Scorza Dragoni type theorems

by

Anna Kucia (Katowice)

Abstract. We give a new proof of the Scorza Dragoni type theorem for functions as well as for set-valued functions. Our method of proof also gives the Baire version of the Scorza Dragoni type theorem.

1. Introduction. Scorza Dragoni [15] proved that if $f: [a, b]\times[c, d]\to R$ is Lebesgue measurable in the first variable and continuous in the second one, then for each $\varepsilon > 0$ there exists a compact set $K\subset[a, b]$ such that the Lebesgue measure of $[a, b]\setminus K$ is less than ε, and the restriction $f|_{K\times[c,d]}$ is a continuous function.

There are many generalizations of this theorem. Usually their proofs are based on Lusin type theorems for functions or set-valued functions (cf. [10]). Our method of proof is similar to the proof of the Baire version of the Lusin theorem (cf. [12; §32.II]).

2. Definitions and some elementary properties. Let X be a topological space. We denote by $\mathscr{C}(X)$ the σ-algebra of all subsets of X which have the Baire property, and by $\mathscr{B}(X)$ the σ-algebra of Borel sets.

Let $(T, \mathfrak{M})$ be a measurable space. We denote the vertical t-section of $B\subset T\times X$ by $B_t = \{x\in X: (t, x)\in B\}$, $t\in T$, and the horizontal x-section by $B^x = \{t\in T: (t, x)\in B\}$, $x\in X$. Let p denote the projection from $T\times X$ onto T, and $\mathfrak{M}\otimes\mathscr{B}(X)$ the product σ-algebra on $T\times X$. We say that $(T, \mathfrak{M}; X)$ has the *projection property* if for each $A\in\mathfrak{M}\otimes\mathscr{B}(X)$ its projection $p(A)$ belongs to $\mathfrak{M}$. In the case of a separable space X and a set with open vertical sections no projection property is needed. More precisely, for each $A\subset T\times X$ such that A_t is open for all $t\in T$ and $A^x\in\mathfrak{M}$ for x in some countable dense set $D\subset X$, the projection $p(A)$ belongs to $\mathfrak{M}$, because $p(A) = \bigcup_{x\in D} A^x$.

By a *set-valued function* from a set Z to Y we mean a function F defined on Z whose values are subsets (possibly empty) of Y. For $A\subset Y$ we put $F^{-1}(A) = \{z\in Z: F(z)\cap A\neq\emptyset\}$. By the domain of F we mean the set $\operatorname{dom} F = \{z\in Z: F(z)\neq\emptyset\} = F^{-1}(Y)$, and F is called a *multifunction* if $\operatorname{dom} F = Z$. The graph of F is defined as $\operatorname{Gr} F = \{(z, y)\in Z\times Y: y\in F(z)\}$. A set-valued function F from X to a topological space Y is called *lower-semicontinuous* (*upper-semicontinuous*), abbreviated l.s.c. (u.s.c), if $F^{-1}(A)\subset X$ is open (closed) whenever $A\subset Y$ is open (closed). Observe that $\operatorname{dom} F$ is open whenever F is l.s.c.

A set-valued function F from $(T, \mathfrak{M})$ to Y is called *$\mathfrak{M}$-measurable*, or simply *measurable*, if $F^{-1}(A)$ belongs to $\mathfrak{M}$ for each open $A \subset Y$. Note that if F is measurable then $\operatorname{dom} F \in \mathfrak{M}$. We say that a function or a set-valued function from X to Y has the *Baire property* if it is $\mathscr{C}(X)$-measurable.

We say that a function or a set-valued function F defined on $S \subset T$ is $\mathfrak{M}$-measurable or simply measurable, if it is measurable with respect to the trace σ-algebra $\mathfrak{M}|_S$. In the case of $S \in \mathfrak{M}$ this means that $F^{-1}(A) \in \mathfrak{M}$ for each open A. In the same way we mean the $\mathfrak{M} \otimes \mathscr{B}(X)$-measurability of F defined on $B \subset T \times X$.

The following lemma is a slight generalization of known results (cf. [12; Th. 31.V.2], [3], [13], [9]).

LEMMA. *Let $(T, \mathfrak{M})$ be a measurable space, X a second-countable topological space and Y perfectly normal. Suppose $B \subset T \times X$ has open vertical sections and $B^x \in \mathfrak{M}$ for x in some countable dense set $D \subset X$. If a function $f\colon B \to Y$ is continuous in the second variable and $f(\cdot, x)$ is measurable for $x \in D$, then f is $\mathfrak{M} \otimes \mathscr{B}(X)$-measurable.*

Proof. Let $\{U_n\colon n \in N\}$ be a base in X and let $F \subset Y$ be closed. Take a sequence of closed subsets $F_n \subset Y$ such that $F = \bigcap_{n\in N} F_n$ and $F \subset \operatorname{int} F_n$ for each $n \in N$. Put $M_{nk} = \{t \in T\colon U_n \cap B_t \subset f(t, \cdot)^{-1}(F_k)\}$. The set $(T \times U_n) \cap B \backslash f^{-1}(F_k)$ has open vertical sections and its horizontal x-sections belong to $\mathfrak{M}$ for $x \in D$. Hence M_{nk} belongs to $\mathfrak{M}$, because $T \backslash M_{nk} = p((T \times U_n) \cap B \backslash f^{-1}(F_k))$ belongs to $\mathfrak{M}$. It is easy to check that $f^{-1}(F) = B \cap \bigcap_{k\in N} \bigcup_{n\in N} M_{nk} \times U_n$.

Let F be a set-valued function with values in a separable metric space (Y, d). It is known that some measurability and continuity properties of F follow from or are equivalent to suitable properties of the distance functions $d(y, F(\cdot))$, $y \in Y$; here we put $d(y, \emptyset) = \infty$. In these relationships it suffices to regard only a countable family of such functions, namely $\{d(y, F(\cdot))\colon y \in D\}$, where $D \subset Y$ is countable dense. A set-valued function F is called *d-continuous* if $d(y, F(\cdot))$ is continuous for each $y \in Y$ or equivalently, for each $y \in D$.

3. Scorza Dragoni type theorems. First we prove the general theorem under assumption of the projection property.

THEOREM 1. *Let T be a topological (Hausdorff) space, $\mathfrak{M}$ a σ-algebra on T, and m a measure on $\mathfrak{M}$ such that for each $M \in \mathfrak{M}$ and $\varepsilon > 0$ there exists a closed (compact) set $K \in \mathfrak{M}$ such that $K \subset M$ and $m(M \backslash K) < \varepsilon$. Assume that X and Y are second-countable topological spaces, $(T, \mathfrak{M}; X)$ has the projection property and $B \in \mathfrak{M} \otimes \mathscr{B}(X)$. Let f be a function or a multifunction from B to Y which is $\mathfrak{M} \otimes \mathscr{B}(X)$-measurable. Assume that all $f(t, \cdot)$, $t \in T$, satisfy one of the following continuity conditions:*

(a) *the function $f(t, \cdot)$ is continuous,*
(b) *$Y = \mathbf{R}$ and the real-valued function $f(t, \cdot)$ is lower-semicontinuous,*
(c) *$Y = \mathbf{R}$ and the real-valued function $f(t, \cdot)$ is upper-semicontinuous,*
(d) *the multifunction $f(t, \cdot)$ is l.s.c.,*
(e) *(Y, d) is a metric space and the multifunction $f(t, \cdot)$ is d-continuous.*

Then for each $\varepsilon > 0$ there exists a closed (compact) set $K \subset T$ such that $m(T \backslash K) < \varepsilon$, and the restriction $f|_{(K\times X)\cap B}$ has the same continuity property as $f(t, \cdot)$.

Proof. In the case of $B = T \times X$ the idea of the proof is simple enough. Namely, any set $A \in \mathfrak{M} \otimes \mathscr{B}(X)$ with open vertical sections is equal to $\bigcup_{n\in N} M_n(A) \times U_n$, where $M_n(A) \in \mathfrak{M}$ and $\{U_n\colon n \in N\}$ is a base in X. Having a countable family $\mathscr{A}$ of such sets A one can take a closed (compact) set $K \subset T$ such that $m(T \backslash K) < \varepsilon$ and $M_n(A) \cap K$ is open in K for each $n \in N$ and $A \in \mathscr{A}$. If $\mathscr{A} = \{f^{-1}(V_n)\colon n \in N\}$, where $\{V_n\colon n \in N\}$ is a base in Y, then all sets $f^{-1}(V_n) \cap (K \times X)$ are open in $K \times X$.

Now we prove the theorem in detail. Assume that a set $A \subset B$ belonging to $\mathfrak{M} \otimes \mathscr{B}(X)$ is such that A_t is open in B_t for each $t \in T$. If $U \subset X$ is open, then $M(A) = \{t \in T\colon B_t \cap U \subset A_t\}$ belongs to $\mathfrak{M}$. In fact,

$$T \backslash M(A) = \{t \in T\colon B_t \cap U \backslash A_t \neq \emptyset\} = p((T \times U) \cap B \backslash A).$$

Of course, $(M(A) \times U) \cap B \subset A$. Let $\{U_n\colon n \in N\}$ be a base in X. Put $M_n(A) = \{t \in T\colon B_t \cap U_n \subset A_t\}$. We obtain $B \cap \bigcup_{n\in N} M_n(A) \times U_n \subset A$. The converse inclusion is also true, because A_t is open in B_t and $\{U_n \cap B_t\colon n \in N\}$ is a base in B_t, $t \in T$. Hence $A = B \cap \bigcup_{n\in N} M_n(A) \times U_n$.

Let $\mathscr{A}$ be a countable family of sets A as above. Enumerate $\{M_n(A)\colon A \in \mathscr{A}, n \in N\}$ in a single sequence $\{M_n\colon n \in N\}$. Observe that for each $M \in \mathfrak{M}$ and $\varepsilon > 0$ there exists a closed (compact) set $F \in \mathfrak{M}$ such that $m(T \backslash F) < \varepsilon$ and $M \cap F$ is open in F. Indeed, let $F_1 \subset M$ and $F_2 \subset T \backslash M$ be closed (compact) and such that $m(M \backslash F_1) < \varepsilon/2$ and $m((T \backslash M) \backslash F_2) < \varepsilon/2$. Put $F = F_1 \cup F_2$. The sets F_1 and F_2 are closed (T being Hausdorff) and disjoint, so they are open in F. Hence, $M \cap F = F_1$ is open in F. Now let $K_n \in \mathfrak{M}$ be closed (compact) such that $m(T \backslash K_n) < \varepsilon/2^n$ and $M_n \cap K_n$ is open in K_n, $n \in \mathbf{N}$. Put $K = \bigcap_{n\in N} K_n$. Of course $M_n \cap K$ is open in K for each $n \in N$. Hence, $A \cap (K \times X) = B \cap \bigcup_{n\in N} (M_n(A) \cap K) \times U_n$ is open in $B \cap (K \times X)$ for each $A \in \mathscr{A}$.

In case (a) or (d) we take $\mathscr{A} = \{f^{-1}(V_n)\colon n \in N\}$, where $\{V_n\colon n \in N\}$ is a base in Y. Then $f|_{(K\times X)\cap B}$ is a continuous function under assumption (a), and a l.s.c. multifunction under assumption (d).

In case (b) or (c) we take, respectively, $\mathscr{A} = \{f^{-1}((q, \infty))\colon q \in \mathbf{R} \text{ is rational}\}$ or $\mathscr{A} = \{f^{-1}((-\infty, q))\colon q \in \mathbf{R} \text{ is rational}\}$.

In case (e) we take $\mathscr{A} = \{f_n^{-1}(V_k)\colon k, n \in N\}$, where $\{V_k\colon k \in N\}$ is a base in $[0, \infty)$, $\{y_n\colon n \in N\}$ is dense in Y and $f_n(t, x) = d(y_n, f(t, x))$.

THEOREM 1′ (the Baire version of the Scorza Dragoni theorem). *Assume that X and Y are second-countable topological spaces, T is a topological space, $(T, \mathscr{C}(T); X)$ has the projection property, and $B \in \mathscr{C}(T) \otimes \mathscr{B}(X)$. Let f be a function or a multifunction from B to Y which is $\mathscr{C}(T) \otimes \mathscr{B}(X)$-measurable and such that all $f(t, \cdot)$ satisfy one of the continuity conditions from Theorem 1. Then there exists a comeager set $K \subset T$ such that $f|_{(K\times X)\cap B}$ has the same continuity property as $f(t, \cdot)$.*

Proof. The proof differs from the previous one only in choosing a set K for the family $\{M_n\colon n \in N\}$ of sets from $\mathfrak{M} = \mathscr{C}(T)$. Each set M_n is equal to $(W_n \cup P_n) \backslash R_n$, where

W_n is open and P_n, R_n are meager sets in T. Put $P = \bigcup_{n\in N} P_n \cup \bigcup_{n\in N} R_n$ and $K = T\setminus P$. Since $M_n \cap K = W_n \cap K$, the set $M_n \cap K$ is open in K, $n\in N$. The rest of the proof remains without change.

In the case of continuous functions or d-continuous multifunctions and a metrizable space Y, the assumption of the projection property is superfluous. Namely, we have the following theorem (cf. [3; Th. I.1.1]).

THEOREM 2. *Let $(T, \mathfrak{M}, m)$ be such as in Theorem 1. Let X be a second-countable topological space and Y separable and metrizable. Assume that $B \subset T\times X$ has open vertical sections and $B^x \in \mathfrak{M}$ for x in some countable dense $D \subset X$. Let f be a function or multifunction from B to Y such that $f(\cdot, x)$ is measurable for $x\in D$ and $f(t, \cdot)$ is a continuous function or a d-continuous multifunction, where d is a metric in Y, $t\in T$. Then there exists a closed (compact) set $K \subset T$ such that $m(T\setminus K) < \varepsilon$ and $f|_{(K\times X)\cap B}$ is a continuous function or, respectively, a d-continuous multifunction.*

Proof. The proof is a combination of the previous one and that of the Lemma. We start with the case when f is a function. Let $\{U_n: n\in N\}$ be a base in X. Now for open $V\subset Y$ we define $M_n(V) = \{t\in T: B_t \cap U_n \subset f(t, \cdot)^{-1}(\mathrm{cl}\, V)\}$. The set $(T\times U_n) \cap B\setminus f^{-1}(\mathrm{cl}\, V)$ has open vertical sections and measurable x-sections for $x\in D$. Hence $M_n(V)$ belongs to $\mathfrak{M}$, because $T\setminus M_n(V) = p(T\times U_n \cap B\setminus f^{-1}(\mathrm{cl}\, V))\in \mathfrak{M}$. Moreover, we obtain

$$f^{-1}(V) \subset B \cap \bigcup_{n\in N} M_n(V)\times U_n \subset f^{-1}(\mathrm{cl}\, V).$$

Let $\mathscr{P} = \{V_n: n\in N\}$ be a countable base in Y. For every open $W\subset Y$ there exists a subfamily $\mathscr{R} \subset \mathscr{P}$ such that $W = \bigcup \mathscr{R}$ and $\mathrm{cl}\, V \subset W$ for $V\in\mathscr{R}$. Hence $f^{-1}(W)$ is equal to $B\cap \bigcup_{V\in\mathscr{R}} \bigcup_{n\in N} M_n(V)\times U_n$. Since $\{M_n(V_m): n, m\in N\}$ is countable, as in the proof of Theorem 1 we can take for each $\varepsilon > 0$ a closed (compact) set $K \subset T$ such that all sets $M_n(V_m)\cap K$ are open in K. Then $f|_{(K\times X)\cap B}$ is continuous.

In the case of a d-continuous multifunction f, we consider the family $\{M_{nk}(V_m): m, n, k\in N\}$, where $\{V_m: m\in N\}$ is a base in $[0, \infty)$, $\{y_k: k\in N\}$ is countable dense in Y, and $M_{nk}(V)$ is defined as above for the function f_k, $f_k(t, x) = d(y_k, f(t, x))$.

In the similar way one can obtain Theorem 2′, the Baire version of Theorem 2. Since its formulation and proof are obvious, we omit them. Instead, let us give the following more interesting corollary to Theorem 2′.

COROLLARY. *Let T be a topological space, X a second-countable topological space and Y separable and metrizable. Let a function f: $T\times X\to Y$ be such that $f(t, \cdot)$ is continuous for each $t\in T$, and $f(\cdot, x)$ has the Baire property for x in some countable dense set $D \subset X$. Then there exists a comeager set $K \subset T$ such that $f|_{K\times X}$ is continuous, and $f(\cdot, x)$ has the Baire property for each $x\in X$.*

4. Comments and remarks. First we compare the assumptions of the theorems of the previous section with familiar results (cf. [16]).

(1) If $\mathfrak{M}$ is closed under the Suslin operation and X is a weakly Suslin space, i.e., a continuous image of a Polish space, then $(T, \mathfrak{M}; X)$ has the projection property.

(2) $\mathfrak{M}$ is closed under the Suslin operation provided one of the following conditions is satisfied:

(i) $\mathfrak{M}$ is complete with respect to a σ-finite measure,
(ii) $\mathfrak{M}$ is the family of all m-measurable sets, where m is an outer measure on T,
(iii) $\mathfrak{M}$ is the σ-algebra of universally measurable sets w.r.t. some σ-algebra on T,
(iv) T is a topological space and $\mathfrak{M} = \mathscr{C}(T)$.

(3) The assumption on the approximation of the measure by closed (compact) sets is satisfied, for instance, if m is a finite regular (Radon) Borel measure on T and $\mathscr{B}(T) \subset \mathfrak{M} \subset \mathscr{B}_m(T)$, where $\mathscr{B}_m(T)$ denotes the completion of $\mathscr{B}(T)$ w.r.t. the measure m. Recall that every finite Borel measure on a metrizable (Polish) space is regular (Radon). The approximation by closed sets also holds in the case of a locally finite Borel measure on a separable and metrizable space T.

Some assumptions in the Scorza Dragoni type theorems can be relaxed:

(4) It is enough to assume the continuity conditions of $f(t, \cdot)$ only for almost all $t\in T$ or all t except some meager set in the Baire version.

(5) If $\mathfrak{M}$ is closed under the Suslin operation and X is weakly Suslin, hence in most interesting cases (cf. (1)–(3)), instead of the requirement $B\in \mathfrak{M}\otimes\mathscr{B}(X)$ in Theorems 1 and 1′ one can assume that B is obtained from $\mathfrak{M}\otimes\mathscr{B}(X)$ by the Suslin operation. The proof is the same.

(6) In the Lemma as well as in Theorems 2 and 2′, instead of the assumptions "B has open vertical sections, $B^x\in\mathfrak{M}$ and $f(\cdot, x)$ is measurable for $x\in D$" one can assume that there exists a countable family of measurable functions p_n: $T\to X$ such that $\{p_n(t): n\in N\} \subset B_t \subset \mathrm{cl}\{p_n(t): n\in N\}$ for each $t\in T$, and the function $t\to f(t, p_n(t))$ is measurable for each $n\in N$. The proofs are the same (in this case $p(T\times U\cap B\setminus f^{-1}(F)) = \bigcup_{n\in N} p_n^{-1}(U)\cap\{t: f(t, p_n(t))\notin F\}$, for open U and closed F).

(7) Let Y be metrizable and $B = T\times X$. Instead of the separability of Y, it is enough to assume that there exists a countable dense subset $D = \{x_n: n\in N\} \subset X$ and a sequence of sets $A_n\in\mathfrak{M}$ with $m(A_n) = 0$ (or meager sets $A_n \subset T$ in the Baire version) such that $f((T\setminus A_n)\times\{x_n\})$ are separable. In fact, if $A = \bigcup_{n\in N} A_n$ and f is a function continuous in x or a multifunction l.s.c. in x, then $f((T\setminus A)\times X) \subset \mathrm{cl}\, f((T\setminus A)\times D)$ and, consequently, $f((T\setminus A)\times X)$ is separable. Hence, for functions continuous in x and for σ-finite measures, it suffices to assume that the weight of Y is less than the first real-valued measurable cardinal (cf. [7; 2.3.6]). Moreover, if m is a finite Radon measure and f is a function or a compact-valued multifunction, then Y is allowed to be an arbitrary metrizable space (cf. [8] for functions and [11] for multifunctions).

(8) Theorems 1 and 1′ for semicontinuous real-valued functions also hold if we admit the values $-\infty$ and $+\infty$. Similarly, our results for multifunctions remain true for set-valued functions. In fact, we need not admit empty values, because we regard multifunctions defined on subsets. Moreover, this point of view gives sometimes stronger results. For example, the assumption "f is a set-valued function from $T\times X$

which is l.s.c. in x" (cf. [1; Th. 2.1]) implies that the sets B_t are open in X, where $B = \operatorname{dom} f$.

(9) The assumption of the second-countability of X in the above theorems cannot be dropped, even if X is normal and a Lusin space (i.e., a continuous and one-to-one image of a Polish space) and $(T, \mathfrak{M}, m)$ is a complete measure space (cf. (1) and (2i)). Namely, we have the following example.

Example. Let X be the set $C(I)$ of all real-valued continuous functions on the interval $I = [0, 1]$, endowed with the topology of pointwise convergence, $T = I$, $\mathfrak{M}$ the σ-algebra of Lebesgue measurable sets and m the Lebesgue measure on $\mathfrak{M}$. Let $f\colon T\times X \to \boldsymbol{R}$ be the evaluation function, i.e., $f(t, x) = x(t)$. The function f is continuous in each variable separately. Moreover, f is Borel measurable. This follows for instance from the Lemma, where as the measurable space we take $(X, \mathscr{B}(X))$. Let $F \subset I$ be of positive measure or a comeager set. There exists a decreasing sequence (t_n) of points of F which converges to $t_0 \in F$. Let $x_n \in X$ be defined by $x_n(t) = 0$ if $t \leqslant t_0$ or $t \geqslant t_n$, $x_n(t_{n+1}) = 1$ and x_n is linear on the intervals $[t_0, t_{n+1}]$ and $[t_{n+1}, t_n]$. The sequence (x_n) converges to $x_0 = 0$ in X. On the other hand, we have $\lim_{n\to\infty} f(t_{n+1}, x_n) = 1 \neq f(t_0, x_0) = 0$. Thus $f|_{F\times X}$ is not continuous for each $F \subset I$ of positive measure or comeager. Observe that (x_n) converges to x_0 also in the weak topology of $C(I)$. Since the weak topology of $C(I)$ is stronger than the topology of pointwise convergence, one can take instead of X the space $C(I)$ with the weak topology. Of course, $C(I)$ with each of these topologies is normal and a Lusin space. Another example of this sort is given in [2; Es. 2]. But in that example X is not Suslin (X is the Sorgenfrey line).

(10) Recall for completeness that condition (d) in Theorems 1 and 1′ cannot be replaced by upper-semicontinuity or by continuity (i.e., l.s.c. and u.s.c.), because the corresponding Lusin type theorem does not hold (cf. [14; Ex. 14]).

Some interesting results can be derived from the Scorza Dragoni type theorems.

(11) In our proof we do not use the classical Lusin theorem. It follows from Theorem 1 by taking one point X. Also the Lusin type theorem for set-valued functions (cf. [10] and [4]) can be derived from Theorem 2. As an example we give the Baire version of this result.

Corollary. *Let T be a topological space and F a set-valued function from T to a separable and metrizable space X. If F has the Baire property, then there exists a comeager set $K \subset T$ such that $F|_K$ is d-continuous, where d is a metric on X. In particular, if F has closed values, then $F|_K$ is l.s.c. and has a closed graph.*

Proof. Let $f\colon T\times X \to Y = [0, \infty]$ be defined by $f(t, x) = d(x, F(t))$. By Theorem 2′, the restriction $f|_{K\times X}$ is continuous for some comeager set $K \subset T$. Since $f(\cdot, x)$ is upper-semicontinuous on K, the restriction $F|_K$ is l.s.c. Since F has closed values and $f|_{K\times X}$ is continuous, $\operatorname{Gr}(F|_K)$ is closed.

(12) The Scorza Dragoni type theorems for multifunctions enable us to prove some results on the existence of Carathéodory selections (cf. [5] and [1]). Also one can obtain some theorems on the extending of a Carathéodory function $f\colon B \to Y$, where $B \subset T\times X$, by reducing this problem to the problem of finding a Carathéodory selection for the multifunction F defined by $F(t, x) = \{f(t, x)\}$ for $(t, x) \in B$, and $F(t, x) = Y$ for $(t, x) \in B$ (cf. also [6]).

References

[1] Z. Artstein and K. Prikry, *Carathéodory selections and the Scorza Dragoni property*, J. Math. Anal. Appl. 127 (1987), 540–547.

[2] D. Averna and A. Fiacca, *Sulla proprietà di Scorza Dragoni*, Atti Sem. Mat. Fis. Univ. Modena 33 (1984), 313–318.

[3] H. Berliocchi et J. M. Lasry, *Intégrandes normales et mesures paramétrées en calcul des variations*, Bull. Soc. Math. France 101 (1973), 129–184.

[4] C. Castaing, *Une nouvelle extension du théorème de Dragoni-Scorza*, C. R. Acad. Sci. Paris Sér. A 271 (1970), 396–398.

[5] —, *A propos de l'existence des sections séparément mesurables et séparément continues d'une multiapplication séparément mesurable et séparément semi-continue inférieurement*, Séminaire d'Analyse Convexe, Montpellier 1976, Exposé n° 6.

[6] F. S. De Blasi and J. Myjak, *On the random Dugundji extension theorem*, J. Math. Anal. Appl. 128 (1987), 305–311.

[7] H. Federer, *Geometric Measure Theory*, Springer-Verlag, New York–Berlin 1969.

[8] D. H. Fremlin, *Measurable functions and almost continuous functions*, Manuscripta Math. 33 (1981), 387–405.

[9] C. J. Himmelberg, *Measurable relations*, Fund. Math. 87 (1975), 53–72.

[10] M. Q. Jacobs, *Measurable multivalued mappings and Lusin's theorem*, Trans. Amer. Math. Soc. 134 (1968), 471–481.

[11] G. Koumoullis and K. Prikry, *The Ramsey property and measurable selections*, J. London Math. Soc. (2) 28 (1983), 203–210.

[12] K. Kuratowski, *Topology*, Vol. I, Pergamon and PWN, New York–Warszawa 1966.

[13] S. J. Leese, *Set-valued functions and selectors*, Thesis, Keele, England, 1974.

[14] S. Łojasiewicz, Jr., *Some theorems of Scorza Dragoni type for multifunctions with application to the problem of existence of solutions for differential multivalued equations*, in: *Mathematical Control Theory*, Banach Center Publ. 14, PWN–Polish Scientific Publishers, Warszawa 1985, 625–643.

[15] G. Scorza Dragoni, *Un teorema sulle funzioni continue rispetto ad una e misurabili rispetto ad un'altra variabile*, Rend. Sem. Mat. Univ. Padova 17 (1948), 102–106.

[16] D. H. Wagner, *Survey of measurable selection theorems*, SIAM J. Control Optim. 15 (1977), 859–903.

INSTITUTE OF MATHEMATICS
SILESIAN UNIVERSITY
Bankowa 14
40-007 Katowice
Poland

Received 20 April 1989;
in revised form 1 August 1990