathscr{B}) \to (I, \mathscr{B}_I)$ is naturally a bijection $f\colon \Omega \to I$ such that the neighborhood diagrams of f and f^{-1} are coded Borel.

Now, let us give the new definition of measure we are going to use:

DEFINITION. Let $(\Omega, \mathscr{B})$ be a coded Borel space. A *regular measure* on $(\Omega, \mathscr{B})$ is a map $\mu\colon \mathscr{M} \to [0, \infty]$ such that $(\mu, \mathscr{M})$ satisfies conditions (a) to (d) above, with Ω instead of $\mathbf{R}$. Say that μ is *nonatomic* when $(\forall x \in \Omega)(\mu(\{x\}) = 0)$.

The essential isomorphism theorem between these measure spaces is still valid (after a suitable reformulation). It can be stated the following way:

PROPOSITION 6. *Let μ be a regular, nonatomic measure on a coded Borel space $(\Omega, \mathscr{B})$, with $\mu(\Omega) = 1$. Then there are $N \subseteq \Omega$, $D \subseteq [0, 1]$ and $f\colon \Omega \to [0, 1]$ such that, if l is Lebesgue measure on $[0, 1]$,*

(i) *$N \in \mathscr{B}$, D is countable, $\mu(N) = l(D) = 0$.*

(ii) *f is a coded Borel isomorphism $\Omega \setminus N \to [0, 1] \setminus D$.*

(iii) *For all B in $\mathscr{B}$, $f[B]$ is coded Borel in $[0, 1]$ and $\mu(B) = l(f[B])$.*

Outline of proof (see [11]). First, notice that by (b) and $\mu(\Omega) = 1$, Ω is uncountable. So, by Proposition 5, without loss of generality, $\Omega = [0, 1]$ and $\mathscr{B}$ is the algebra of coded Borel subsets of $[0, 1]$. Then define $f\colon [0, 1] \to [0, 1]$ by $f(x) = \mu([0, x])$. Then D is just $\{y \in [0, 1]\colon f^{-1}\{y\}$ has nonempty interior$\}$ and N is $f^{-1}[D]$. (iii) is proven by induction on a code of B, and it uses nonatomicity of μ. ■

Now, Proposition 6 has an immediate corollary:

COROLLARY 1. *Let μ be a regular, nonatomic measure on a coded Borel space* $(\Omega, \mathscr{B})$, *with* $\mu(\Omega) \neq 0$. *Then the following are equivalent*:

(i) *Every subset of Ω is μ-measurable.*

(ii) *Every subset of* $[0, 1]$ *is Lebesgue measurable.* ■

(To prove (i)$\Rightarrow$(ii), one has to use σ-finiteness, nonatomicity of μ and $\mu(\Omega) \neq 0$; for (ii)$\Rightarrow$(i), use characterization (c) above of μ-measurability.)

In particular, every subset of $\boldsymbol{R}^n$ ($n \geqslant 1$) is Lebesgue measurable iff every subset of $[0, 1]$ is Lebesgue measurable (which is well known in the classical theory using countable choice). Let LM be the latter statement.

Now, define Lebesgue measure v_n on S^n as being the image under $x \mapsto x/\|x\|$ of Lebesgue measure on $B^{n+1}\setminus\{0\}$, where B^{n+1} is the Euclidean closed ball of $\boldsymbol{R}^{n+1}$ of volume 1.

COROLLARY 2. *LM implies $IM(S^n, SO_{n+1})$ for all $n \geqslant 1$.*

Proof. If LM holds, then v_n is defined on $\mathscr{P}(S^n)$ by the previous corollary; so v_n witnesses $IM(S^n, SO_{n+1})$. ■

More precisely, the result would be the same with a rotation invariant extension of Lebesgue measure on $\mathscr{P}(S^2)$; thus, the results of the previous section imply for example that *the Hahn–Banach theorem implies nonexistence of a rotation invariant extension of Lebesgue measure to a (finitely additive) measure on* $\mathscr{P}(\boldsymbol{R}^3)$.

Further notes. Theorem 4 could be formulated as follows: *If G is a nonamenable group acting freely on a set Ω and if μ is a G-invariant finitely additive probability measure defined on a G-invariant subalgebra of $\mathscr{P}(\Omega)$, then Ω has nonmeasurable subsets (w.r.t. μ).* Now, while this paper was printed, the second author showed, under the same hypotheses, that in the G-equidecomposability type semigroup of Ω (see [14]), $n[\Omega] = (n+1)[\Omega]$ for some integer n, effectively computable from the number of pieces necessary to a paradoxical decomposition of G. For the action of F_2 described above, we can get $n = 5$, which is somewhat disappointing since it is not known whether the cancellation law (see [14]) follows from HB (it follows from BPI). But independently, J. Pawlikowski proved, using ideas from this paper, that one can actually take $n = 1$, that is, $[\Omega] = 2[\Omega]$; thus, HB implies the Banach–Tarski paradox. See [8] for more details.

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DEPARTMENT OF MATHEMATICS
OHIO STATE UNIVERSITY
Columbus, OH 43210
U.S.A.

CNRS
UNIVERSITÉ DE CAEN
France

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