Some productive classes of maps which are related to confluent maps

by

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Abstract. We define some productive classes of maps between OM-maps and (weakly) confluent maps. Essential maps from continua onto a simple closed curve are characterized by some of these maps. Relations to span zero continua are also studied.

1. Introduction. A *continuum* means a compact connected metric space. All maps are assumed to be continuous *surjections*. Let $f\colon X \to Y$ be a map between continua. The map f is called *confluent* (*weakly confluent*, resp.) if for each subcontinuum K of Y, each (some, resp.) component C of $f^{-1}(K)$ satisfies $f(C) = K$. The map f is called an *OM map* if $f = g \circ m$ for some open map g and monotone map m. Let M be a class of maps (we do not specify domains and ranges). M is said to be *productive* if for each $f, g \in M$, $f \times g \in M$. In general, the class of all confluent maps and the class of all weakly confluent maps are not productive (see [11] and [8] for examples), while the class of all OM maps is productive. L. Oversteegen [16] and the author [5] have shown that the productivity of (weakly) confluent maps is related to the preservation of the property of having span zero under (weakly) confluent maps.

In this paper, we define some new classes of maps between OM maps and (weakly) confluent maps, which are productive. We also study the properties of these maps between arc-like and circle-like continua.

DEFINITIONS and NOTATIONS 1.1. Let X be a continuum and let d be a metric of X. The ε-neighbourhood of a set $A \subset X$ is denoted by $N(A, \varepsilon)$. The Hausdorff metric induced by d is denoted by d_H. A finite sequence of points $a_1, \ldots, a_n$ is called an *ε-chain* if $d(a_i, a_{i+1}) < \varepsilon$ for each $i = 1, \ldots, n-1$.

Let $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ be ε-chains and $a_m = b_1$. The ε-chain defined by $a_1, \ldots, a_m = b_1, \ldots, b_n$ is denoted by $(a_1, \ldots, a_m) + (b_1, \ldots, b_n)$.

Let $f, g\colon Y \to X$ be maps. $f \underset{\varepsilon}{=} g$ denotes $d(f, g) < \varepsilon$.

X is called *arc-like* (*circle-like*, *tree-like*, resp.) if X is the limit of an inverse sequence of arcs (circles, trees, resp.).

1980 *Mathematics Subject Classification*: 54C10, 54F20.

Key words and phrases: (weakly) confluent map, chain lifting property, span zero, pseudo-arc.

A function $\varphi\colon \{1, \ldots, m\} \to \{1, \ldots, n\}$ is called a *pattern* if $|\varphi(i) - \varphi(i+1)| \leqslant 1$ for each $i = 1, \ldots, m-1$. In this paper, all patterns are assumed to be *surjective*. Moreover, if $|\varphi(m) - \varphi(1)| \leqslant 1$, then φ is called a *cyclic pattern*. A pattern $\varphi\colon \{1, \ldots, m\} \to \{1, \ldots, n\}$ is said to be *monotone* if, for each $i = 1, \ldots, m$, $\varphi^{-1}(i) = \{a_i, a_i+1, \ldots, b_i\}$ for some $1 \leqslant a_i \leqslant b_i \leqslant m$. A *monotone cyclic pattern* is similarly defined.

A map $r\colon X \to Y$ is called *refinable* if for each $\varepsilon > 0$, there exists an ε-map $r_\varepsilon\colon X \to Y$ such that $r \underset{\varepsilon}{=} r_\varepsilon$.

Let P be a polyhedron with a triangulation τ, and let A be a subset of P. The collection $\mathrm{st}(A, \tau)$ is defined by $\mathrm{st}(A, \tau) = \{s \mid s \in \tau \text{ and } s \cap A \neq \emptyset\}$, and $\mathrm{st}(A, \tau)^* = \bigcup \mathrm{st}(A, \tau)$.

2. Definitions, basic properties and examples

DEFINITIONS 2.1. Let $f\colon X \to Y$ be a map. It is said to have the *chain lifting property* (abbreviated to CLP) if for each $\varepsilon > 0$ and for each $\zeta > 0$, there exists an $\eta > 0$ such that

[CLP(ε, ζ)]: for each η-chain $a_1, \ldots, a_s$ in Y there exist a ζ-chain $c = c_1, \ldots, c_t$ in X and a monotone pattern $\varphi\colon \{1, \ldots, t\} \to \{1, \ldots, s\}$ such that $\varphi(1) = 1$ and $d(f(c_i), a_{\varphi(i)}) < \varepsilon$ for each $i = 1, \ldots, t$.

The map f is said to have the *based chain lifting property* (abbreviated to BCLP) if for each $\varepsilon > 0$ and for each $\zeta > 0$, there exists an $\eta > 0$ such that

[BCLP(ε, ζ)]: for each η-chain $a_1, \ldots, a_s$ in Y and for each $c \in X$ with $d(f(c), a_1) < \eta$, there exist a ζ-chain $c = c_1, \ldots, c_t$ in X and a monotone pattern $\varphi\colon \{1, \ldots, t\} \to \{1, \ldots, s\}$ such that $\varphi(1) = 1$ and $d(f(c_i), a_{\varphi(i)}) < \varepsilon$ for each $i = 1, \ldots, t$.

The map $f\colon X \to Y$ is said to have the *weak chain lifting property* (abbreviated to WCLP) if for each $\varepsilon > 0$ and for each $\zeta > 0$, there exists an $\eta > 0$ such that

[WCLP(ε, ζ)]: for each η-chain $a_1, \ldots, a_s$ in Y, there exist a ζ-chain $c_1, \ldots, c_t$ in X and a pattern $\varphi\colon \{1, \ldots, t\} \to \{1, \ldots, s\}$ such that $d(f(c_i), a_{\varphi(i)}) < \varepsilon$ for each $i = 1, \ldots, t$.

The class of all maps which have CLP, of all maps which have BCLP, and of all maps which have WCLP are denoted by CL, BCL, and WCL respectively.

These terms were suggested by the editorial board.

PROPOSITION 2.2. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be maps. Let $M =$ CL, or BCL, or WCL.*

(i) *If $f, g \in M$, then $g \circ f \in M$.*
(ii) *If $g \circ f \in M$, then $g \in M$.*

The proofs are easy and will be omitted.

The relationships between these classes and (weakly) confluent maps are as follows.

THEOREM 2.3.

$$\begin{array}{ccccccc}
 & & \mathrm{OM} & \xrightarrow[\text{(a)}]{} & \textit{based chain lifting property} & \xrightarrow[\text{(b)}]{} & \textit{confluent} \\
 & & & & \downarrow \text{(c)} & & \downarrow \\
\textit{refinable} & \xrightarrow[\text{(d)}]{} & & & \textit{chain lifting property} \xrightarrow[\text{(e)}]{} \textit{weak chain lifting property} & \xrightarrow[\text{(f)}]{} & \textit{weakly confluent}
\end{array}$$

None of the reverse implications hold.

Proof. We will prove the implications (a)–(f). The examples which indicate that the reverse implications do not hold will be given later.

Proof of (a). By Proposition 2.2, it is sufficient to show that any open map and any monotone map have the BCLP.

(a1) Let $f\colon X \to Y$ be an open map. Take any $\varepsilon > 0$ and $\zeta > 0$. There exists an $\eta > 0$ such that

$$d_H(f^{-1}(p), f^{-1}(q)) < \zeta \quad \text{for all } p, q \in Y \text{ with } d(p, q) < \eta.$$

It is easy to see that this η is the desired one.

To prove that any monotone map has the BCLP, we need the following lemma.

LEMMA 2.4 ([15], Lemma 2.2). *Let $f\colon X \to Y$ be a monotone map between continua. There exist sequences $(X_i)_{i \geqslant 0}$ and $(Y_i)_{i \geqslant 0}$ of Peano continua which satisfy the following conditions.*

(i) $X_i \supset X_{i+1} \supset \bigcap_{i \geqslant 0} X_i = X$ *and* $Y_i \supset Y_{i+1} \supset \bigcap_{i \geqslant 0} Y_i = Y$.
(ii) *There exists a monotone extension $F\colon X_0 \to Y_0$ of f such that, for each $i \geqslant 0$, $F|X_i\colon X_i \to Y_i$ is a monotone onto map.*

(a2) Let $f\colon X \to Y$ be a monotone map.

Case 1. First we assume that Y is a Peano continuum. Take any $\varepsilon > 0$ and any $\zeta > 0$. There exists an $\eta > 0$ such that for all $p, q \in Y$ with $d(p, q) < \eta$, there exists an arc A from p to q such that $\operatorname{diam} A < \varepsilon/2$.

Take any chain $a_1, \ldots, a_s$ in Y and a point $c \in X$ with $d(f(c), a_1) < \eta$. Let $a_0 = f(c)$. There exist arcs $A_0, A_1, \ldots, A_{s-1}$ such that A_i is an arc from a_i to a_{i+1} and $\operatorname{diam} A_i < \varepsilon/2$, $0 \leqslant i \leqslant s-1$. Notice that $f^{-1}(A_i)$ is a continuum. Hence for each i, we can choose a ζ-chain $c_{i1}, \ldots, c_{ik_i}$ in $f^{-1}(A_i)$ from $f^{-1}(a_i)$ to $f^{-1}(a_{i+1})$, where $c_{01} = c$. Then $c = c_{01}, c_{02}, \ldots, c_{11}, \ldots, c_{1k_1}, c_{21}, \ldots, c_{sk_s}$ is the required chain.

Case 2. *General case.* We take (X_i), (Y_i), and $F\colon X_0 \to Y_0$ as in Lemma 2.4. Given any $\varepsilon > 0$ and any $\zeta > 0$, there exists a $\delta > 0$ such that $\delta < \zeta/4$ and

$$d(F(x), F(y)) < \varepsilon/4 \quad \text{for all } x, y \in X_0 \text{ with } d(x, y) < \delta.$$

Take a large i so that $X_i \subset N(X, \delta)$. Since Y_i is a Peano continuum and $F_i = F|X_i$: $X_i \to Y_i$ is monotone, F_i has the BCLP. So there exists an $\eta > 0$ which satisfies BCLP($\varepsilon/4$, $\zeta/4$) for F_i. This η is the required one.

For any η-chain $a_1, \ldots, a_s$ in Y and for any $c \in X$ with $d(f(c), a_1) < \eta$, there exist a $\zeta/4$-chain $c = c'_1, \ldots, c'_t$ in X_i and a monotone pattern φ such that $d(F_i(c'_j), a_{\varphi(j)}) < \varepsilon/4$, $j = 1, \ldots, t$. We can find a point $c_j \in X$ such that $d(c_j, c'_j) < \delta$, where $c = c_1$. Then $c_1, \ldots, c_t$ is a ζ-chain and $d(f(c_j), a_{\varphi(j)}) < \varepsilon$ for each j. This completes the proof.

Proof of (b) and (f). These are immediate consequences of the following result.

PROPOSITION 2.5. *Let f: $X \to Y$ be a map.*

(i) *The following statements are equivalent.*

(i1) *f is confluent.*

(i2) *For each $\varepsilon > 0$ and for each $\zeta > 0$, there exists an $\eta > 0$ such that*

[C(ε, ζ)]: *for each η-chain $a_1, \ldots, a_s$ in Y and for each $c \in X$ with $d(f(c), a_1) < \eta$, there exists a ζ-chain $c = c_1, \ldots, c_t$ in X such that*

$$d_H(f(\{c_1, \ldots, c_t\}), \{a_1, \ldots, a_s\}) < \varepsilon.$$

(ii) *The following statements are equivalent.*

(ii1) *f is weakly confluent.*

(ii2) *For each $\varepsilon > 0$ and for each $\zeta > 0$, there exists an $\eta > 0$ such that*

[WC(ε, ζ)]: *for each η-chain $a_1, \ldots, a_s$ in Y, there exists a ζ-chain $c_1, \ldots, c_t$ in X such that*

$$d_H(f(\{c_1, \ldots, c_t\}), \{a_1, \ldots, a_s\}) < \varepsilon.$$

Proof. We only prove (i).

(i1)$\to$(i2). Suppose that (i2) does not hold. Then there are an ε_0 and a ζ_0 such that for each $n \geqslant 0$, there exist $1/n$-chains A_n: $a_1^n, \ldots, a_{s_n}^n$ in Y and points $c_n \in X$ such that $d(f(c_n), a_1^n) < 1/n$ and A_n and c_n do not satisfy [C(ε_0, ζ_0)].

We may assume that $A_n \to A$, $a_1^n \to a$, and $c_n \to c$. Then A is a continuum and $a = f(c) \in A$. Let K be the component of $f^{-1}(A)$ which contains c. By (i1), $f(K) = A$. Take a sufficiently large n such that $d_H(A, A_n) < \varepsilon_0/4$ and $1/n < \zeta_0/4$. Since $f(K) = A$, we can take a ζ_0-chain $C \subset K$ from c such that $d_H(f(C), A) < \varepsilon_0/4$. Then we can easily see that $d_H(f(\{c\} \cup C), A_n) < \varepsilon_0$. As $\{c\} \cup C$ is also a ζ_0-chain, we have a contradiction.

The proof of (i2)$\to$(i1) is easy and will be omitted.

Proof of (c) and (e). These are trivial.

Proof of (d). Let r: $X \to Y$ be a refinable map and take any $\varepsilon > 0$ and $\zeta > 0$. There exists a $\zeta/2$-map r_0: $X \to Y$ such that $d(r, r_0) < \varepsilon/2$. We can take an $\eta > 0$ such that that $\operatorname{diam} r_0^{-1}(S) < \zeta$ for each subset $S \subset Y$ with $\operatorname{diam} S < \eta$. It is easy to see that the η is the required number.

This completes the proof of Theorem 2.3.

Let X_1 and X_2 be continua. The metric of $X_1 \times X_2$ is given by

$$d((x_1, x_2), (y_1, y_2)) = \max_{i=1,2} d(x_i, y_i).$$

THEOREM 2.6. *Let M = CL, or BCL, or WCL, and let f_i: $X_i \to Y_i$ be maps, $i = 1, 2$. Then $f_1 \times f_2 \in M$ if and only if $f_i \in M$, $i = 1, 2$.*

Proof. We prove the case M = WCL. Assume that $f_1, f_2 \in M$. We prove that $f_1 \times f_2 \in M$. First, we prove that

(1) For each map $(f: X \to Y) \in M$ and each continuum Z, $(f \times \mathrm{id}_Z: X \times Z \to Y \times Z) \in M$.

For any $\varepsilon > 0$ and for any $\zeta > 0$, there exists an $\eta > 0$ which satisfies [WCLP(ε, ζ)] for f. Take any η-chain $x_1, \ldots, x_s$ in $Y \times Z$ and let $x_i = (y_i, z_i)$. There exists a ζ-chain $c_1, \ldots, c_t$ in X and an onto pattern φ: $\{1, \ldots, t\} \to \{1, \ldots, s\}$ such that $d(f(c_i), y_{\varphi(i)}) < \varepsilon$, $1 \leqslant i \leqslant t$. Define $w_i = (c_i, z_{\varphi(i)})$. Then $w_1, \ldots, w_t$ is a ζ-chain and $d(f \times \mathrm{id}_Z(w_i), x_{\varphi(i)}) < \varepsilon$, for each $i = 1, \ldots, t$. This proves (1).

The general case can be obtained by using Proposition 2.2 and the equality $f_1 \times f_2 = (f_1 \times \mathrm{id}_{Y_2}) \circ (\mathrm{id}_{X_1} \times f_2)$.

The proof of the reverse implication is easy and we will omit it.

Next, we will give some examples which indicate that none of the implications in Theorem 2.3 can be reversed.

EXAMPLE 2.7 (A map which has the BCLP, and is not an OM map). Lelek and Read ([10], Example 3.6) produce a map which is confluent and is not an OM map. We prove that their example has the BCLP.

The map f: $X \to Y$ is a retraction indicated in Fig. 1. Let $A_i = \overline{P_i P_{i+1}}$ (= the segment from p_i to p_{i+1}). Then $\operatorname{Lim} A_i = L_0$ and $f|L$: $L \to L_0$ is a homeomorphism. For simplicity, we assume that $(d(p_i, p_{i+2}))_{i \geqslant 0}$ is decreasing and converges to 0.

Take any $\varepsilon > 0$ and $\zeta > 0$ and let $\nu = \min(\varepsilon, \zeta)$. Take a small $\eta_1 > 0$ such that

$$\bigcup_{l \geqslant m-1} A_l \subset N(L_0, \nu), \quad \text{where } m = \min\{k \mid d(p_k, p_{k+2}) < \eta_1\}.$$

Notice that the number of the arc components of $\mathrm{cl}(N(L_0, \nu)) - N(p, \nu) \cup N(q, \nu) \cup \bigcup_{l \geqslant m} A_l$ is finite. Let $\{B_1, \ldots, B_k\}$ be the components. Take a small $\eta > 0$ such that

$$0 < \eta < \eta_1, \min\{d(B_i, B_j) \mid i \neq j\}.$$

Then we can see that η is the desired number. (See Figure 1 for an example how to "cover" an η-chain by a ζ-chain.)

EXAMPLE 2.8. Clearly, each retraction has the CLP. Hence a retraction which is neither confluent nor refinable gives an example which shows that the implications (c) and (d) in Theorem 2.3 cannot be reversed.

EXAMPLE 2.9 (A confluent map which does not have the WCLP). Maćkowiak ([12], (5.37)) has given an example of a confluent map f: $X \to Y$ such that

$f \times \mathrm{id}_I$: $X \times I \to Y \times I$ is not weakly confluent. This example shows that the implications (b) and (f) cannot be reversed.

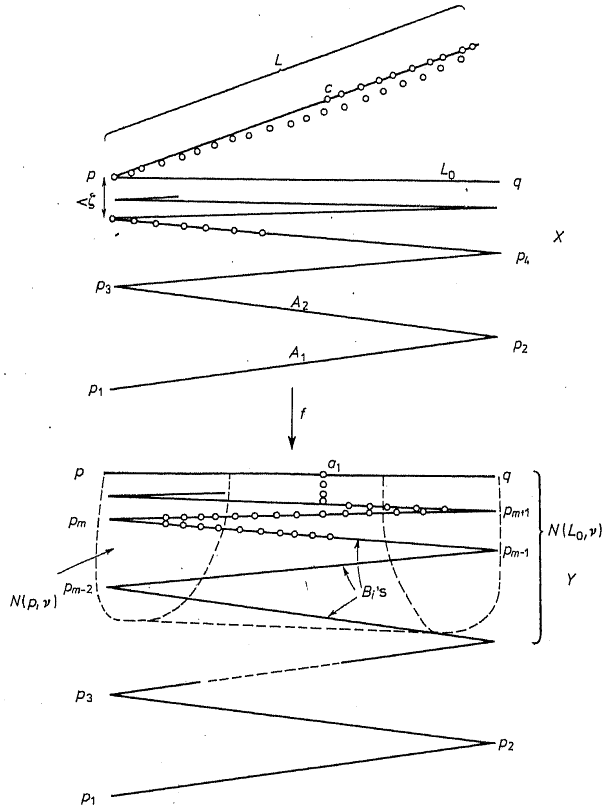

Fig. 1

EXAMPLE 2.10 (A map which has the WCLP and does not have the CLP). Let P be the pseudo-arc. Any map f: $P \to I$ onto an arc has the WCLP by Theorem 3.1 below. But f does not have the CLP. This follows from Proposition 2.11.

PROPOSITION 2.11. *Let* f: $X \to Y$ *be a map which has the* CLP. *If* X *is hereditarily indecomposable, then so is* Y.

Proof. Suppose that Y contains a decomposable continuum $A \cup B$, where A and B are subcontinua of Y. Take points $a \in A - B$, $b \in B - A$, and $p \in A \cap B$. There exists an $\eta_n > 0$ which satisfies the condition [CLP$(1/n, 1/n)$]. For each n, we can take an η_n-chain α_n from a to p in A and an η_n-chain β_n from p to b in B, such that α_n (β_n, resp.) is $1/n$-dense in A (B, resp.). There exists a sequence (γ_n) of $1/n$-chains in X such that γ_n satisfies [CLP$(1/n, 1/n)$] for $\alpha_n + \beta_n$. Each γ_n can be decomposed as $\gamma_n = a_n + b_n$ so that $d_H(f(a_n), \alpha_n)$, $d_H(f(b_n), \beta_n) < 1/n$. We may assume that $a_n \to K$ and $b_n \to L$, where K and L are subcontinua of X. Then $f(K) = A$ and $f(L) = B$; so $K \not\subset L$ and $L \not\subset K$. This contradicts X being hereditarily indecomposable.

THEOREM 2.12. *Let* r: $X \to Y$ *be a refinable map. If either* X *or* Y *is homogeneous, then* r *has the* BCLP.

To prove this theorem, we consider the following property.

DEFINITION 2.13. A continuum X is said to have the *property* $(*)$ if for each $\varepsilon > 0$ and for each point $a \in X$, there exists a $\delta > 0$ which satisfies the following condition: for each $\xi > 0$, there exists an $\eta > 0$ such that

[$*(\varepsilon, \delta, \xi, \eta)$] for each $b \in X$ with $d(a, b) < \delta$ and for each η-chain $a = a_1, \ldots, a_s$ in X, there exists a ξ-chain $b = b_1, \ldots, b_t$ in X and a monotone pattern φ: $\{1, \ldots, t\} \to \{1, \ldots, s\}$ such that $\varphi(1) = 1$ and $d(b_i, a_{\varphi(i)}) < \varepsilon$ for $1 \leqslant i \leqslant t$.

The motivation of the above definition comes from [3].

The proof of Theorem 2.12 is divided into three steps.

Step 1. If a continuum is homogeneous, then it has the property $(*)$ (cf. [3]).
Step 2. Each refinable map preserves the property $(*)$ (cf. [2] (2.1)).
Step 3. If r: $X \to Y$ is a refinable map and Y has the property $(*)$, then r has the BCLP (cf. [2], (2.3)).

Proof of Step 1. For any $\varepsilon > 0$, let $\delta > 0$ be the Effros number for $\varepsilon > 0$. Using compactness, it is easy to see that this δ is the required one.

Proof of Step 2. Let r: $X \to Y$ be a refinable map and suppose that X has the property $(*)$. We will prove that Y has the property $(*)$.

Let r_i: $X \to Y$ be an $1/i$-map such that $r_i \rightrightarrows r$ (uniform convergence). Take any $\varepsilon > 0$ and $p \in Y$. We may assume that $r_i^{-1}(p) \to a$ as $i \to \infty$. Take δ_1, δ_2, N and δ as follows.

(1) If $d(x, y) < \delta_1$, then $d(r(x), r(y)) < \varepsilon/4$.

(2) $0 < \delta_2 < \delta_1$ and δ_2 satisfies the property $(*)$ for $\varepsilon = \delta_1/2$ and a. That is, for each $\xi > 0$, there exists an $\eta > 0$ such that the condition [$*(\delta_1/2, \delta_2, \xi, \eta)$] holds.

(3) For each $n \geqslant N$, r_n is a $\delta_2/4$-map such that $r \underset{\varepsilon/4}{=} r_n$, and $d_H(r_n^{-1}(p), a) < \delta_2/2$.

(4) If $\operatorname{diam} S < \delta$, $S \subset Y$, then $\operatorname{diam} r_N^{-1}(S) < \delta_2/2$.

This δ is the required number for ε. To see this, fix a $\xi > 0$. Take α, β, i and η as follows.

(5) If $d(x, y) < \alpha$, then $d(r_N(x), r_N(y)) < \xi$.

(6) β satisfies [$*(\delta_1/2, \delta_2, \alpha, \beta)$].

(7) r_i is a $\beta/2$-map and $i \geqslant N$ (hence, $d_H(r_i^{-1}(p), a) < \beta/2$ and $r_i \underset{\varepsilon/4}{=} r$).

(8) For each $S \subset Y$ with $\operatorname{diam} S < \eta$, $\operatorname{diam} r_i^{-1}(S) < \beta$.

This η satisfies $[*(\varepsilon, \delta, \xi, \eta)]$ for Y. To show this, take any η-chain $p = p_1, \ldots, p_s$ in Y and $q \in Y$ with $d(p, q) < \delta$. Let $a_N \in r_N^{-1}(p)$ and $b_N \in r_N^{-1}(q)$. Then $d(a_N, b_N) < \delta_2/2$ by (4). Hence, by (3) and (2),

$$d(a, a_N) < \delta_2/2 < \delta_1/2 \quad \text{and} \quad d(a, b_N) < \delta_2. \tag{9}$$

Define a chain $a = a_0, a_1, \ldots, a_s$ in X by $a_l \in r_i^{-1}(p_l)$, for $l \geq 1$. By (8) and (7), it is a β-chain. By (6) and (9), there is an α-chain $b_N = b_1, \ldots, b_m$ in X and a monotone pattern $\varphi\colon \{1, \ldots, m\} \to \{0, 1, \ldots, s\}$ such that $\varphi(1) = 0$ and $d(b_l, a_{\varphi(l)}) < \delta_2/2$, $1 \leq l \leq m$. The set $\{q_1, \ldots, q_m\}$ defined by $q_l = r_N(b_l)$ is a ξ-chain from q by (5). And for each $l \in \varphi^{-1}(\{1, \ldots, s\})$, we have

$$\begin{aligned} d(q_l, p_{\varphi(l)}) &= d\big(r_N(b_l), r_i(a_{\varphi(l)})\big) \\ &\leq d\big(r_N(b_l), r(b_l)\big) + d\big(r(b_l), r(a_{\varphi(l)})\big) + d\big(r(a_{\varphi(l)}), r_i(a_{\varphi(l)})\big) < \varepsilon \\ &\qquad ((3),\ (1) \text{ and } (7)). \end{aligned}$$

Moreover,

$$\text{if } l \in \varphi^{-1}(0), \text{ then } d(b_l, a) = d(b_l, a_0) < \delta_1/2. \tag{10}$$

So we have

$$\begin{aligned} d(q_l, p_1) &= d\big(r_N(b_l), r_N(a_N)\big) \\ &\leq d\big(r_N(b_l), r(b_l)\big) + d\big(r(b_l), r(a_N)\big) + d\big(r(a_N), r_N(a_N)\big) \\ &< d\big(r(b_l), r(a_N)\big) + \varepsilon/2 \quad ((3)) \\ &\leq d\big(r(b_l), r(a)\big) + d\big(r(a), r(a_N)\big) + \varepsilon/2 \\ &< \varepsilon/4 + \varepsilon/4 + \varepsilon/2 \quad ((1),\ (3),\ (9) \text{ and } (10)) \\ &= \varepsilon. \end{aligned}$$

Now define a pattern $\psi\colon \{1, \ldots, m\} \to \{1, \ldots, s\}$ by $\psi|\varphi^{-1}(0) = 1$, and $\psi|\varphi^{-1}(\{1, \ldots, s\}) = \varphi$. Then $q_1, \ldots, q_m$ and ψ are the desired ones.

For the proof of Step 3, we need the following lemma.

LEMMA 2.14. *Let $f\colon X \to Y$. Suppose that for each $\varepsilon > 0$, each $\zeta > 0$, each $a \in Y$ and each $c \in f^{-1}(a)$, there exists an $\eta > 0$ which satisfies the condition* [BCLP(ε, ζ)] *for any chain $a_1, \ldots, a_s$ with $a_1 = a$ and for the point c. Then f has the* BCLP.

The above lemma can be easily proved using the compactness of X and Y.

Proof of Step 3. Let $r\colon X \to Y$ be a refinable map and suppose that Y has the property $(*)$. We verify that r satisfies the hypothesis of Lemma 2.14. Take any $\varepsilon > 0$, any $\zeta > 0$, and a point $c \in r^{-1}(a)$. There exists a $\delta > 0$ such that $\delta < \varepsilon$ and the property $(*)$ is satisfied for $\varepsilon/2$ and a. There exists a $\zeta/2$-map r_0 such that $r \underset{\delta/2}{=} r_0$. Let $\xi > 0$ be such that $\operatorname{diam} r_0^{-1}(S) < \zeta$ for each $S \subset Y$ with $\operatorname{diam} S < \xi$. Since Y has the property $(*)$, we can find an $\eta > 0$ which satisfies the condition $[*(\varepsilon/2, \delta, \xi, \eta)]$. It can be seen that the η is the required number.

This completes the proof of Theorem 2.12.

3. Maps onto arc-like and circle-like continua

DEFINITION 3.1. Let X be a continuum and let $C = \{C_1, \ldots, C_n\}$ be a finite open cover of X.

C is called a *chain cover* of X provided $C_i \cap C_j \neq \emptyset$ if and only if $|i-j| \leq 1$. A chain cover C is said to be *taut* provided $\operatorname{cl} C_i \cap \operatorname{cl} C_j \neq \emptyset$ if and only if $|i-j| \leq 1$. In this paper, *all chains are assumed to be taut*. Each member of C is called a *link*.

C is called a *circular chain* provided $C_i \cap C_j \neq \emptyset$ if and only if $|i-j| \pmod{n} \leq 1$. A *taut circular chain* and a *link* of C are defined similarly.

Let C be a chain or a circular chain. For each link C_k of C, $i(C_k)$ denotes the set $C_k - \bigcup_{l \neq k} \operatorname{cl} C_l$.

Let $D = \{D_1, \ldots, D_m\}$ be a finite open cover of X which is a refinement of C. We say that D is a *proper refinement* of C if, for each $k = 1, \ldots, n$, there exists a j $(1 \leq j \leq m)$ such that $D_j \subset i(C_k)$.

THEOREM 3.2. *Any map from any continuum onto an arc-like continuum has the* WCLP.

Proof. Let $f\colon Y \to X$ be a map from a continuum Y onto an arc-like continuum X and take any $\varepsilon > 0$ and $\zeta > 0$. There exists a chain cover $C = \{C_1, \ldots, C_n\}$ of X such that $\operatorname{mesh} C < \varepsilon$.

Let $D = f^{-1}(C)$ and take a finite open cover E of Y which is a proper refinement of D with $\operatorname{mesh} E < \zeta$. We can number the members of E (admitting repetitions) so that $E = \{E_1, \ldots, E_m\}$ is a weak chain (i.e. $E_i \cap E_{i+1} \neq \emptyset$ for each $i = 1, \ldots, m-1$). Since D is a chain cover of Y, a pattern $\varphi\colon \{1, \ldots, m\} \to \{1, \ldots, n\}$ is defined by $E_i \subset D_{\varphi(i)}$ for $1 \leq i \leq m$. By the choice of E, φ is surjective.

Let $\eta > 0$ be the Lebesgue number of C such that

$$0 < \eta < \min\{d(\operatorname{cl} C_i, \operatorname{cl} C_j) \mid |i-j| \geq 2\}.$$

We prove that this η is the required one. Take any η-chain $a_1, \ldots, a_s$ in X. By adding a suitable η-chain $a_{s+1}, \ldots, a_t$, we can assume that the set $\{a_1, \ldots, a_s, \ldots, a_t\}$ intersects each $i(C_j)$. By the choice of η, a surjective pattern $\psi\colon \{1, \ldots, t\} \to \{1, \ldots, n\}$ is defined by $a_i \in C_{\psi(i)}$, $1 \leq i \leq t$. By the uniformization theorem [14], there are two patterns $h\colon \{1, \ldots, l\} \to \{1, \ldots, m\}$ and $k\colon \{1, \ldots, l\} \to \{1, \ldots, t\}$ such that $\varphi \circ h = \psi \circ k$. For each $i = 1, \ldots, l$, take a point $c_i \in E_{h(i)}$. Then $c_1, \ldots, c_l$ is a ζ-chain and

$$f(c_i) \in f(E_{h(i)}) \subset f(D_{\varphi\circ h(i)}) = C_{\varphi\circ h(i)} = C_{\psi\circ k(i)} \ni a_{k(i)}.$$

Hence $d(f(c_i), a_{k(i)}) < \varepsilon$. We can find a "subinterval" $\{j, j+1, \ldots, j+u\}$ such that $k(\{j, \ldots, j+u\}) = \{1, \ldots, s\}$. The desired ζ-chain is $c_j, \ldots, c_{j+u}$. This completes the proof.

THEOREM 3.3. *Each map from any continuum onto the pseudo-arc has the* BCLP.

Proof. Let $f\colon X \to P$ be a map from a continuum X onto the pseudo-arc P. We verify that f satisfies the hypothesis of Lemma 2.14. Given any $\varepsilon > 0$, $\zeta > 0$ and a point $c \in f^{-1}(a)$, take a chain cover $C = \{C_1, \ldots, C_n\}$ of P such that $\operatorname{mesh} C < \varepsilon$ and $a \in i(C_1)$. Let $D = f^{-1}(C)$. There exists a finite open cover E of X which is a proper refinement of D and mesh $E < \zeta$. We can number the elements of E, admitting repetitions, so that $E = \{E_1, \ldots, E_m\}$ is a weak chain and $c \in E_1 \subset D_1$. Since D is a chain and E is a proper refinement of D, a surjective pattern φ: $\{1, \ldots, m\} \to \{1, \ldots, n\}$ is defined by $\varphi(1) = 1$ and $E_i \subset D_{\varphi(i)}$ for $i = 1, \ldots, m$. By [17], Theorem 3, there exists a chain cover $F = \{F_1, \ldots, F_m\}$ of P which follows φ in C and $a \in F_1$.

Let $\eta > 0$ be the Lebesgue number of F such that

$$0 < \eta < \min\{d(\operatorname{cl} F_i, \operatorname{cl} F_j) \mid |i-j| \geq 2\}.$$

To see that the η is the required number, let $a = a_1, \ldots, a_s$ be any η-chain in Y. A pattern ψ: $\{1, \ldots, s\} \to \{1, \ldots, m\}$ is defined by $a_i \in F_{\psi(i)}$, $1 \leq i \leq s$. Take points $c = c_1, \ldots, c_s$ in X such that $c_i \in E_{\psi(i)}$. Then $c_1, \ldots, c_s$ is a ζ-chain and

$$f(c_i) \in f(E_{\psi(i)}) \subset f(D_{\varphi\circ\psi(i)}) = C_{\varphi\circ\psi(i)}, \quad a_i \in F_{\psi(i)} \subset C_{\varphi\circ\psi(i)}.$$

So $d(f(c_i), a_i) < \varepsilon$, and $c_1, \ldots, c_s$ is the desired chain.

Corollary 3.4. *Let X be an arc-like continuum. Then the following statements are equivalent.*

(i) $X = P$.
(ii) *Each map onto X has the* BCLP.
(iii) *Each map onto X is confluent.*
(iv) *Each map onto X has the* CLP.

This follows from Theorem 3.2 and Proposition 2.11.

Next, we consider maps onto S^1.

Theorem 3.5. *Let X be a continuum and let f: $X \to S^1$ be a map. Then $f \not\simeq 0$ if and only if f has the* WCLP.

For the proof, we need some lemmas.

Lemma 3.6. *Let f: $S^1 \to S^1$ be a map. If $f \not\simeq 0$, then f has the* WCLP.

Proof. Let p: $\boldsymbol{R} \to S^1$ be the universal covering. We may assume that p is a local isometry. Take any $\varepsilon > 0$ and $\zeta > 0$. There is a PL-map f': $S^1 \to S^1$ which is sufficiently close to f. For simplicity of notation, we let $f' = f$.

There exist triangulations (S^1, σ), (S^1, τ) of S^1 such that

(1) mesh $\sigma < \zeta$ and mesh $\tau < \varepsilon/4$,
(2) f: $(S^1, \sigma) \to (S^1, \tau)$ is simplicial.

There exist triangulations $(\boldsymbol{R}, \tilde{\sigma})$, $(\boldsymbol{R}, \tilde{\tau})$ of $\boldsymbol{R}$ and a simplicial map $\tilde{f}$: $(\boldsymbol{R}, \tilde{\sigma}) \to (\boldsymbol{R}, \tilde{\tau})$ such that

(3) p: $(\boldsymbol{R}, \tilde{\sigma}) \to (S^1, \sigma)$ and p: $(\boldsymbol{R}, \tilde{\tau}) \to (S^1, \tau)$ are simplicial,
(4) p is isometric on the union of any two adjacent simplexes of $\tilde{\sigma}$ and $\tilde{\tau}$.
(5) $p \circ \tilde{f} = f \circ p$.

Take an $\eta > 0$ sufficiently small so that any two points $x, y \in S^1$ with $d(x, y) < \eta$ belong to a common open star of a vertex of τ.

To see that this η is the required one, we take any η-chain $a_1, \ldots, a_s$ in (S^1, τ). There is an η-chain $\tilde{a}_1, \ldots, \tilde{a}_s$ in $(\boldsymbol{R}, \tilde{\tau})$ such that $p(\tilde{a}_i) = a_i$, $1 \leq i \leq s$. Let $A = \operatorname{st}(\{\tilde{a}_1, \ldots, \tilde{a}_s\}, \tilde{\tau})^*$, which is a subinterval of $(\boldsymbol{R}, \tilde{\tau})$ by the choice of η and (4). Since $f \not\simeq 0$, $\tilde{f}(\boldsymbol{R}) = \boldsymbol{R}$. Further, $\tilde{f}$: $(\boldsymbol{R}, \tilde{\sigma}) \to (\boldsymbol{R}, \tilde{\tau})$ is a simplicial map. Hence there exists a (finite) subinterval C of $(\boldsymbol{R}, \tilde{\sigma})$ such that $\tilde{f}(C) = A$. Let $C = \bigcup_{i=1}^{u} \tilde{s}_i$, where $\tilde{s}_i$'s are 1-simplexes of τ arranged according to the natural order of $\boldsymbol{R}$. Let $\operatorname{st}(\{\tilde{a}_1, \ldots, \tilde{a}_s\}, \tilde{\tau}) = \{\tilde{t}_1, \ldots, \tilde{t}_v\}$, where $\tilde{t}_i$'s are also arranged according to the natural order of $\boldsymbol{R}$. By the choice of η and (4), a pattern ψ: $\{1, \ldots, s\} \to \{1, \ldots, v\}$ is defined by $\tilde{a}_i \in \tilde{t}_{\psi(i)}$, $1 \leq i \leq s$. Since $\tilde{f}|C$: $C \to A$ is simplicial, a pattern φ: $\{1, \ldots, u\} \to \{1, \ldots, v\}$ is defined by $\tilde{f}(\tilde{s}_i) = \tilde{t}_{\varphi(i)}$, $1 \leq i \leq u$ (notice that each $\tilde{f}(\tilde{s}_i)$ is a 1-simplex of $\tilde{\tau}$).

By the uniformization theorem [14], we get two patterns h: $\{1, \ldots, k\} \to \{1, \ldots, s\}$ and g: $\{1, \ldots, k\} \to \{1, \ldots, u\}$ such that $\psi \circ h = \varphi \circ g$. Constructing $c_1, \ldots, c_m$ as in Theorem 3.1, we see that $d(\tilde{f}(\tilde{c}_i), \tilde{a}_{\varphi(i)}) < \varepsilon$ for each $i = 1, \ldots, k$. Let $c_i = p(\tilde{c}_i)$. Since p is local isometry, we have $d(f(c_i), a_{\varphi(i)}) < \varepsilon$, $1 \leq i \leq k$.

Lemma 3.7. *If a map f: $X \to S^1$ is null homotopic, then f does not have the* WCLP.

Proof. Assume that f: $X \to S^1$ has the WCLP and $f \simeq 0$. Let $\tilde{f}$: $X \to \boldsymbol{R}$ be a lift of f. As $p \circ \tilde{f} = f$, by Proposition 2.2, $p|\tilde{f}(X)$: $\tilde{f}(X) \to S^1$ has the WCLP. But it is easy to see that the map does not have the WCLP (see also [8], Example, p. 51).

Proof of Theorem 3.5. First, assume that $f \not\simeq 0$. Let $X = \varprojlim(P_n, p_{n,n+1})$, where P_n is a compact polyhedron and $p_{n,n+1}$: $P_{n+1} \to P_n$. Let p_n: $X \to P_n$ be the projection. We can assume that $X \cup \bigcup_{n \geq 1} P_n$ is contained in the Hilbert cube Q, with a metric d, so that each p_n is a $1/2^n$-translation (i.e. $d(x, p_n(x)) < 1/2^n$ for each $x \in X$). There exists a map f_n: $P_n \to S^1$ such that $f \underset{1/2^n}{=} f_n \circ p_n$. We may also assume that $f \cup \bigcup_{n \geq 1} f_n$: $X \cup \bigcup_{n \geq 1} P_n \to S^1$ is continuous.

Since $f \not\simeq 0$, there is an $N > 0$ such that $f_n \not\simeq 0$ for each $n \geq N$. As $\pi_i(S^1) = 0$ $(i \geq 2)$, we have $f_n|P_n^{(1)} \not\simeq 0$ ($P_n^{(1)}$ denotes the 1-skeleton of P_n). Hence there is a simple closed curve $S_n \subset P_n^{(1)}$ such that $f_n|S_n \not\simeq 0$. Notice that $\operatorname{Lim} S_n \subset X$.

To see that f has the WCLP, take any $\varepsilon > 0$ and $\zeta > 0$. Take sufficiently large n so that $S_n \subset N(X, \zeta/4)$, $1/2^n < \varepsilon/4, \zeta/4$. As $f_n|S_n \not\simeq 0$, by Lemma 3.6 there exists an $\eta > 0$ satisfying [WCLP($\varepsilon/4$, $\zeta/4$)] for $f_n|S_n$.

Noticing that p_n: $X \to P_n$ is a $\zeta/4$-translation, we can see that the above η satisfies [WCLP(ε, ζ)] for f.

The reverse implication follows from Lemma 3.7. This completes the proof.

Combining Theorem 3.5 and Proposition 2.2, we have

Theorem 3.8. *Any map which has the* WCLP *induces a monomorphism of the first Čech cohomology groups with integer coefficients.*

Corollary 3.9. *Let f: $X \to Y$ be a map from a continuum X onto a proper circle-like continuum Y. The following are equivalent.*

(1) *f has the* WCLP.
(2) *f^*: $\check{H}^1(Y) \to \check{H}^1(X)$ is a monomorphism.*
(3) $f^* \neq 0$.

Proof. (1)⇒(2) follows from Theorem 3.8. (2)⇒(3) is trivial.

(3)⇒(1). Let $Y = \varprojlim(S_n, q_{n,n+1})$, where each S_n is a simple closed curve and $q_{n,n+1}\colon S_{n+1}\to S_n \not\simeq 0$. Since $f^* \neq 0$, $q_n \circ f \not\simeq 0$ for sufficiently large n. To obtain (1), take any $\varepsilon > 0$ and $\zeta > 0$. Take a large n so that the projection $q_n\colon Y\to S_n$ is an $\varepsilon/4$-map and $q_n \circ f \not\simeq 0$. There exists a $\delta > 0$ such that $q_n^{-1}(A) < \varepsilon/2$ for each subset $A \subset S_n$ with $\operatorname{diam} A < \delta$. By Theorem 3.5, $q_n \circ f$ has the WCLP, so there exists a $\xi > 0$ satisfying [WCLP(δ, ζ)] for $q_n \circ f$. Finally, take an $\eta > 0$ such that if $d(x, y) < \eta$, $x, y \in Y$, then $d(q_n(x), q_n(y)) < \xi$. Then the η satisfies [WCLP(ε, ζ)] for f.

There exists a (topologically unique) circle-like continuum S which can be mapped onto any circle-like and arc-like continuum. The continuum S is hereditarily indecomposable and has the inverse limit representation $S = \varprojlim(S_n, p_{n,n+1})$, where each $S_n = S^1$ and $p_{n,n+1}\colon S_{n+1}\to S_n$. Further, for each prime number p, there exist infinitely many n's such that $p \mid \deg p_{n,n+1}$ (see [18]).

THEOREM 3.10. *Let S be the circle-like continuum as above. Each map from any continuum onto S has the* CLP.

Proof. Let $f\colon X\to S$ be a map and take any $\varepsilon > 0$ and any $\zeta > 0$. We proceed as in Theorem 3.3. Take a circular chain cover $C = \{C_1, \ldots, C_n\}$ of S with $\operatorname{mesh} C < \varepsilon$. Then $D = f^{-1}(C)$ is a circular chain cover of X. Let E be a finite open cover of X which is a proper refinement of D and $\operatorname{mesh} E < \zeta$. We can number the members of E, admitting repetitions, so that $E = \{E_1, \ldots, E_m\}$ is a circular weak chain (i.e. $E_i \cap E_j \neq \emptyset$ if $|i-j| \,(\mathrm{mod}\, m) \leqslant 1$). A cyclic pattern $\varphi\colon \{1, \ldots, m\}\to\{1, \ldots, n\}$ is defined by $E_i \subset D_{\varphi(i)}$.

Notice that $f^*\colon \check{H}^1(S)\to\check{H}^1(X)$ is a monomorphism because f is confluent ([7]). So taking a sufficiently small refinement of D if necessary, we may assume that $\deg\varphi \neq 0$ (the pattern φ is regarded as a simplicial map between simple closed curves). By the property of S which was stated above, there exist infinitely many pairs $m_i < n_i$ of integers such that $\deg\varphi \mid \deg q_{n_i m_i}$.

Hence by [1], Theorem 3.1 (or [4], Theorem 7), we can take a circular chain $F = \{F_1, \ldots, F_l\}$ which follows φ in C. Let $\eta > 0$ be the Lebesgue number of F such that

$$0 < \eta < \min\{d(\operatorname{cl} F_i, \operatorname{cl} F_j) \mid |i-j| \,(\mathrm{mod}\, l) \geqslant 2\}.$$

In the same way as in Theorem 3.3, we see that η is the required number.

4. Relations to span zero continua. First we recall the following results.

THEOREM 4.1 ([16], Theorem 7). *Let $f\colon X\to Y$ be a confluent map between hereditarily indecomposable continua X and Y. Suppose that $\sigma(X) = 0$. Then $\sigma(Y) = 0$ if and only if $f\times f\colon X\times X\to Y\times Y$ is confluent.*

THEOREM 4.2 ([5], Theorem 3.3). *Let $f\colon X\to Y$ be a map between continua and suppose that $\sigma(X) = 0$. Then the following statements are equivalent.*

(i) $\sigma(Y) = 0$.

(ii) *For each subcontinuum K of X, $(f|K)\times \mathrm{id}_P\colon K\times P\to f(K)\times P$ and $(f|K)\times\mathrm{id}_Y\colon K\times Y\to f(K)\times Y$ are weakly confluent.*

By the above results and Theorem 2.5, we have

THEOREM 4.3. *Let $f\colon X\to Y$ be a map and suppose that $\sigma(X) = 0$.*

(i) *If X is hereditarily indecomposable and f has the* BCLP, *then $\sigma(Y) = 0$.*

(ii) *If $f|K\colon K\to f(K)$ has the* WCLP *for each subcontinuum K of X, then $\sigma(Y) = 0$.*

Let G be a graph. The set of all branch points of G and the set of all end points of G are denoted by $B(G)$ and $E(G)$ respectively. If G is a tree, $B(G)\cup E(G)$ determines a natural triangulation of G which is denoted by T_G.

THEOREM 4.4. *Let $f\colon X\to Y$ be a map from a continuum X onto a tree-like continuum Y which has the* BCLP. *Let*

$$X = \varprojlim(X_n, p_{n,n+1}\colon X_{n+1}\to X_n) \quad \text{and} \quad Y = \varprojlim(Y_n, q_{n,n+1}\colon Y_{n+1}\to Y_n)$$

be inverse limit representations satisfying the following conditions.

(1) *Each X_n is a polyhedron.*

(2) *Each Y_n is a tree and there exists an integer $M > 0$ such that for each Y_n, there exists an arc $A_n \subset Y_n$ such that $\mathrm{st}^M(A_n, T_{Y_n})^* = Y_n$.*

Then for each Y_m and for each $\varepsilon > 0$, there exist an X_l, $n > m$, maps $f_{ml}\colon X_l\to Y_m$ and $s\colon Y_n\to X_l$ such that

(3) $q_m \circ f \underset{\varepsilon}{=} f_{ml}\circ p_l$,

(4) $q_{mn} \underset{\varepsilon}{=} f_{ml}\circ s$,

where $p_n\colon X\to X_n$ and $q_n\colon Y\to Y_n$ denote the projections.

Proof. Take any $\varepsilon > 0$ and m. There exist an X_l and a map $f_{ml}\colon X_l\to Y_m$ satisfying (3). By the simplicial approximation theorem, we may assume that $f_{ml}\colon (X_l, \sigma_l)\to(Y_m, \tau_m)$ is a simplicial map for suitable triangulations σ_l of X_l, τ_m of Y_m such that $\operatorname{mesh}\tau_m < \varepsilon/4$.

Define ξ, μ, ζ, and δ as follows.

(5) For all $x, y\in X_l$ with $d(x, y) < \xi$, there exist simplexes $s_1, s_2 \in \sigma_l$ such that $x\in s_1$, $y\in s_2$ and $s_1\cap s_2 \neq \emptyset$.

(6) For all $x, y\in Y_m$ with $d(x, y) < \mu$, there exist simplexes $t_1, t_2\in\tau_m$ such that $x\in t_1$, $y\in t_2$ and $t_1\cap t_2\neq\emptyset$.

(7) For all $a, b\in X$ with $d(a, b) < \zeta$, $d(p_l(a), p_l(b)) < \xi$.

(8) For all $a, b\in Y$ with $d(a, b) < \delta$, $d(q_m(a), q_m(b)) < \min(\mu, \varepsilon/4)$.

Inductively define $\eta_1, \eta_2, \ldots, \eta_M, \eta_{M+1}$ as follows.

(9)
$$\begin{aligned} &0 < \eta_1 < \delta \text{ and } \eta_1 \text{ satisfies [BCLP}(\delta, \zeta)],\\ &0 < \eta_2 < \eta_1 \text{ and } \eta_2 \text{ satisfies [BCLP}(\eta_1, \zeta)],\\ &0 < \eta_3 < \eta_2 \text{ and } \eta_2 \text{ satisfies [BCLP}(\eta_2, \zeta)],\\ &\vdots\\ &0 < \eta_{M+1} < \eta_M \text{ and } \eta_{M+1} \text{ satisfies [BCLP}(\eta_M, \zeta)].\end{aligned}$$

Take an $n > m$ such that $q_n\colon Y \to Y_n$ is an $\eta_{M+1}/4$-map. There exists a $\lambda > 0$ such that

(10) $\operatorname{diam} q_n^{-1}(S) < \eta_{M+1}/2$ for each $S \subset Y_n$ with $\operatorname{diam} S < \lambda$, and if $d(x, y) < \lambda$, $x, y \in Y_n$, then $d(q_{mn}(x), q_{mn}(y)) < \varepsilon/4$.

We now define a map $s\colon Y_n \to X_l$. Take the arc A_n as in the hypothesis (2). We may assume that A_n is a maximal arc.

Step 1. Take a λ-chain $\boldsymbol{y}$: $y_1 < y_2 < \ldots < y_{s-1} < y_s$ in A_n such that $B(Y_n) \cap A_n \subset \boldsymbol{y}$ and $E(A_n) = \{y_1, y_s\}$ ($<$ denotes a natural order on A_n). For each i, let $a_i \in q_n^{-1}(y_i)$; then $\boldsymbol{a}$: $a_1, \ldots, a_s$ is an η_{M+1}-chain in Y by (10). By (9), there exists a ζ-chain $\boldsymbol{c}$: $c_{11}, c_{12}, \ldots, c_{1k_1}, c_{21}, \ldots, c_{sk_s}$ in X such that

(11) $d(f(c_{ij}), a_i) < \eta_M < \delta$ for each $1 \leqslant i \leqslant s$ and $1 \leqslant j \leqslant k_i$.

By (7), $\boldsymbol{x}$: $x_{11}, x_{12}, \ldots, x_{sk_s}$ defined by $x_{ij} = p_l(c_{ij})$ is a ξ-chain. Let $\boldsymbol{y}'$: $y_1 = y_{11} < y_{12} < \ldots < y_{1k_1-1} < y_{1k_1} < y_2 = y_{21} < y_{22} < \ldots < y_{s-1,k_{s-1}} < y_s$ be a "refinement" of $\boldsymbol{y}$. Let $s|\boldsymbol{y}'$: $\boldsymbol{y}' \to X_l$ be defined by $s(y_{ij}) = x_{ij}$. Then

$$\begin{aligned} d(f_{ml}\circ s(y_{ij}), q_{mn}(y_{ij})) &= d(f_{ml}\circ p_l(c_{ij}), q_{mn}(y_{ij})) \\ &\leqslant d(f_{ml}\circ p_l(c_{ij}), q_m\circ f(c_{ij})) + d(q_m\circ f(c_{ij}), q_{mn}(y_{ij})) \\ &\leqslant \varepsilon/4 + d(q_m\circ f(c_{ij}), q_m(a_i)) + d(q_{mn}(y_i), q_{mn}(y_{ij})) \qquad ((5)) \\ &\leqslant \varepsilon/4+\varepsilon/4+\varepsilon/4 = 3\varepsilon/4 \qquad ((8), (10), (11)). \end{aligned}$$

Extend s linearly to A_n. Then by (9) and since $\operatorname{mesh}\tau_m < \varepsilon/4$, we see that $q_{mn} \underset{2\varepsilon}{=} f_{ml}\circ s|A_n$.

Step 2. Take any edge e of Y_n such that $e \cap A_n \neq \emptyset$. By the choice of $\boldsymbol{y}$, $e \cap A_n = \{y_{i(e)}\}$ for some $i(e)$. Take a λ-chain $\boldsymbol{y}^e$: $y_{i(e)} = y_1^e < y_2^e < \ldots < y_{s(e)}^e$ in e such that $E(e) = \{y_{i(e)}, y_{s(e)}^e\}$. Let $a_j^e \in q_n^{-1}(y_j^e)$, where $a_1^e = a_{i(e)} \in \boldsymbol{a}$. We have an η_{M+1}-chain $(\eta_{M+1} < \eta_M)$ $\boldsymbol{a}^e$: $a_{i(e)} = a_1^e, \ldots, a_{s(e)}^e$ in Y and the point $c_{i(e)1} \in X$ satisfies $d(f(c_{i(e)1}), a_1^e) < \eta_M$. By (9), there exists a ζ-chain $\boldsymbol{c}^e$: $c_{i(e)1} = c_{11}^e, c_{12}^e, \ldots, c_{1k_1}^e, c_{21}^e, \ldots$ $\ldots, c_{s(e)k_{s(e)}}^e$ in X such that $d(f(c_{ij}^e), a_i^e) < \eta_{M-1}$ for $1 \leqslant i \leqslant s(e)$, $1 \leqslant j \leqslant k_i$. By (7), $\boldsymbol{x}^e$: $x_{11}, \ldots, x_{tk_t}$ defined by $x_{ij} = p_l(c_{ij})$ is a ξ-chain in X_l.

The map $s|e$: $e \to X_l$ is defined as in step 1 and satisfies $f_{ml}\circ(s|e) \underset{2\varepsilon}{=} q_{mn}|e$.

Continuing the above process, we can define s on $\operatorname{st}(A_n, T_{Y_n})^*$ such that

$$f_{ml}\circ s|\operatorname{st}(A_n, Y_n)^* \underset{2\varepsilon}{=} q_{mn}|\operatorname{st}(A_n, Y_n)^*.$$

Since $\operatorname{st}^M(A_n, T_{Y_n}) = Y_n$, we can define s by repeating the above steps at most M times in which the condition (9) can be applied. This completes the proof.

PROPOSITION 4.5. *Let $f\colon X \to Y$ be a map onto a tree-like continuum Y which has the* BCLP. *Suppose that the inverse limit representation $Y = \varprojlim(Y_n, q_{n,n+1}\colon Y_{n+1} \to Y_n)$ by trees Y_n satisfies the condition* (2) *in the hypothesis of Theorem* 4.4.

(1) *If X is arc-like, then so is Y.*

(2) *If $\sigma(X) = 0$, then $\sigma(Y) = 0$.*

Proof. (1) Let $X = \varprojlim X_n$ be an inverse limit representation of X by arcs X_n. Applying Theorem 4.4 yields that for each Y_m and for each $\varepsilon > 0$, there exist an X_l, $n > m$, and maps $f_{ml}\colon X_l \to Y_m$, $s\colon Y_n \to X_l$ such that $q_{mn} \underset{\varepsilon}{=} f_{ml}\circ s$. From this fact, it is easy to see that Y is arc-like.

(2) Assume that $\sigma(X) = 0$ and let $X = \varprojlim X_n$ be an inverse limit representation of X by trees X_n. We may assume that both of $X \cup \bigcup_{n\geqslant 1} X_n$ and $Y \cup \bigcup_{n\geqslant 1} Y_n$ are imbedded in the Hilbert cube Q (with a metric d) such that

(3) Each projection $p_n\colon X \to X_n$ and $q_n\colon Y \to Y_n$ is a $1/n$-translation in Q (i.e. $d(p_n(x), x) < 1/n$ for each $x \in X$ etc.).

Then, by [9], Theorem 3.1, we have

(4) $\lim_n \sigma(X_n) = \sigma(X) = 0$.

To prove that $\sigma(Y) = 0$, we take an arbitrary pair of maps $\alpha, \beta\colon C \to Y$ from any continuum C to Y such that $\alpha(C) = \beta(C)$ and take any $\varepsilon > 0$. There exists an integer m such that

(5) $q_m\colon Y \to Y_m$ is an $\varepsilon/20$-translation.

There exists a $\delta > 0$ such that

(6) $d(f(x), f(y)) < \varepsilon/20$ for $x, y \in X$ with $d(x, y) < \delta$.

By Theorem 4.4 applied to $\varepsilon/20$ and m, there exist X_l, $n > m$ and maps $f_{ml}\colon X_l \to Y_m$, $s\colon Y_n \to X_l$ such that

(7) $q_m\circ f \underset{\varepsilon/20}{=} f_{ml}\circ p_l$,

(8) $q_{mn} \underset{\varepsilon/20}{=} f_{ml}\circ s$.

By the proof of Theorem 4.4, we may assume that

(9) $\sigma(X_l) < \delta/8$ and $p_l\colon X \to X_l$ is a $\delta/8$-translation in Q.

Note that

(10) $d(f_{ml}(x), f_{ml}(y)) < \varepsilon/4$ for $x, y \in X_l$ with $d(x, y) < \delta/8$.

Consider the maps $s\circ q_n\circ\alpha$ and $s\circ q_n\circ\beta\colon C \to X_l$. By (9), there exists a $p \in C$ such that $d(s\circ q_n\circ\alpha(p), s\circ q_n\circ\beta(p)) < \delta/8$. Then we have

$$\begin{aligned} d(\alpha(p), \beta(p)) &\leqslant d(\alpha(p), q_m\circ\alpha(p)) + d(q_m\circ\alpha(p), q_m\circ\beta(p)) + d(q_m\circ\beta(p), \beta(p)) \\ &< \varepsilon/20 + d(q_{mn}\circ q_n\circ\alpha(p), q_{mn}\circ q_n\circ\beta(p)) + \varepsilon/20 \qquad ((5)) \\ &< \varepsilon/10 + d(q_{mn}\circ q_n\circ\alpha(p), f_{ml}\circ s\circ q_n\circ\alpha(p)) \\ &\quad + d(f_{ml}\circ s\circ q_n\circ\alpha(p), f_{ml}\circ s\circ q_n\circ\beta(p)) \\ &\quad + d(f_{ml}\circ s\circ q_n\circ\beta(p), q_{mn}\circ q_n\circ\beta(p)) \\ &< \varepsilon/10+\varepsilon/20+\varepsilon/4+\varepsilon/20 < \varepsilon \qquad ((8) \text{ and } (10)). \end{aligned}$$

Since ε was arbitrarily chosen, $\sigma(Y) = 0$.

Finally, we consider the images of the pseudo-arc P under these maps.

THEOREM 4.6. *Let* $M = \mathrm{CL}$, *or* BCL, *or* WCL. *Let* X *be a continuum such that*

(1) *there exists a map* $f\colon P \to X$ *which belongs to* M *and*

(2) f *is irreducible, or* $\sigma(X) = 0$.

Then each map from any continuum onto X *belongs to* M.

Proof. Case 1. First we assume that f is irreducible. Let $a\colon Y \to X$ be any map and consider $H = \{(a(y), y) | y \in Y\}$. Since $f \times \mathrm{id}_Y\colon P \times Y \to X \times Y$ is weakly confluent by Theorem 2.6, there exists a continuum $K \subset P \times Y$ such that $f \times \mathrm{id}_Y(K) = H$. Clearly $\pi_P(K) = P$ and by the irreducibility of f, $\pi_Y(K) = Y$ (π_P and π_Y denote the projections). Further, $f \circ \pi_P = a \circ \pi_Y$ on K. Since $\pi_P \in \mathrm{BCL}$ by Theorem 3.3, $f \circ \pi_P \in M$ and by Proposition 2.2(ii), $a \in M$.

Case 2. Next we assume that $\sigma(X) = 0$. We will prove that for each subcontinuum $Q \subset P$ such that $f(Q) = X$, $f|Q\colon Q \to X$ also belongs to M. The required conclusion follows from this fact and Case 1.

Consider the maps $f|Q\colon Q \to X$ and $f\colon P \to X$. By [5], Theorem 1.3, there are a continuum Z and maps $a\colon Z \to Q$, $b\colon Z \to P$ such that $(f|Q) \circ a = f \circ b$. Since $b \in \mathrm{BCL}$, $f|Q \in M$ as in Case 1.

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Received 7 November 1989;
in revised form 5 June 1990