Power stability of k-spaces and compactness

by

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Abstract. It is proved that a topological space X is compact if $X^{\mathfrak{m}}$ is a k-space for each cardinal number $\mathfrak{m}$.

0. Introduction. A subset A of a topological space is called *k-closed* if $A \cap K$ is a closed subset of K for each compact subset K of X. X is called a *k-space* if each k-closed subset is closed. k-Spaces are the quotients of locally compact spaces [1]. The product of two k-spaces need not be a k-space. However, if one of the factors is locally compact then the product is also a k-space [1]. This is the best possible result, since if X is not locally compact, one can find a k-space Y such that $X \times Y$ is not a k-space [2].

We consider the following questions: What can we say about a family of topological spaces when their product is a k-space, and which topological spaces have powers so that all of them are k-spaces? It turns out that if $\prod \mathscr{F}$ is a k-space then a large part of the family $\mathscr{F}$ must be a certain type of compact spaces and if $X^{\mathfrak{m}}$ is a k-space for every cardinal number $\mathfrak{m}$ then X is a compact space.

A topological space X is called *$(\mathfrak{m}, \mathfrak{n})$-compact* if every open cover of X of cardinality $\leqslant \mathfrak{m}$ has a subcover of cardinality $< \mathfrak{n}$.

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1. Family of topological spaces whose product is a k-space. The following theorem is a generalization of Proposition 5.5 in [3].

THEOREM 1. *Let $\mathscr{F}$ be a family of topological spaces and let $\mathfrak{m}$ be a cardinal number. If $\operatorname{card} \mathscr{F} \geqslant 2^{\mathfrak{m}}$ and each $X \in \mathscr{F}$ is not $(\mathfrak{m}, \aleph_0)$-compact but $(\mathfrak{n}, \aleph_0)$-compact for each $\mathfrak{n} < \mathfrak{m}$, then $\prod \mathscr{F}$ is not a k-space.*

Proof. Let $\mathscr{U}$ be a free ultrafilter on $(0, \mathfrak{m})$ such that $\sup A = \mathfrak{m}$ for each $A \in \mathscr{U}$. Choose a subfamily $\mathscr{H}$ of $\mathscr{F}$ with $\operatorname{card} \mathscr{H} = 2^{\mathfrak{m}}$. We can index $\mathscr{H}$ by the elements of $\mathscr{U}$, $\mathscr{H} = \{X_M \mid M \in \mathscr{U}\}$. Since X_M is not $(\mathfrak{m}, \aleph_0)$-compact but $(\mathfrak{l}, \aleph_0)$-compact for each $\mathfrak{l} < \mathfrak{m}$, there is a family of closed subsets $(F_{M,\alpha})_{\alpha<\mathfrak{m}}$ of X_M for each $M \in \mathscr{U}$ such that $\emptyset \neq F_{M,\alpha} \subsetneq F_{M,\beta}$ for each $\beta < \alpha < m$ and $\bigcap_{\alpha<\mathfrak{m}} F_{M,\alpha} = \emptyset$. Choose $w_{M,\alpha} \in F_{M,\alpha} \setminus F_{M,\alpha+1}$ and define $\varphi_\alpha \in \prod \mathscr{H}$ for $0 \leqslant \alpha < \mathfrak{m}$ by

$$\varphi_\alpha(M) = \begin{cases} w_{M,\alpha} & \text{if } \alpha \notin M, \\ w_{M,0} & \text{if } \alpha \in M, \end{cases} \quad \text{for each } M \in \mathscr{U}.$$

We will prove that $\varphi_0 \in \overline{\{\varphi_\alpha \mid 0 < \alpha < m\}}$ and $\overline{\{\varphi_\alpha \mid 1 \leqslant \alpha < m\}} \setminus [\varphi_0] \cap K$ is closed for each compact subset K of $\prod \mathscr{H}$. This means that the quotient space $\prod \mathscr{H}$ of $\prod \mathscr{F}$ is not a k-space. Hence $\prod \mathscr{F}$ is not a k-space. We shall give the proof in several steps. We set

$$A_\alpha = \{\varphi_\beta \mid 0 < \beta < \alpha\} \quad \text{and} \quad B_\alpha = \{\varphi_\beta \mid \alpha \leqslant \beta < \mathfrak{m}\}.$$

We consider the following statements:

(i) $\varphi_0 \notin \bar{A}_\alpha$ for each $\alpha < \mathfrak{m}$.

(ii) $\varphi_0 \in \bar{B}_\alpha$ for each $\alpha < \mathfrak{m}$.

(iii) If $\Theta \in \bar{B}_\alpha$ and $\Theta \neq \varphi_0$ then there is $\beta < \mathfrak{m}$ such that $\Theta \in \overline{A_\beta \cap B_\alpha}$.

(iv) For each compact subset K of $\prod \mathscr{H}$ there is $\alpha < \mathfrak{m}$ such that $K \cap \bar{B}_\alpha \setminus \{\varphi_0\} = \emptyset$.

Proof of these statements. (i) Let $\alpha < \mathfrak{m}$ be given. Since $\mathscr{U}$ is an ultrafilter we have either $[1, \alpha) \in \mathscr{U}$ or $[\alpha, \mathfrak{m}) \in \mathscr{U}$. But $[1, \alpha) \notin \mathscr{U}$ because of $\sup[1, \alpha) = \alpha < \mathfrak{m}$. So $[\alpha, \mathfrak{m}) \in \mathscr{U}$. Set $M = [\alpha, \mathfrak{m})$. Then $\varphi_\beta(M) = w_{M,\beta}$ for each $1 \leqslant \beta < \alpha$. Hence $P_M(A_\alpha) = \{\varphi_\beta(M) \mid 1 \leqslant \beta < \alpha\} \subset F_{M,1}$. We have

$$P_M(\bar{A}_\alpha) \subset \overline{P_M(A_\alpha)} \subset F_{M,1}$$

and $P_M \varphi_0 = w_{M,0} \in F_{M,0} \setminus F_{M,1}$. So $\varphi_0 \notin \bar{A}_\alpha$.

(ii) Let $M_1, \ldots, M_n \in \mathscr{U}$ and $\alpha < \mathfrak{m}$ be given. Choose $\beta \in \bigcap_{i=1}^n M_i$ with $\beta > \alpha$. Then $\varphi_\beta(M_i) = w_{M_i,0}$ for each $1 \leqslant i \leqslant n$, which shows $\varphi_0 \in \bar{B}_\alpha$.

(iii) Let $\Theta \in \bar{B}_\alpha$ with $\Theta \neq \varphi_0$. Then there is $M \in \mathscr{U}$ such that $\Theta(M) \neq w_{M,0}$. Since $\bigcap_{\alpha < \mathfrak{m}} F_{M,\alpha} = \emptyset$ there is $\alpha < \varrho < m$ such that $\Theta(M) \notin F_{M,\varrho}$. Since $B_\alpha = (B_\alpha \cap A_\varrho) \cup B_\varrho$ and $\Theta \in \bar{B}_\alpha$ we have either $\Theta \in \overline{B_\alpha \cap A_\varrho}$ or $\Theta \in \bar{B}_\varrho$. Suppose $\Theta \in \bar{B}_\varrho$. Then

$$\{\varphi(M) \mid \varphi \in \bar{B}_\varrho\} \subset \overline{\{\varphi_\beta(M) \mid \beta \geqslant \varrho\}} \subset F_{M,\varrho} \cup \{w_{M,0}\}.$$

But this gives $\Theta(M) \in F_{M,\varrho}$ or $\Theta(M) = w_{M,0}$, which cannot be true. So $\Theta \in \overline{B_\alpha \cap A_\varrho}$.

(iv) Assume the contrary. Let K be a compact subset of $\prod \mathscr{H}$ such that $\bar{B}_\alpha \cap K \setminus \{\varphi_0\} \neq \emptyset$ for each $\alpha < m$. By using $\bar{B}_\alpha \setminus \{\varphi_0\} \cap K \neq \emptyset$ and step (iii) alternately we can find a cofinal subset A of $(0, \mathfrak{m})$ and $\varrho_\alpha \in (0, \mathfrak{m})$ for each $\alpha \in A$ such that $\varrho_\alpha < \beta$ and $\overline{B_\alpha \cap A_{\varrho_\alpha}} \cap K \neq \emptyset$ whenever $\alpha < \beta$ and $\alpha, \beta \in A$. Let $B \subset A$ such that B and $A \setminus B$ are cofinal subsets of $(0, \mathfrak{m})$. We set $M_1 = \bigcup_{\alpha \in B} [\alpha, \varrho_\alpha)$ and $M_2 = \bigcup_{\alpha \in A \setminus B} [\alpha, \varrho_\alpha)$. Since $\mathscr{U}$ is an ultrafilter there is $M \in \mathscr{U}$ such that either $M_1 \cap M = \emptyset$ or $M_2 \cap M = \emptyset$. We have

$$P_M\big((\overline{B_\alpha \cap A_{\varrho_\alpha}}) \cap K\big) \subset P_M(\overline{B_\alpha \cap A_{\varrho_\alpha}}) \cap P_M(K) \subset \overline{P_M(B_\alpha \cap A_{\varrho_\alpha})} \cap P_M(K).$$

Since $\overline{B_\alpha \cap A_{\varrho_\alpha}} \cap K \neq \emptyset$ for each $\alpha \in A$, we have $\overline{P_M(B_\alpha \cap A_{\varrho_\alpha})} \cap P_M(K) \neq \emptyset$ for each $\alpha \in A$. If $M \cap M_1 = \emptyset$ then $P_M(B_\alpha \cap A_{\varrho_\alpha}) \subset F_{M,\alpha}$ for each $\alpha \in B$. If $M \cap M_2 = \emptyset$ then $P_M(B_\alpha \cap A_{\varrho_\alpha}) \subset F_{M,\alpha}$ for each $\alpha \in A \setminus B$. In both cases we have $F_{M,\alpha} \cap P_M(K) \neq \emptyset$ for each $\alpha \in C$, some cofinal subset of $(0, \mathfrak{m})$. Since $P_M(K)$ is compact and $(F_{M,\alpha})_{\alpha \in C}$ are closed subsets of X_M which are well-ordered by inclusion we have $\bigcap_{\alpha < \mathfrak{m}} F_{M,\alpha} \neq \emptyset$, which is a contradiction. This completes the proof of the fourth statement.

Now it is easy to see that the non-closed set $\overline{(\varphi_\alpha)_{\alpha < \mathfrak{m}}} \setminus \{\varphi_0\}$ has a closed intersection with each compact subset of $\prod \mathscr{H}$. Let a compact subset $K \subset \prod \mathscr{H}$ be given. Find $\alpha < \mathfrak{m}$ for this K as in (iv). We have

$$\overline{(\varphi_\alpha)_{\alpha < \mathfrak{m}}} \setminus \{\varphi_0\} = \bar{A}_\alpha \cup \bar{B}_\alpha \setminus \{\varphi_0\} = \bar{A}_\alpha \cup (\bar{B}_\alpha \setminus \{\varphi_0\})$$

since $\varphi_0 \notin \bar{A}_\alpha$ by (i); hence $\big(\overline{(\varphi_\alpha)_{\alpha < \mathfrak{m}}} \setminus \{\varphi_0\}\big) \cap K = \bar{A}_\alpha \cap K$, which is a closed subset of K.

Corollary 1. *Let X be a topological space which is not $(\mathfrak{m}, \aleph_0)$-compact. Then $X^{2^{\mathfrak{m}}}$ is not a k-space.*

Proof. Since X is not $(\mathfrak{m}, \aleph_0)$-compact there is a smallest cardinal $\mathfrak{l} \leqslant \mathfrak{m}$ such that X is not $(\mathfrak{l}, \aleph_0)$-compact. This means X is $(\mathfrak{k}, \aleph_0)$-compact for each $\mathfrak{k} < \mathfrak{l}$. Hence by the above theorem $X^{2^{\mathfrak{l}}}$ is not a k-space and therefore $X^{2^{\mathfrak{m}}}$ is a not a k-space.

Corollary 2. *If $X^{\mathfrak{m}}$ is a k-space for each cardinal $\mathfrak{m}$ then X is compact.*

Proof. By the above corollary X is $(\mathfrak{l}, \aleph_0)$-compact for each cardinal $\mathfrak{l}$. This means X is compact.

The following is a direct consequence of the theorem and therefore it needs no proof.

Corollary 3. *Let $\mathscr{F}$ be a family of topological spaces such that $\prod \mathscr{F}$ is a k-space. Then the subfamily $G = \{X \in \mathscr{F} \mid X$ is not $(\mathfrak{m}, \aleph_0)$-compact but $(\mathfrak{l}, \aleph_0)$-compact $\forall \mathfrak{l} < \mathfrak{m}\}$ has cardinality less than $2^{\mathfrak{m}}$.*

Theorem 1 can be improved in the following way:

Theorem 1′. *Let $\mathfrak{m}$ be a cardinal number such that $\mathfrak{n} < \mathfrak{m}$ implies $2^{\mathfrak{n}} \leqslant \mathfrak{m}$. If $\mathscr{F}$ is a family of topological spaces which are not $(\mathfrak{m}, \aleph_0)$-compact and $\operatorname{card} \mathscr{F} \geqslant 2^{\mathfrak{m}}$ then $\prod \mathscr{F}$ is not a k-space.*

Proof. Let $\mathscr{F}_{\mathfrak{n}} = \{X \mid X$ is not $(\mathfrak{n}, \aleph_0)$-compact but $(\mathfrak{l}, \aleph_0)$-compact for each $\mathfrak{l} < \mathfrak{n}\}$. If there is $\mathfrak{n} < \mathfrak{m}$ such that $\operatorname{card} \mathscr{F}_{\mathfrak{n}} \geqslant 2^{\mathfrak{n}}$ then by Theorem 1, $\prod \mathscr{F}$ is not a k-space. Otherwise if $\operatorname{card} \mathscr{F}_{\mathfrak{n}} < 2^{\mathfrak{n}}$ for each $\mathfrak{n} < \mathfrak{m}$ then $\operatorname{card} \bigcup_{\mathfrak{n} < \mathfrak{m}} \mathscr{F}_{\mathfrak{n}} \leqslant \mathfrak{m} < 2^{\mathfrak{m}}$. Hence $\mathscr{F} \setminus \bigcup_{\mathfrak{n} < \mathfrak{m}} \mathscr{F}_{\mathfrak{n}}$ is a family of topological spaces which has cardinality at least $2^{\mathfrak{m}}$ and each element of it is $(\mathfrak{n}, \aleph_0)$-compact for each $\mathfrak{n} < \mathfrak{m}$ but not $(\mathfrak{m}, \aleph_0)$-compact. So by Theorem 1, $\prod \mathscr{F} \setminus \bigcup_{\mathfrak{n} < \mathfrak{m}} \mathscr{F}_{\mathfrak{n}}$ is not a k-space.

If we assume the generalized continuum hypothesis (GCH) we have

Corollary 4. (GCH) *Let $\prod \mathscr{F}$ be a k-space for a family $\mathscr{F}$ of topological spaces. Then $\operatorname{card}\{X \in \mathscr{F} \mid X$ is not $(\mathfrak{m}, \aleph_0)$-compact$\} \leqslant \mathfrak{m}$.*

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