The Hahn–Banach theorem implies
the existence of a non-Lebesgue measurable set

by

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Abstract. In this paper we show that the Axioms of Zermelo–Fraenkel set theory together with the Hahn–Banach theorem imply the existence of a non-Lebesgue measurable set. Our construction does not make any use of the Axiom of Choice.

§ 0. Introduction. Few methods are known to construct non-Lebesgue measurable sets of reals; most standard ones start from a well-ordering of \( \mathbb{R} \), or from the existence of a nontrivial ultrafilter over \( \omega \), and thus need the axiom of choice AC or at least the Boolean Prime Ideal theorem (BPI, see [5]). In this paper we present a new way for proving the existence of nonmeasurable sets using a convenient operation of a discrete group on the Euclidean sphere. The only choice assumption used in this construction is the Hahn–Banach theorem, a weaker hypothesis than BPI (see [9]). Our construction proves that the Hahn–Banach theorem implies the existence of a nonmeasurable set of reals. This answers questions in [9], [10]. (Since we do not even use the countable axiom of choice, we cannot assume the countable additivity of Lebesgue measure, e.g. the real numbers could be a countable union of countable sets.)

In fact we prove (under the Hahn–Banach theorem) that there is a finitely additive, rotation invariant extension of Lebesgue measure to \( \mathcal{P}(\mathbb{R}) \). Recall that the Hahn–Banach theorem implies the existence of a finitely additive, isometry invariant extension of Lebesgue measure to \( \mathcal{P}(\mathbb{R}) \) (see [14]).

We use standard set-theoretical notation and terminology. For example, if \( X \) is any set, \( \mathcal{P}(X) \) is the power set of \( X \). If \( A \subseteq X \) and \( f : X \rightarrow Y \) is a map, then \( f(A) \) is the image of \( A \) under \( f \).

We assume ZF throughout this paper; no choice assumption (even countable) is made.

§ 1. Definitions. First, let us give one of the many equivalent statements of the Hahn–Banach theorem. We use the version [11]:

The Hahn–Banach Theorem. Let \( E \) be a vector space over the reals, let \( S \) be a subspace of \( E \), and \( f \) be a linear functional on \( S \). Let \( p : E \rightarrow \mathbb{R} \) such that whenever \( x, y \in E \) and \( \lambda > 0 \), we have \( p(\lambda x) = \lambda p(x) \) and \( p(x + y) \leq p(x) + p(y) \) and for all
\[ x \in S \Longleftrightarrow f(x) \leq p(x). \] Then there is a linear functional \( \tilde{f} \) on \( E \), extending \( f \), such that \( (\forall x \in E) \tilde{f}(x) \leq p(x) \).

**Definition.** If \( B \) is a Boolean algebra, a finitely additive probability measure on \( B \) (from now on a measure) is a map \( \mu : B \to [0, 1] \) such that \( \mu(\emptyset) = 1 \) and \( \mu(x y) = \mu(x) + \mu(y) \) whenever \( x \cdot y = 0 \).

It is known that \( \mathcal{F}^+ \)-Hahn–Banach implies that every Boolean algebra has a measure (actually in \( \mathcal{F}^+ \) without choice, this last statement is equivalent to the Hahn–Banach theorem, see [7, 15]). It also yields the following statement for collections of Boolean algebras:

**Proposition 1 (ZF + Hahn–Banach).** Let \( \langle B_i; i \in I \rangle \) be a sequence of Boolean algebras (with \( I \) not necessarily well-orderable). Then there exists \( \langle \mu_i; i \in I \rangle \) such that for each \( i \in I, \mu_i \) is a measure on \( B_i \).

**Proof.** Let \( (B, e)_{\mu,i} \) be the direct sum of \( (B, e)_{\mu,i} \) in the category of Boolean algebras: so, for every \( i \in I, e_i \) is an isomorphism \( B_i \to B \) (elements of \( B \) are formal Boolean combinations of elements of the \( B_i \) with no other relations than those from the \( B_i \): one can prove that \( e_i = 1 \) in each one). By the Hahn–Banach theorem there is a measure \( \mu \) on \( B \). Put \( \mu_i = \mu ℗ e_i \).

**Definition.** A universally measured space is an ordered pair \( (\Omega, \mu) \) where \( \Omega \) is a set and \( \mu \) is a measure on the Boolean algebra \( \mathcal{P}(\Omega) \). A group \( G \) is said to act by measure preserving transformations on \( (\Omega, \mu) \) when \( G \) acts on \( \Omega \) and \( \mu(gA) = \mu(A) \) for all \( g \in G \) and \( A \in \mathcal{P}(\Omega) \).

We are going to be mainly concerned about the following measure existence statement:

**Definition.** Let a group \( G \) act on a set \( \Omega \). \( IM(\Omega, G) \) is the statement “there is a \( G \)-invariant measure on \( \mathcal{P}(\Omega) \).”

In the case of a group acting on itself, we get the following classical definition:

**Definition.** A group \( G \) is amenable when there is a measure \( \mu \) on \( \mathcal{P}(G) \) such that \( \mu(gA) = \mu(A) \) for all \( g \in G \), \( A \in \mathcal{P}(G) \).

Assuming the Hahn–Banach theorem many groups are amenable, including finite groups, solvable groups, and their extensions. The best known nonamenable group is the free group on two generators.

**Proposition 2 (classical, [14]).** The free group on two generators, \( F_2 \), is not amenable.

For all integers \( n \geq 1 \), denote by \( O_n \) the isometry group of \( S^{n-1} \) (with Euclidean norm), \( SO_n = \{ x \in O_n; \det(x) = 1 \} \), where \( S^n = \{ x \in \mathbb{R}^{n+1}; \| x \| = 1 \} \) is the \( n \)-dimensional Euclidean sphere. One can prove in ZFC that \( IM(S^n, SO_n) \) holds for all \( n \geq 1 \), and thus \( SO_n \) is not amenable (see [14]). On the other hand, in [10] and [13], the authors construct models of ZF + DC in which \( IM(S^n, O_n) \) holds for all \( n \geq 1 \) (in [13], the measure is just normalized Lebesgue measure).

A group \( G \) acts on a set \( \Omega \) freely when for all \( g \in G, x \in \Omega, gx = x \) implies \( g = 1 \).

**§ 2. The main results.** We start with a classical result.

**Proposition 3.** Assume \( IM(S^2, SO_3) \). Then there is a free measure preserving action of \( F_2 \) on some universally measured space \((\Omega, \mu)\).

**Proof.** Consider a subgroup of \( SO_3 \) isomorphic to \( F_2 \) [14] and \( D \) the subset of \( S^2 \) consisting of the fixed points of elements of \( F_2 \backslash \{ 1 \} \). \( D \) is countable since each orbit is effectively countable and it is easy to distinguish fixed points of elements of \( F_2 \) acting on \( S^2 \). Hence \( D \) is the image of a function with domain \( [0, 1] \times F \times F \). (Recalling we do not know that a countable union of countable sets is countable.) Let \( \mu \) be the witness to \( IM(S^2, SO_3) \). Since \( F_2 \) acts freely on \( S^2 \setminus D \), we will be done if we can show \( \mu(D) = 0 \).

In [14] it is shown that every \( SO_3 \)-invariant finitely additive measure on \( S^2 \) gives every countable set measure zero. We paraphrase the proof given there and check that it works without AC.

It clearly suffices to find a rotation \( g \) such that for all \( k \in \mathbb{N} \), \( g^k D \cap D = \emptyset \), since \( (g^k D; k \in \mathbb{N}) \) is an infinite collection of pairwise disjoint subsets of \( S^2 \) of the same \( \mu \)-measure. Let \( (A_k; n \in \mathbb{N}) \) be an enumeration of \( D \). Let \( l \) be a line through the origin missing \( D \). Let \( A_n = \{ x \in SO(3); g \text{ is a rotation about } l \text{ and for some } i \neq j \in \mathbb{N}, g^i A_i = A_j \} \). Then \( A_n \) is countable in a canonical way, since each \( g \in A_n \) is determined by \( a_n, a_j \). Hence \( \bigcup A_n \) is countable. Choose a rotation \( g \) about \( l \) such that \( g \notin \bigcup A_n \) and \( g \) has finite order. Then for all \( n \geq 1 \), \( g^n D \cap D = \emptyset \).

Another example is \( IM(S^2, G) \) where \( G \) is the Cantor space with its canonical metric and \( G \) is the group of isometries (see [12]).

Our main theorem is:

**Theorem 4 (ZF + Hahn–Banach).** Let a group \( G \) act freely and measure preserving on a universally measured space \((\Omega, \mu)\). Then \( G \) is amenable.

**Proof.** (Note the similarity to [13].) Denote by \( \Omega/G \) the set of orbits of \( \Omega \) modulo \( G \). By Proposition 1, there is a sequence \( \langle \mu_n; [x] \in \Omega/G \rangle \) such that for each \( [x] \in \Omega/G \), \( \mu_n \) is a measure on \( \mathcal{P}([x]) \). For each \( A \subseteq G \), let \( \alpha_A : \Omega \to [0, 1] \) be the following function:

\[
\alpha_A(x) = \mu_n(A \cdot x); \text{define } \lambda : \mathcal{P}(\Omega) \to [0, 1] \text{ by } \lambda(A) = \int \alpha_A(x) \mu_n(dx). \text{ Note that } x \mapsto \alpha_A(x) \text{ is a measurable function since } (\Omega, \mu) \text{ is a universally measured space; the integration here is essentially Lebesgue integration, and it does not appeal to any choice (no limit theorems are needed).}
\]

We claim that \( \lambda \) is a measure on \( \mathcal{P}(\Omega) \), invariant under right translation.

Note that \( \lambda \) takes values in \([0, 1]\) and \( \lambda(G) = 1 \). If \( A, B \) are two disjoint subsets of \( G \) and \( a, b, c \) are the functions corresponding to \( A, B, A \cup B \) respectively, then \( (\forall x \in \Omega) (\lambda(x) = \alpha_A(x) + \alpha_B(x) \text{ or } \alpha_A(x) \text{ or } \alpha_B(x)) \). Hence \( \lambda(A \cup B) = \lambda(A) + \lambda(B) \).

Finally, if \( B = A_g \) for some \( g \in G \) and \( a, b \) are the functions corresponding to \( A, B \) respectively, then \( (\forall x \in \Omega) (\lambda(x) = \alpha_A(x) + \alpha_{A_g}(x)) \). Hence \( \lambda(\emptyset) = \int \beta(x) \mu_n(dx) = \int \alpha(x) \mu_n(dx) = \lambda(A) \text{ since } g \in \mu \text{-measure preserving.} \)

**Corollary 1.** ZF + Hahn–Banach implies not \( IM(S^2, SO_3) \). Thus, there is a non-Lebesgue measurable subset of \( S^2 \).

**Proof.** Propositions 2, 3 and Theorem 4.\)
Note that in the last part of the statement above, $S^3$ could be replaced by any other spaces, like $R^2$, $n \geq 1$. (See §3 for details.)

Corollary 2. If $H$ is generic for the partial ordering adding $\omega_1$ random reals to a model $\mathfrak{M}$ of ZFC and $V(\mathfrak{M})$ is the smallest model of set theory containing $V$ and reals of $V[\mathfrak{M}]$, then $V(\mathfrak{M})$ does not satisfy the Hahn–Banach theorem.

Proof. $V(\mathfrak{M})$ is the model considered by D. Pincus and R. Solovay in [10]. It satisfies $IM(S^n, SO_{n+1})$ for all $n \geq 1$, and thus $IM(S^5, SO_3)$ we conclude by Corollary 1.

Another way to see Corollary 1 is the following:

Corollary 3. If $F_2$ acts freely on $\Omega = S^3 \setminus D$ as in the proof of Proposition 3) by rotations, and if $\langle \mu_0; \{x\} \in \Omega \rangle F_2$ is any assignment of finitely additive probability measures $\mu_0(x)$ on $\mathcal{P}([x])$, then there are $A \subseteq F_2$ and $x \in [0, 1]$ such that $\{x; \mu_0(A \cap x) < a\}$ is not Lebesgue measurable. Further, the set $A$ can be isolated explicitly (see [14]).

§ 3. Appendix. Lebesgue measure without countable choice. Ordinarily, the theory of Lebesgue measure is developed with use of $AC_\omega$. The use of $AC_\omega$ allows one to use arbitrary Borel sets. In this section we explore how to use "coded" Borel sets to eliminate the necessity of $AC_\omega$ in many applications. For example, we would still like the existence of nonmeasurable set to be independent of the reference space (here, $S^3$). The aim of this section is to show how to adapt the proofs of the "classical" theory (with $AC_\omega$) to the study of Lebesgue measure in a totally choiceless context. The ideas here date from [13].

In order to get as many measurable as possible, the classical outer measure construction (see [4]) seems convenient enough. This construction, which we will sketch in $R$, works as well in $R^n$ or in much more abstract spaces.

Define the outer measure of $A \subseteq \mathcal{R}$ by the lower bound of all sums $\sum_{i=1}^n \lambda(I_i)$ where $I_i$ are intervals, and $A \subseteq \bigcup_{i=1}^n I_i$; call it $\mu^*(A)$. Say that $A$ is Lebesgue measurable when for all $X \subseteq \mathcal{R}$, $\mu^*(X) = \mu^*(X \cap A) + \mu^*(X \setminus A)$. Set $\mathcal{M} = \{A \subseteq \mathcal{R}; A$ is Lebesgue measurable\}, $\mu = \mu^*|\mathcal{M}$. It is still possible to prove that $\mathcal{M}$ is a Boolean subalgebra of $\mathcal{P}(\mathcal{R})$ and that $\mu$ is a finite additive function $\mathcal{M} \to [0, \infty]$, and that $\mathcal{M}$ contains all open sets. But one cannot prove any more that $\mathcal{M}$ is a $\sigma$-algebra (since $R$ can be a countable union of countable sets, see [5]). So, instead of considering Borel subsets of $\mathcal{R}$, consider those which have a code, as e.g. in [12]; a Borel code is essentially a real encoding of the "construction" of some Borel set. Similarly, say that $(A_\alpha)_\alpha$ is a coded sequence of Borel sets when there is a sequence of codes for each $n$, $c_\alpha$ is a code for $A_\alpha$. And then, we can prove the following properties of $(\mu, \mathcal{M})$: 

(a) $\mathcal{M}$ is a Boolean subalgebra of $\mathcal{P}(\mathcal{R})$, containing all coded Borel subsets of $\mathcal{R}$.
(b) $\mu$ is a finite additive map $\mathcal{M} \to [0, \infty]$, and whenever $(A_\alpha)_\alpha$ is a disjoint coded sequence of Borel sets, we have $\mu(\bigcup_{\alpha} A_\alpha) = \sum_{\alpha} \mu(A_\alpha)$.

(c) A subset $A \subseteq \mathcal{R}$ is in $\mathcal{M}$ if for all $\varepsilon > 0$ and all coded Borel $B$ with $\mu(B) < \varepsilon$, there are coded Borel $F$ and $U$ such that $F \subseteq A \setminus B \subseteq U$ and $\mu(U \setminus F) < \varepsilon$.

(Actually, it is enough to check when $B$ is the bounded interval, and $U$ can be chosen as an open set, $F$ as a closed set.)

(d) $\mu$ is $\sigma$-finite: there is a coded sequence $(A_\alpha)_\alpha$ of Borel sets such that $\mathcal{R} = \bigcup_{\alpha} A_\alpha$ and $(\forall \alpha, \beta \in \mathcal{N}) (\mu(A_\alpha) < \varepsilon)$. (Take $\alpha_0 = [0, \varepsilon].$

The precautions needed by elimination of $AC_\omega$ in the classical proof of (a) and (d) above (see [4]) make the proof somewhat more lengthy, but not really difficult. Note that in (c), the assumption $\mu(B) < \varepsilon$ does not seem to removable without countable choice.

Let us call the $\mu$ above the Lebesgue measure on $\mathcal{R}$; a similar construction yields Lebesgue measure on $R$, for all $n \geq 1$.

More generally, let us set the following definition:

Definition. A coded Borel space is an ordered pair $(\Omega, \mathfrak{B})$ where $\Omega$ is a coded Borel subset of the Hilbert cube $[0, 1]^\omega$ and $\mathfrak{B}$ is the algebra of coded Borel subsets of $\Omega$.

We can naturally extend this definition by taking all isomorphic images; this way, all usual spaces of analysis, like $R^n$, $S^1$, or $S^2$, together with their coded Borel subsets, become coded Borel spaces. Anyway, even without using countable choice, it turns out that the following is true:

Proposition 5. Let $(\Omega, \mathfrak{B})$ be an uncountable coded Borel space. Then there is a coded Borel isomorphism from $(\Omega, \mathfrak{B})$ onto $(I, \mathfrak{B}_I)$, where $I = [0, 1]$ and $\mathfrak{B}_I$ is the algebra of coded Borel subsets of $I$.

Here, a coded Borel isomorphism $(\Omega, \mathfrak{B}) \to (I, \mathfrak{B}_I)$ is naturally a bijection $\beta: \Omega \to I$ such that the neighborhood diagrams of $\beta$ and $\beta^{-1}$ are coded Borel.

Now, let us give the new definition of measure we are going to use:

Definition. Let $(\Omega, \mathfrak{B})$ be a coded Borel space. A regular measure on $(\Omega, \mathfrak{B})$ is a map $\mu: \mathcal{M} \to [0, \infty]$ such that $\mu(A)$ satisfies conditions (a) to (d) above, with $\mathcal{M}$ instead of $\mathcal{R}$. Say that $\mu$ is nonatomic when $(\forall x \in D)(\mu(x)) = 0$.

The essential isomorphism theorem between these measure spaces is still valid (after a suitable reformulation). It can be stated the following way:

Proposition 6. Let $\mu$ be a regular, nonatomic measure on a coded Borel space $(\Omega, \mathfrak{B})$, with $\mu(D) = 1$. Then there are $N \subseteq \Omega, D \subseteq [0, 1]$ and $\mu: \Omega \to [0, 1]$ such that, if $I$ is Lebesgue measure on $[0, 1]$,

(i) $N \in \mathfrak{B}$, $D$ is countable, $\mu(N) = 0$, $\mu(D) = 1$.
(ii) $f$ is a coded Borel isomorphism $\Omega, N \to [0, 1]$.
(iii) For all $B$ in $\mathfrak{B}$, $f(B)$ is coded Borel in $[0, 1]$ and $\mu(B) = \mu(f[B])$.

Outline of proof (see [11]). First, notice that by (b) and $\mu(D) = 1$, $\Omega$ is uncountable. So, by Proposition 5, without loss of generality, $\Omega = [0, 1]$ and $\mathfrak{B}$ is the algebra of coded Borel subsets of $(0, 1]$. Then define $f: [0, 1] \to [0, 1]$ by $f(x) = \mu(D, x)$. Then $D$ is just $\{x \in [0, 1]: f^{-1}(y)$ has nonempty interior $\}$ and $N$ is $f^{-1}(D)$, (iii) is proven by induction on a code of $B$, and it uses nonatomicity of $\mu$. 

2 — Fundamenta Mathematicae 138.1
Now, Proposition 6 has an immediate corollary:

**Corollary 1.** Let $\mu$ be a regular, nonatomic measure on a coded Borel space $(\Omega, \mathcal{B})$, with $\mu(0) \neq 0$. Then the following are equivalent:

(i) Every subset of $\Omega$ is $\mu$-measurable.
(ii) Every subset of $[0, 1]$ is Lebesgue measurable.

(To prove (i)$\Rightarrow$(ii), one has to use $\sigma$-finiteness, nonatomicity of $\mu$ and $\mu(0) \neq 0$; for (ii)$\Rightarrow$(i), use characterization (c) above of $\mu$-measurability.)

In particular, every subset of $\mathbb{R}^n$ ($n \geq 1$) is Lebesgue measurable if every subset of $[0, 1]$ is Lebesgue measurable (which is well known in the classical theory using countable choice). Let $LM$ be the latter statement.

Now, define Lebesgue measure $v_n$ on $S^n$ as being the image under $x \mapsto x/\|x\|$ of Lebesgue measure on $\mathbb{R}^{n+1}\setminus\{0\}$, where $\mathbb{R}^{n+1}$ is the Euclidean closed ball of $\mathbb{R}^{n+1}$ of volume 1.

**Corollary 2.** $LM$ implies $IM(S^n, SO_{n+1})$ for all $n \geq 1$.

**Proof.** If $LM$ holds, then $v_n$ is defined on $\mathcal{B}(S^n)$ by the previous corollary; so $v_n$ witnesses $IM(S^n, SO_{n+1})$.

More precisely, the result would be the same with a rotation invariant extension of Lebesgue measure on $\mathcal{B}(S^n)$; thus, the results of the previous section imply for example that the Hahn–Banach theorem implies nonexistence of a rotation invariant extension of Lebesgue measure of a (finitely additive) measure on $\mathcal{B}(\mathbb{R}^n)$.

Further notes. Theorem 4 could be formulated as follows: If $G$ is a nonamenable group acting freely on a set $\Omega$ and if $\mu$ is a $G$-invariant finitely additive probability measure defined on a $G$-invariant subalgebra of $\mathcal{P}(\Omega)$, then $\Omega$ has nonmeasurable subsets (w.r.t. $\mu$).

Now, while this paper was printed, the second author showed, under the same hypotheses, that in the G-equidecomposability type semigroup of $\Omega$ (see [14]), $n[\omega] = (n+1)[\omega]$ for some integer $n$, effectively computable from the number of pieces necessary to a paradoxical decomposition of $G$. For the action of $F_2$ described above, we can get $n = 5$, which is somewhat disappointing since it is not known whether the cancellation law (see [14]) follows from HB (it follows from BPI). But independently, J. Pawlikowski proved, using ideas from this paper, that one can actually take $n = 1$, that is, $[\omega] = 2[\omega]$; thus, HB implies the Banach–Tarski paradox. See [3] for more details.

**References**


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