We close with some questions:
1. Does Corollary 15 hold with no set-theoretic assumptions?
2. Does Corollary 15 hold under MA + ¬CH?
3. Is there a generalization of Corollary 15 to higher cardinals?
4. Given an index set $I$ and an ultrafilter $\mathcal{U}$ on $I$, is it possible to determine the cardinality of the set of prime ideals between $\langle \mathcal{U} \rangle$ and $\langle \mathcal{F} \rangle$ (perhaps assuming some additional set-theoretic hypothesis)?

References


On some subclasses of Darboux functions

by

J. M. Jastrzębski (Gdańsk), J. M. Jędrzejewski and T. Natkaniec (Bydgoszcz)

Abstract. The maximal additive, multiplicative and lattice-like classes for some classes of real functions are computed.

1. Introduction. We shall consider mainly real functions of a real variable, however, some of the considered functions will be defined on different sets, and sometimes the range sets will be different. Let us settle some of the notations to be used in the article.

$\mathcal{C}onst$ — the class of constant functions,
$\mathcal{C}on$ — the class of connected functions,
$\mathcal{C}$ — the class of continuous functions,
$\mathcal{A}$ — the class of almost continuous functions,
$\mathcal{D}$ — the class of Darboux functions,
$\mathcal{D} \cap \mathcal{B}$ — the class of Darboux functions of the first class of Baire,
$\mathcal{F}$ — the class of functionally connected functions ([5]),
$\mathcal{L}(suc)$ — the class of lower (upper) semicontinuous functions,
$\mathcal{M}$ — the class of Darboux functions $f$ with the following property: if $x_0$ is a right-hand (left-hand) point of discontinuity of $f$, then $f(x_0) = 0$ and there is a sequence $(x_n)$ converging to $x_0$ such that $x_n > x_0$ ($x_n < x_0$) and $f(x_n) = 0$,
$\mathcal{D}_0$ — the class of all functions $f$ such that for each $x$ from the domain $f(x) \in L^-(f, x) \cap L^+(f, x)$ and the sets $L^-(f, x), L^+(f, x)$ are closed intervals.

The symbols $L^-(f, x), L^+(f, x)$ denote the cluster sets from the left and right, respectively, of the function $f$ at the point $x$.

Notice that if $f \in \mathcal{M}$, then the set $E$ of all points of discontinuity of $f$ is nowhere dense and $f(x) = 0$ for each $x$ in $E$. Consequently, $f$ is a function of the first class of Baire, hence $\mathcal{M} \subseteq \mathcal{D} \cap \mathcal{B}$. Since $\mathcal{D} \cap \mathcal{B} = \mathcal{D} \cap \mathcal{B} = \mathcal{D} \cap \mathcal{B} = \mathcal{D} \cap \mathcal{B}$, we have $\mathcal{M} \subseteq \mathcal{D}$. Thus for the classes of real functions defined on an interval we have

$\mathcal{C} \subseteq \mathcal{A} \subseteq \mathcal{C} \subseteq \mathcal{C} \subseteq \mathcal{D} \subseteq \mathcal{D} \subseteq \mathcal{D} \subseteq \mathcal{D} \subseteq \mathcal{D} \subseteq \mathcal{D}$.

Let $\mathcal{F}$ be a class of real functions. The maximal additive (multiplicative, lattice-like, respectively) class for $\mathcal{F}$ is defined to be the class of all $f \in \mathcal{F}$ for which $f + g \in \mathcal{F}$ (if $g \in \mathcal{F}$, $\max(f, g)$ and $\min(f, g) \in \mathcal{F}$, respectively) whenever $g \in \mathcal{F}$. The respective classes are denoted by $\mathcal{M}_a(\mathcal{F}), \mathcal{M}_m(\mathcal{F}), \mathcal{M}(\mathcal{F})$. 
Moreover, let
\[
\mathcal{M}_{\min}(\mathcal{F}) = \{ f \in \mathcal{F} : \text{ if } g \in \mathcal{F}, \text{ then min}(f, g) \in \mathcal{F} \},
\]
\[
\mathcal{M}_{\max}(\mathcal{F}) = \{ f \in \mathcal{F} : \text{ if } g \in \mathcal{F}, \text{ then max}(f, g) \in \mathcal{F} \}.
\]
Then \( \mathcal{M}(\mathcal{F}) = \mathcal{M}_{\max}(\mathcal{F}) \cap \mathcal{M}_{\min}(\mathcal{F}) \).

For real functions of a real variable we know that
\[
\mathcal{M}_{\min}(\mathbb{R}_{\text{mon}}) = \mathcal{M}_{\min}(\mathcal{F}) = \mathcal{M}_{\min}(\mathcal{F} \cap \mathcal{B}) = \mathcal{C}, \quad \text{(see [2], [5]),}
\]
\[
\mathcal{M}_{\max}(\mathbb{R} \cap \mathcal{B}) = \mathcal{C}, \quad \mathcal{M}_{\max}(\mathcal{F}) = \mathcal{M}_{\max} = \mathcal{C}, \quad \text{[9].}
\]

In the present article we shall prove the following:
\[
\mathcal{M}_{\min}(\mathcal{F}) = \mathcal{M}_{\min}(\mathbb{R}_{\text{mon}}) = \mathcal{M}_{\min}(\mathcal{F}) = \mathcal{C},
\]
\[
\mathcal{M}_{\max}(\mathcal{F}) = \mathcal{M}_{\max}(\mathcal{F}) = \mathcal{C},
\]
\[
\mathcal{M}_{\max}(\mathcal{F}) = \mathcal{C}. \quad \text{(compare [3]).}
\]

We omit some proofs that can be left to the reader.

2. Some basic lemmas

**Lemma 2.1.** Let \( \Phi \) be some property of functions and \( X \) a topological space. Let \( \mathcal{F} \) be the class of all functions \( f : X \to \mathbb{R} \) with property \( \Phi \) and let \( \mathcal{F}_1 \) be the class of all functions \( g : X \to \mathbb{R} \times \mathbb{R} \) with property \( \Phi \). Suppose the classes \( \mathcal{F}_1, \mathcal{F}_2 \) fulfill the following conditions:

1. (i) If \( g \in \mathcal{F}_1 \) and \( k \in \mathcal{B} \), then \( h \circ g \in \mathcal{F}_1 \),

2. (ii) If \( f \in \mathcal{F}_1 \), \( g \in \mathcal{F}_2 \), \( X \to \mathbb{R} \), then \( k = (f, g) \in \mathcal{F}_2 \), where \( k(x) = (f(x), g(x)) \) for \( x \in X \).

Then \( \mathcal{F} \subseteq \mathcal{M}_{\min}(\mathcal{F}_1) \cap \mathcal{M}_{\max}(\mathcal{F}_2) \).

**Lemma 2.2.** Let \( \mathcal{F}_1 \subseteq \mathcal{D}_0 \), fulfill the following conditions:

1. (i) If \( f : \mathbb{I} \to \mathbb{R}, f \in \mathcal{F}_1 \), \( J \) is a subinterval of an interval \( I \), then \( f|J \in \mathcal{F}_1 \),

2. (ii) If \( h : (a, b) \to \mathbb{R}, h \in \mathcal{L}^+(h, a), z \in L^{-}(h, b) \) then the functions \( h_1 : [a, b] \to \mathbb{R}, h_2 : (a, b) \to \mathbb{R} \) and \( h_3 : [a, b] \to \mathbb{R} \) belong to \( \mathcal{F} \), where \( h_1 = h \cup \{(a, y)\}, h_2 = h \cap \{(b, z)\}, h_3 = h_1 \cup h_2 \),

3. (iii) If \( I \subseteq \mathbb{R} \) is an interval, \( a \in \mathbb{D} \), and \( f|I \cap (-\infty, a) \in \mathcal{F}, f|I \cap [a, +\infty) \in \mathcal{F} \), then \( f \in \mathcal{F} \).

Then

(i) \( \mathcal{M}(\mathcal{F}) \subseteq \mathcal{C} \),

(ii) \( \mathcal{M}_{\min}(\mathcal{F}) \subseteq \mathcal{D} \cap \mathrm{lsc} \) and \( \mathcal{M}_{\max}(\mathcal{F}) \subseteq \mathcal{D} \cap \mathrm{usc} \) (hence \( \mathcal{M}(\mathcal{F}) \subseteq \mathcal{C} \)).

If moreover the class \( \mathcal{F} \) fulfills the additional condition

(2.5) \( f : \mathbb{I} \to (0, \infty) \) and \( f \in \mathcal{F} \), then \( 1/f \in \mathcal{F} \),

then also

(iii) \( \mathcal{M}(\mathcal{F}) \subseteq \mathcal{C} \).

Proof. (i) Assume that \( f \in \mathcal{F} \) and suppose that \( f \) is not right-continuous at \( x_0 \). Since \( \mathcal{F} \subseteq \mathcal{D}_0 \) we have

\[
f(x_0) \in \mathcal{L}^{-}(f, x_0) \cap \mathcal{L}^{+}(f, x_0).
\]

Suppose that

\[
f(x_0) < \limsup_{x \to x_0} f(x).
\]

Let

\[
c = \begin{cases} f(x_0) + \limsup_{x \to x_0} f(x)/2 & \text{if } \limsup_{x \to x_0} f(x) < \infty, \\ f(x_0) + 1 & \text{if } \limsup_{x \to x_0} f(x) = \infty, \end{cases}
\]

\[
g(x) = \begin{cases} f(x) & \text{for } x \leq x_0, \\ 2c - f(x) & \text{for } x > x_0. \end{cases}
\]

Notice that in view of (2.4) the function \( g \) is \( x_0, \infty \) belongs to \( \mathcal{F} \). Since \( f(x_0) \in \mathcal{L}^{+}(g, x_0) \), according to (2.2) and (2.3) we infer that \( g \in \mathcal{F} \). Moreover, \( f + g(x_0) = 2f(x_0) < 2c \) and \( f + g(x_0) = 2c \) for \( x > x_0 \). Thus \( f + g \notin \mathcal{D}_0 \).

If \( f(x_0) > \liminf_{x \to x_0} f(x) \) the proof is quite the same.

(ii) Assume that \( f \) is not upper semicontinuous at \( x_0 \) from the right, i.e. \( f(x_0) \in \limsup_{x \to x_0} f(x) \), for some \( x_0 \in \mathbb{R} \). Let \( c \) be defined by (**) and define \( g \) by (**) as above. Then \( \max(f, g)(x_0) = f(x_0) < c \) and \( \max(f, g)(x) \geq c \) for \( x > x_0 \), hence \( \max(f, g) \notin \mathcal{D}_0 \) and consequently \( f \notin \mathcal{M}(\mathcal{F}) \).

In the same way we can prove that \( \mathcal{M}_{\min}(\mathcal{F}) \subseteq \mathcal{D}_0 \cap \mathrm{lsc}, \) and since \( \mathcal{D}_0 \cap \mathcal{F} = \mathcal{D} \cap \mathcal{F} \) (see e.g. [2]), \( \mathcal{M}_{\max}(\mathcal{F}) \subseteq \mathcal{D} \cap \mathrm{lsc}. \)

(iii) Let \( f \in \mathcal{F} \) and suppose that \( f \) is discontinuous at some point \( x_0 \), say from the right. Assume that there exists \( d > 0 \) for which \( f(x) \neq 0 \) for \( x \in (x_0, x_0 + d) \). Let \( c \neq 0 \), \( c \neq f(x_0) \), be a point from \( \mathcal{L}^{-}(f, x_0) \). Define \( g : \mathbb{R} \to \mathbb{R} \) by

\[
g(x) = \begin{cases} 1/c & \text{for } x \leq x_0, \\ 1/f(x) & \text{for } x \in (x_0, x_0 + d), \\ 1/(f(x_0 + d) & \text{for } x > x_0 + d. \end{cases}
\]

Then \( 1/c \in \mathcal{L}^{+}(1/f, x_0) \). We infer from (2.5) that \( g|\mathcal{F} \in \mathcal{F} \), and from (2.2), (2.3) that \( g \in \mathcal{F} \).
Notice that $fg(x_0) = f(x_0)/c \neq 1$ and $fg(x) = 1$ for $x \in [x_0, x_0 + d]$. Thus $fg \notin \mathcal{M}_0$ and hence $f \notin \mathcal{M}_0$. Thus there exists a sequence $(x_n)$ such that $x_n > x_0$, $x_n \to x_0$, and $f(x_n) = 0$. Let $f(x_0) = a$ and suppose that $a \neq 0$. Then we define
\[
g(x) = \begin{cases} 2a - f(x) & \text{for } x > x_0, \\ 2a & \text{for } x \leq x_0. \end{cases}
\]
It follows from (2.2)–(2.4) that $g \notin \mathcal{I}$. On the other hand, $fg(x_0) = 2a^2$ and, for $x > x_0$, \[
fg(x) = (2a - f(x)) x < a^2 < 2a^2.
\]

3. Almost continuity. If $X, Y$ are two topological spaces, then a function $f : X \to Y$ is almost continuous if each open subset $G$ of $X \times Y$ containing the function $f$ contains a function continuous on $X$ (here a function and its graph coincide).

**Lemma 3.1.** If $f : X \to Y$ is continuous and $g : X \to Z$ is almost continuous, then $h : X \to Y \times Z$ defined by $h = (f, g)$ is almost continuous.

(See [3] for metric spaces. The proof for topological spaces needs no change.)

**Lemma 3.2.** If $f : X \to Y$ is almost continuous and $g : Y \to Z$ is continuous, then $g 
 f$ is almost continuous.

**Corollary 3.1.** If $f : X \to Y$ is almost continuous and $g : S \to Y$ and $F : X \times Y \to Z$ are continuous, then $F f$ : $f$ is almost continuous. Since addition, multiplication, min, max are continuous functions of two variables, this implies $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A}) \cap \mathcal{M}(\mathcal{A}) \cap \mathcal{M}(\mathcal{A}) \cap \mathcal{M}(\mathcal{A})$.

**Remark.** There are almost continuous functions $f : R \to R$, $g : R \to R$ such that $h = (f, g)$ from $R$ into $R^2$ is not almost continuous.

**Proof.** By [7] there are almost continuous functions $f$ and $g$ with $f + g$ not almost continuous. For such pairs of functions the transformation $(f, g)$ is not almost continuous.

**Lemma 3.3.** If $I = [a, b]$, $f : I \to R$ is almost continuous, and $G \subseteq I \times R$ is an open neighbourhood of $f$, then there is a continuous function $g : I \to R$ such that $g \subseteq G$, $g(a) = f(a)$, and $g(b) = f(b)$.

**Proof.** Let $U_a \times V_a \subseteq G$, $U_b \times V_b \subseteq G$ be neighbourhoods of $(a, f(a))$, $(b, f(b))$, respectively. Choose $x_0 \in U_a \setminus \{a\}$, $x_1 \in U_b \setminus \{b\}$ such that $f(x_1) \in V_a$, $f(x_2) \in V_b$. Then $G_1 = G_1 \setminus \{f(x_1) \times (R, V_b) \cup \{(x_2) \times (R, V_a)\}\}$

is an open neighbourhood of $f$, hence there is a continuous function $h : I \to R$ such that $h \subseteq G_1$. Now define $g$ as follows:
\[
g(x) = \begin{cases} h(x) & \text{for } x \in [x_1, x_2], \\ f(x) & \text{for } x \in [a, b], \end{cases}
\]
linear on each of the intervals $[a, x_1]$, $[x_2, b]$.

This function fulfills the requirements of the lemma.

**Lemma 3.4.** If $h : (a, b) \to R$ is almost continuous, $y \in \mathcal{L}^n(h, a) \subseteq \mathcal{L}^n(h, b)$, then the functions $h_1 = h \cup \{(a, y)\}$, $h_2 = h \cup \{(b, y)\}$, $h_3 = h_1 \cup h_2$, are almost continuous.

**Proof.** Let $I = [a, b]$. Let $G \subseteq I \times R$ be an open neighbourhood of $f$. In view of Lemma 3.3, for each $n$ there is a continuous function $g_n : I \to R$ such that $g_n \subseteq G \cap \{(a, y)\}$, $g_n(a) = f(a)$, $g_n(b) = f(b)$.

Then $g = \bigcup_{n=1}^{\infty} g_n$ is a function that is continuous and contained in $G$.

**Theorem 3.1.** $\mathcal{M}_0(\mathcal{A}) \subseteq \mathcal{I}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})$.

**Proof.** By Corollary 3.1 we have
\[
\mathcal{M}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A}) \cap \mathcal{M}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A}) \cap \mathcal{M}(\mathcal{A}) \cap \mathcal{M}(\mathcal{A}) \cap \mathcal{M}(\mathcal{A})
\]

Notice that the class of the functions fulfills the conditions (2.1)–(2.5) from Lemma 2.2. Hence $\mathcal{M}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A}) \subseteq \mathcal{M}(\mathcal{A})
\]

therefore $\mathcal{M}(\mathcal{A}) = \mathcal{M}(\mathcal{A})$.

It now suffices to prove that $\mathcal{M} = \mathcal{M}(\mathcal{A})$.

Let $f \in \mathcal{M}$. In view of Lemma 3.5 one can assume $f$ to be defined on a closed interval $I = [a, b]$, where $f(a) \neq b \neq f(b)$. Let $g : I \to R$ be an almost continuous function and $G$ an open set in $I \times R$ such that $f \subseteq G$. Put $E = \{x \in I : f(x) = 0\}$. This set is closed and contains the set of all points of discontinuity of $f$. For each $x \in E$ we take intervals $U_x \subseteq I$ and $V_x \subseteq R$ such that
\[
(x, 0) \subseteq U_x \times V_x \subseteq U \times \{0\} \subseteq G.
\]

There is a finite sequence of intervals $U_{x_1}, \ldots, U_{x_n}$ such that $E \subseteq \bigcup_{i=1}^{n} U_{x_i}$. Now let $V = \bigcup_{i=1}^{n} V_{x_i}$. Let $W_1, \ldots, W_n$ be the components of the set $\bigcup_{i=1}^{n} U_{x_i}$ such that $x \in W_i$ for $i < j$ then $x < y$. Notice that $E \subseteq \bigcup_{i=1}^{n} W_i \cap W_i \cap \{0\} \subseteq \emptyset$ for $i \neq j$. Moreover, the end-points of the intervals $W_i$ do not belong to $E$.

For each $i = 1, \ldots, n$ choose a closed interval $[a_{x_i}, a_{x_{i+1}}] \subseteq W_i$ such that $E \cap \bigcup_{i=1}^{n} W_i \cap \{a_{x_i}, a_{x_{i+1}}\}$ is continuous. Since $f_{[a_{x_i}, a_{x_{i+1}}]}$ is almost continuous, the product $f_{[a_{x_i}, a_{x_{i+1}}]}$ is almost continuous too. Since $f_{[a_{x_i}, a_{x_{i+1}}]} \subseteq G$ there exist continuous functions $h_i : [a_{x_i}, a_{x_{i+1}}] \to R$ such that $h_i \subseteq G$, $h_i(a_{x_i}) = h_i(a_{x_{i+1}}) = 0$. Therefore the function $h : I \to R$ defined by
\[
h = \bigcup_{i=1}^{n} h_i \cup \bigcup_{i=1}^{n} (a_{x_i}, a_{x_{i+1}}) \times \{0\}
\]

is contained in $G$ and continuous, which completes the proof.
4. Connectivity. If $X, Y$ are two topological spaces, then a function $f : X \to Y$ is called connected provided $f$ is a connected subset of $X \times Y$. The next two lemmas follow from the fact that the continuous image of a connected set is connected.

**Lemma 4.1.** Let $X, Y, Z$ be topological spaces. If $f : X \to Y$ is a connected function and $g : Y \to Z$ is continuous, then $g \circ f$ is connected.

**Lemma 4.2.** If $f : X \to Y$ is a connected function, $g : X \to Z$ is a continuous function, then $h = (f; g) : X \times Y \to Z$ is connected.

**Corollary 4.1.** If $F : X \times Y \to Z$ is a continuous function, $g : S \to Y$ is connected, $f : S \to X$ is continuous, then $F_0(f; g) : S \to Z$ is a connected function.

In particular, the following inclusions hold:

$\mathcal{M}_n(\mathcal{M}_n) \subseteq \mathcal{M}_n(\mathcal{M}_n) \cap \mathcal{M}_\infty(\mathcal{M}_n) \cap \mathcal{M}_\infty(\mathcal{M}_n)$

**Lemma 4.3.** If $h : (a, b) \to \mathbb{R}$ is a connected function, $y \in L^+(h, a), z \in L^-(h, b)$, then the functions $h_1, h_2, h_3$ are connected, where $h_1 = h \cup \{a, y\}, h_2 = h \cup \{b, z\}, h_3 = h_1 \cup h_2$.

**Lemma 4.4.** If $I \subseteq \mathbb{R}$ is an interval and $L$ is the union of a finite or countable sequence $(I_k)_k$ of closed intervals such that $\bigcup_{k \in \mathbb{N}} I_k$ is connected for every $k$, and a function $f|_I$ is continuous for every $n$, then $f|_I$ is connected.

**Theorem 4.1.** $\mathcal{M}_n(\mathcal{M}_n) = \mathcal{M}_n(\mathcal{M}_n) \subseteq \mathcal{M}_\infty(\mathcal{M}_n) \subseteq \mathcal{M}_\infty(\mathcal{M}_n) \subseteq \mathcal{M}_\infty(\mathcal{M}_n)$

Proof. First, notice that the class $\mathcal{M}_n$ fulfills the conditions (2.1)–(2.5) of Lemma 2.2. Thus $\mathcal{M}_n(\mathcal{M}_n) \subseteq \mathcal{M}_\infty(\mathcal{M}_n) \subseteq \mathcal{M}_\infty(\mathcal{M}_n) = \mathcal{M}_\infty(\mathcal{M}_n)$

According to Lemma 4.1 we infer that $\mathcal{M}_n(\mathcal{M}_n) = \mathcal{M}_n(\mathcal{M}_n)$ (compare [5]). Only the inclusion $\mathcal{M}_\infty(\mathcal{M}_n) \subseteq \mathcal{M}_\infty(\mathcal{M}_n)$ has to be proved. By Lemma 4.4 it suffices to consider functions $f_0, \mathcal{M}_n$ defined on a closed interval $I = [a, b]$ with $a, b \notin D$, where $D$ denotes the set of all points of discontinuity of $f$.

Suppose that there exists a connected function $g : I \to \mathbb{R}$ such that $g|_I$ is not connected. Let $(A, B)$ be a partition of $g|_I$ and suppose that $(a, f(a)) \in A, (b, f(b)) \in B$. Let now

$A_1 = \{x \in I; x, f(x) \in A\}, B_1 = \{x \in I; x, f(x) \in B\}$

and $c = \sup A_1$. Of course, $c < b$. Notice that if $J$ is a component of $\bigcap I \setminus D$, then either $J \subseteq A_1$, or $J \subseteq B_1$, for if not then $g|_J$ would not be connected. Thus $c \in D$. Moreover, $(c, f(c)) \notin D \cap B$. Indeed, either $c$ is an end-point of a component of the set $\bigcap I \setminus D$ that is contained in $A_1$, and hence $c \notin A_1$, or there exists a sequence $(J_n)$ of components of the set $\bigcap I \setminus D$ that are contained in $A_1$ and such that $\bigcap J_n = A_1$. Therefore there exists a sequence $(x_n)$ of points $x_n \in A_1$ such that $f(x_n) = 0$ and $x_n \to c$. Then $(x_n) \subset A$ for $n = 1, 2, \ldots$, and $(c, 0) \notin A$. Similarly one can prove that $(A \cap B) \cup (A \cap B) = \emptyset$.

5. Functional connectedness. Assume now that $I$ is an interval. A function $f : I \to \mathbb{R}$ is called functionally connected provided for each $[a, b] \subseteq I$ and each continuous function $g : [a, b] \to \mathbb{R}$, if $(f - g)(0) > 0$, then there exists a point $c \in (a, b)$ for which $f(c) = g(c)$.

**Lemma 5.1.** If $I$ is a finite or countable union of a sequence $(I_n)_n$ of closed intervals such that $\bigcup_{n \in \mathbb{N}} I_n$ is connected for every $k$, and a function $f|_I$ is functionally connected, then $f|_I$ is functionally connected.

**Lemma 5.2.** If $I$ is a finite or countable union of a sequence $(I_n)_n$ of closed intervals such that $\bigcup_{n \in \mathbb{N}} I_n$ is connected for every $k$, and a function $f|_I$ is functionally connected, then $f|_I$ is functionally connected.

**Lemma 5.3.** If $h : (a, b) \to \mathbb{R}$ is functionally connected, $y \in L^+(h, a), z \in L^-(h, b)$, then the functions $h_1, h_2, h_3$ are functionally connected, where $h_1 = h \cup \{a, y\}, h_2 = h \cup \{b, z\}, h_3 = h_1 \cup h_2$.

**Lemma 5.4.** (a) If $f : I \to \mathbb{R}$ is functionally connected and both $y : I \to (0, +\infty)$ and $h : I \to (0, +\infty)$ are continuous, then $f_h, f_y$ are functionally connected.

(b) If $f : I \to (0, +\infty)$ is functionally connected, then so is $1/f$.

**Lemma 5.5.** Let $C$ be a nowhere dense subset of an interval $I$. Let $(I_n)_n$ be the sequence of all components of the set $\overline{C}$.

**Theorem 5.1.** If $f : I \to \mathbb{R}$ fulfills the conditions

(i) $f|_I$ is functionally connected for each $n \in \mathbb{N}$,

(ii) $f(x) = 0$ on $L^0(f, x) \cap L^1(f, x)$ for $x \in \mathbb{C}$,

then $f$ is functionally connected.

Proof. Let $a, b \in I$ and let $h : I \to \mathbb{R}$ be a continuous function such that $h(a) < f(a)$ and $f(b) < h(b)$.

By Lemma 5.2, with no loss of generality we can assume that $a, b \in C$. Hence $h(a) < f(a) = 0 = f(b) < h(b)$.

Notice that $(a, b)$ is a non-empty compact set.

If $h(x) = 0$ for some $x \in C \cap (a, b)$, then we are done. Assume now that $h(x) \neq 0$ for all $x \in C \cap (a, b)$.

Then $\bigcup_{n \in \mathbb{N}} I_n = \mathbb{C}$ and the sequence $(I_n \cap (a, b); h(x) = 0)$ is a finite set. Let $I_{n_1}, \ldots, I_{n_m}$ be the sequence of...
all components which meet the set \( \{ x \in \mathbb{A} : h(x) = 0 \} \) and for which
\[
\sup I_m < \inf I_m \quad \text{if } i < j.
\]
Let \( I_m = (a_m, b_m) \) for \( m \leq k \) and \( n_0 = \min \{ m \leq k : h(b_m) > 0 \} \). Notice that \( h(b_m) > 0 \) and \( h(a_m) < 0 \). Since \( f\big|_{I_{n_0}} \) is functionally connected, so is \( f\big|_{I_m} \) and consequently \( h(x) = f(x) \) for some \( x \in I_{n_0} \), which completes the proof.

**Theorem 5.1.** \( \mathcal{M}(\mathcal{F}) = \mathcal{G} = \mathcal{M}(\mathcal{F}_0) \), \( \mathcal{M}_\alpha(\mathcal{F}) = \mathcal{M}_\beta(\mathcal{F}) = \mathcal{D} \cap \text{lsc} \), \( \mathcal{M}_\text{max}(\mathcal{F}) = \mathcal{D} \cap \text{usc} \).

**Proof.** For the equality \( \mathcal{M}(\mathcal{F}) = \mathcal{G} \) see [5]. To prove the other equalities notice that \( \mathcal{F} \) fulfills the conditions (2.1)--(2.5) of Lemma 2.2. Thus
\[
\mathcal{M}_\alpha(\mathcal{F}) \subseteq \mathcal{M}, \quad \mathcal{M}_\text{min}(\mathcal{F}) \subseteq \mathcal{D} \cap \text{lsc}, \quad \mathcal{M}_\text{max}(\mathcal{F}) \subseteq \mathcal{D} \cap \text{usc}.
\]
First we shall show that \( \mathcal{M}_\text{max}(\mathcal{F}) = \mathcal{D} \cap \text{lsc} \) and \( \mathcal{M}_\text{min}(\mathcal{F}) = \mathcal{D} \cap \text{usc} \). Let \( f, g, h \) be real functions defined on an interval \( I \) such that \( f \in \mathcal{F} \), \( g \in \mathcal{D} \cap \text{lsc} \), \( h \in \mathcal{F} \) and
\[
(\max(f, g)(a) - h(a))(\max(f, g)(b) - h(b)) < 0,
\]
for some \( a, b \in I \). Since \( \max(f, g) - h = \max(f-h, g-h) \) and \( g, h \in \mathcal{D} \cap \text{usc} \), the function \( k = \max(f, g) - h \) has the Darboux property (see [5]). Since \( k(a)k(b) < 0 \), there exists \( c \in [a, b] \) such that \( k(c) = 0 \). Hence \( \max(f, g)(c) = h(c) \) and, consequently, \( \max(f, g) \) is functionally connected.

In the analogous way we prove that \( \mathcal{D} \cap \text{usc} \subseteq \mathcal{M}_\text{min}(\mathcal{F}) \). Hence
\[
\mathcal{M}_\text{min}(\mathcal{F}) = \mathcal{D} \cap \text{usc}, \quad \mathcal{M}_\text{max}(\mathcal{F}) = \mathcal{D} \cap \text{lsc}, \quad \mathcal{M}(\mathcal{F}) = \mathcal{G}.
\]
Now we shall prove that \( \mathcal{M} \subseteq \mathcal{M}(\mathcal{F}) \). Let \( f, g \) be real functions defined on \( I \) such that \( f \in \mathcal{F} \) and \( g \in \mathcal{M} \). Then the set \( E \) of all points of discontinuity of \( g \) is nowhere dense and the set
\[
B = \{ x \in I : g(x) = 0 \}
\]
is closed. Let the sequence \( (I_m) \) be the union of the set of all components of \( \text{Int}(B) \) and the set of all components of \( I \setminus B \); moreover, let \( C = I \setminus \bigcup I_m \). Thus \( C \) is a closed, nowhere dense set contained in \( B \) and all the assumptions of Lemma 5.6 for the function \( fg \) are satisfied. Consequently, \( fg \) is functionally connected.

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**References**


**Institute of Mathematics**

**University of Gdańsk**

Wita Stwosza 57, 80-952 Gdańsk

**Pedagogical University of Bydgoszcz (WSP)**

Chelmowskiego 30, 85-064 Bydgoszcz

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