We close with some questions:

1. Does Corollary 15 hold with no set-theoretic assumptions?
2. Does Corollary 15 hold under MA+$\neg$CH?
3. Is there a generalization of Corollary 15 to higher cardinals?
4. Given an index set I and an ultrafilter $\mathscr{U}$ on I, is it possible to determine the cardinality of the set of prime ideals between $(\mathscr{U})$ and $(\mathscr{F})$ (perhaps assuming some additional set-theoretic hypothesis)?

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On some subclasses of Darboux functions

by

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Abstract. The maximal additive, multiplicative and lattice-like classes for some classes of real functions are computed.

1. Introduction. We shall consider mainly real functions of a real variable, however, some of the considered functions will be defined on different sets, and sometimes the range sets will be different. Let us settle some of the notations to be used in the article.

$\mathscr{C}onst$ – the class of constant functions,
$\mathscr{C}on$ – the class of connected functions,
$\mathscr{C}$ – the class of continuous functions,
$\mathscr{A}$ – the class of almost continuous functions,
$\mathscr{D}$ – the class of Darboux functions,
$\mathscr{D}\cap\mathscr{B}_1$ – the class of Darboux functions of the first class of Baire,
$\mathscr{F}$ – the class of functionally connected functions ([5]),
lsc (usc) – the class of lower (upper) semicontinuous functions,
$\mathscr{M}$ – the class of Darboux functions f with the following property: if x_0 is a right-hand (left-hand) point of discontinuity of f, then $f(x_0)=0$ and there is a sequence (x_n) converging to x_0 such that $x_n>x_0$ $(x_n<x_0)$ and $f(x_n)=0$,
$\mathscr{D}_0$ – the class of all functions f such that for each x from the domain $f(x)\in L^-(f,x)\cap L^+(f,x)$ and the sets $L^-(f,x)$, $L^+(f,x)$ are closed intervals.

The symbols $L^-(f,x)$, $L^+(f,x)$ denote the cluster sets from the left and right, respectively, of the function f at the point x.

Notice that if $f\in\mathscr{M}$, then the set E of all points of discontinuity of f is nowhere dense and $f(x)=0$ for each x in $\bar{E}$. Consequently, f is a function of the first class of Baire, hence $\mathscr{M}\subseteq\mathscr{D}\cap\mathscr{B}_1$. Since $\mathscr{A}\cap\mathscr{B}_1=\mathscr{C}on\cap\mathscr{B}_1=\mathscr{D}\cap\mathscr{B}_1$ ([1]), we have $\mathscr{M}\subseteq\mathscr{A}$. Thus for the classes of real functions defined on an interval we have

$$\mathscr{C}\subseteq\mathscr{M}\subseteq\mathscr{A}\subseteq\mathscr{C}on\subseteq\mathscr{F}\subseteq\mathscr{D}\subseteq\mathscr{D}_0.$$

Let $\mathscr{X}$ be a class of real functions. The *maximal additive* (*multiplicative*, *lattice-like*, respectively) *class* for $\mathscr{X}$ is defined to be the class of all $f\in\mathscr{X}$ for which $f+g\in\mathscr{X}$ $(fg\in\mathscr{X}$, $\max(f,g)$ and $\min(f,g)\in\mathscr{X}$, respectively) whenever $g\in\mathscr{X}$. The respective classes are denoted by $\mathscr{M}_a(\mathscr{X})$, $\mathscr{M}_m(\mathscr{X})$, $\mathscr{M}_l(\mathscr{X})$.

Moreover, let

$$\mathcal{M}_{\min}(\mathcal{X}) = \{f \in \mathcal{X};\ \text{if } g \in \mathcal{X}, \text{ then } \min(f, g) \in \mathcal{X}\},$$

$$\mathcal{M}_{\max}(\mathcal{X}) = \{f \in \mathcal{X};\ \text{if } g \in \mathcal{X}, \text{ then } \max(f, g) \in \mathcal{X}\}.$$

Then $\mathcal{M}_l(\mathcal{X}) = \mathcal{M}_{\max}(\mathcal{X}) \cap \mathcal{M}_{\min}(\mathcal{X})$.

For real functions of a real variable we know that

$$\mathcal{M}_a(\mathcal{C}on) = \mathcal{M}_a(\mathcal{F}) = \mathcal{M}_a(\mathcal{D} \cap \mathcal{B}_1) = \mathcal{C} \quad \text{(see [2], [5])},$$

$$\mathcal{M}_m(\mathcal{D} \cap \mathcal{B}_1) = \mathcal{M} \quad \text{([4])}, \qquad \mathcal{M}_a(\mathcal{D}) = \mathcal{M}_m(\mathcal{D}) = \mathcal{C}onst \quad \text{([9])}.$$

In the present article we shall prove the following:

$$\mathcal{M}_m(\mathcal{A}) = \mathcal{M}_m(\mathcal{C}on) = \mathcal{M}_m(\mathcal{F}) = \mathcal{M},$$

$$\mathcal{M}_l(\mathcal{A}) = \mathcal{M}_l(\mathcal{C}on) = \mathcal{M}_l(\mathcal{F}) = \mathcal{C},$$

$$\mathcal{M}_a(\mathcal{A}) = \mathcal{C} \quad \text{(compare [8])}.$$

We omit some proofs that can be left to the reader.

2. Some basic lemmas

LEMMA 2.1. *Let Φ be some property of functions and X a topological space. Let $\mathcal{X}_1$ be the class of all functions $f\colon X \to \mathbf{R}$ with property Φ and let $\mathcal{X}_2$ be the class of all functions $g\colon X \to \mathbf{R} \times \mathbf{R}$ with property Φ. Suppose the classes $\mathcal{X}_1$, $\mathcal{X}_2$ fulfil the following conditions:*

(1.1) *if $g \in \mathcal{X}_2$ and $h \in \mathcal{C}$, $h\colon \mathbf{R}^2 \to \mathbf{R}$, then $h \circ g \in \mathcal{X}_1$,*

(1.2) *if $f \in \mathcal{X}_1$, $g \in \mathcal{C}$, $g\colon X \to \mathbf{R}$, then $k = (f, g) \in \mathcal{X}_2$, where $k(x) = (f(x), g(x))$ for $x \in X$.*

Then $\mathcal{C} \subseteq \mathcal{M}_a(\mathcal{X}_1) \cap \mathcal{M}_m(\mathcal{X}_1) \cap \mathcal{M}_l(\mathcal{X}_1)$.

LEMMA 2.2. *Let $\mathcal{X} \subseteq \mathcal{D}_0$ fulfil the following conditions:*

(2.1) *if $f\colon I \to \mathbf{R}$, $f \in \mathcal{X}$, J is a subinterval of an interval I, then $f|J \in \mathcal{X}$,*

(2.2) *if $h\colon (a, b) \to \mathbf{R}$, $h \in \mathcal{X}$, $y \in L^+(h, a)$, $z \in L^-(h, b)$ then the functions $h_1\colon [a, b) \to \mathbf{R}$, $h_2\colon (a, b] \to \mathbf{R}$ and $h_3\colon [a, b] \to \mathbf{R}$ belong to $\mathcal{X}$, where $h_1 = h \cup \{(a, y)\}$, $h_2 = h \cup \{(b, z)\}$, $h_3 = h_1 \cup h_2$,*

(2.3) *if $I \subseteq \mathbf{R}$ is an interval, $a \in I$ and $f|(I \cap (-\infty, a]) \in \mathcal{X}$, $f|(I \cap [a, +\infty)) \in \mathcal{X}$, then $f \in \mathcal{X}$,*

(2.4) *$\mathcal{C}onst \subseteq \mathcal{M}_a(\mathcal{X})$, $-1 \in \mathcal{M}_m(\mathcal{X})$.*

Then

(i) *$\mathcal{M}_a(\mathcal{X}) \subseteq \mathcal{C}$,*

(ii) *$\mathcal{M}_{\min}(\mathcal{X}) \subseteq \mathcal{D} \cap lsc$ and $\mathcal{M}_{\max}(\mathcal{X}) \subseteq \mathcal{D} \cap usc$ (hence $\mathcal{M}_l(\mathcal{X}) \subseteq \mathcal{C}$).*

If moreover the class $\mathcal{X}$ fulfils the additional condition

(2.5) *if $f\colon I \to (0, \infty)$ and $f \in \mathcal{X}$, then $1/f \in \mathcal{X}$,*

then also

(iii) *$\mathcal{M}_m(\mathcal{X}) \subseteq \mathcal{M}$.*

Proof. (i) Assume that $f \in \mathcal{X}$ and suppose that f is not right-continuous at x_0. Since $\mathcal{X} \subseteq \mathcal{D}_0$ we have

$$f(x_0) \in L^-(f, x_0) \cap L^+(f, x_0).$$

Suppose that

$$f(x_0) < \limsup_{x \to x_0^+} f(x).$$

Let

$$(*) \qquad c = \begin{cases} (f(x_0) + \limsup_{x \to x_0^+} f(x))/2 & \text{if } \limsup_{x \to x_0^+} f(x) < \infty, \\ f(x_0) + 1 & \text{if } \limsup_{x \to x_0^+} f(x) = \infty, \end{cases}$$

$$(**) \qquad g(x) = \begin{cases} f(x) & \text{for } x \leqslant x_0, \\ 2c - f(x) & \text{for } x > x_0. \end{cases}$$

Notice that in view of (2.4) the function $g|(x_0, \infty)$ belongs to $\mathcal{X}$. Since $f(x_0) \in L^+(g, x_0)$, according to (2.2) and (2.3) we infer that $g \in \mathcal{X}$. Moreover, $(f+g)(x_0) = 2f(x_0) < 2c$ and $(f+g)(x) = 2c$ for $x > x_0$. Thus $f + g \notin \mathcal{D}_0$.

If $f(x_0) > \liminf_{x \to x_0^+} f(x)$ the proof is quite the same.

(ii) Assume that f is not upper semicontinuous at x_0 from the right, i.e. $f(x_0) < \limsup_{x \to x_0^+} f(x)$, for some $x_0 \in \mathbf{R}$. Let c be defined by $(*)$ and define $g \in \mathcal{X}$ by $(**)$. Then $\max(f, g)(x_0) = f(x_0) < c$ and $\max(f, g)(x) \geqslant c$ for $x > x_0$, hence $\max(f, g) \notin \mathcal{D}_0$ and consequently $f \notin \mathcal{M}_{\max}(\mathcal{X})$.

In the same way we can prove that $\mathcal{M}_{\min}(\mathcal{X}) \subseteq \mathcal{D}_0 \cap lsc$, and since $\mathcal{D}_0 \cap \mathcal{B}_1 = \mathcal{D} \cap \mathcal{B}_1$ (see e.g. [2]), $\mathcal{M}_{\min}(\mathcal{X}) \subseteq \mathcal{D} \cap lsc$.

(iii) Let $f \in \mathcal{X}$ and suppose that f is discontinuous at some point x_0, say from the right. Assume that there exists $d > 0$ for which $f(x) \neq 0$ for $x \in (x_0, x_0 + d]$. Let $c \neq 0$, $c \neq f(x_0)$, be a point from $L^+(f, x_0)$. Define $g\colon \mathbf{R} \to \mathbf{R}$ by

$$g(x) = \begin{cases} 1/c & \text{for } x \leqslant x_0, \\ 1/f(x) & \text{for } x \in (x_0, x_0 + d], \\ 1/f(x_0 + d) & \text{for } x > x_0 + d. \end{cases}$$

Then $1/c \in L^+(1/f, x_0)$. We infer from (2.5) that $g|(x_0, x_0 + d] \in \mathcal{X}$, and from (2.2), (2.3) that $g \in \mathcal{X}$.

Notice that $fg(x_0) = f(x_0)/c \neq 1$ and $fg(x) = 1$ for $x \in (x_0, x_0+d]$. Thus $fg \notin \mathcal{D}_0$ and hence $f \notin \mathcal{M}_m(\mathcal{X})$.

Thus there exists a sequence (x_n) such that $x_n > x_0$, $x_n \to x_0$, and $f(x_n) = 0$. Let $f(x_0) = a$ and suppose that $a \neq 0$. Then we define

$$g(x) = \begin{cases} 2a - f(x) & \text{for } x > x_0, \\ 2a & \text{for } x \leqslant x_0. \end{cases}$$

It follows from (2.2)–(2.4) that $g \in \mathcal{X}$. On the other hand, $fg \notin \mathcal{D}_0$ since $fg(x_0) = 2a^2$ and, for $x > x_0$,

$$fg(x) = (2a - f(x))\, f(x) < a^2 < 2a^2.$$

3. Almost continuity. If X, Y are two topological spaces, then a function $f\colon X \to Y$ is *almost continuous* iff each open subset G of $X \times Y$ containing the function f contains a function continuous on X (here a function and its graph coincide).

LEMMA 3.1. *If $f\colon X \to Y$ is continuous and $g\colon X \to Z$ is almost continuous, then $h\colon X \to Y \times Z$ defined by $h = (f, g)$ is almost continuous.*

(See [8] for metric spaces. The proof for topological spaces needs no change.)

LEMMA 3.2 ([7]). *If $f\colon X \to Y$ is almost continuous and $g\colon Y \to Z$ is continuous, then $g \circ f$ is almost continuous.*

COROLLARY 3.1. *If $f\colon S \to X$ is almost continuous, and $g\colon S \to Y$ and $F\colon X \times Y \to Z$ are continuous, then $F \circ (f, g)\colon S \to Z$ is almost continuous. Since addition, multiplication,* min, max *are continuous functions of two variables, this implies*

$$\mathcal{C} \subseteq \mathcal{M}_a(\mathcal{A}) \cap \mathcal{M}_m(\mathcal{A}) \cap \mathcal{M}_{\min}(\mathcal{A}) \cap \mathcal{M}_{\max}(\mathcal{A}).$$

Remark. There are almost continuous functions $f\colon \boldsymbol{R} \to \boldsymbol{R}$, $g\colon \boldsymbol{R} \to \boldsymbol{R}$ such that $h = (f, g)$ from $\boldsymbol{R}$ into $\boldsymbol{R}^2$ is not almost continuous.

Proof. By [7] there are almost continuous functions f and g with $f+g$ not almost continuous. For such pairs of functions the transformation (f, g) is not almost continuous.

LEMMA 3.3. *If $I = [a, b]$, $f\colon I \to \boldsymbol{R}$ is almost continuous, and $G \subseteq I \times \boldsymbol{R}$ is an open neighbourhood of f, then there is a continuous function $g\colon I \to \boldsymbol{R}$ such that $g \subseteq G$, $g(a) = f(a)$, and $g(b) = f(b)$.*

Proof. Let $U_a \times V_a \subseteq G$, $U_b \times V_b \subseteq G$ be neighbourhoods of $(a, f(a))$, $(b, f(b))$, respectively. Choose $x_1 \in U_a \setminus \{a\}$, $x_2 \in U_b \setminus \{b\}$ such that $f(x_1) \in V_a$, $f(x_2) \in V_b$. Then

$$G_1 = G \setminus [(\{x_1\} \times (\boldsymbol{R} \setminus V_a)) \cup (\{x_2\} \times (\boldsymbol{R} \setminus V_b))]$$

is an open neighbourhood of f, hence there is a continuous function $h\colon I \to \boldsymbol{R}$ such that $h \subseteq G_1$. Now define g as follows:

$$g(x) = \begin{cases} h(x) & \text{for } x \in [x_1, x_2], \\ f(x) & \text{for } x \in \{a, b\}, \\ \text{linear on each of the intervals } [a, x_1],\ [x_2, b]. \end{cases}$$

This function fulfils the requirements of the lemma.

LEMMA 3.4. *If $h\colon (a, b) \to \boldsymbol{R}$ is almost continuous, $y \in L^+(h, a)$, $z \in L^-(h, b)$ then the functions $h_1 = h \cup \{(a, y)\}$, $h_2 = h \cup \{(b, z)\}$, $h_3 = h_1 \cup h_2$ are almost continuous.*

LEMMA 3.5. *If $I \subseteq \boldsymbol{R}$ is an interval and I is the union of a finite or countable sequence (I_n) of closed intervals such that $\bigcup_{n \leqslant k} I_n$ is connected for every k, and a function $f|I_n$ is almost continuous for every n, then $f|I$ is almost continuous.*

Proof. Let $I_n = [a_n, b_n]$. Let $G \subseteq I \times \boldsymbol{R}$ be an open neighbourhood of f. In view of Lemma 3.3, for each n there is a continuous function $g_n\colon I_n \to \boldsymbol{R}$ such that

$$g_n \subseteq G \cap (I_n \times \boldsymbol{R}), \quad g_n(a_n) = f(a_n), \quad g_n(b_n) = f(b_n).$$

Then $g = \bigcup_{n=1}^{\infty} g_n$ is a function that is continuous and contained in G.

THEOREM 3.1. *$\mathcal{M}_a(\mathcal{A}) = \mathcal{M}_l(\mathcal{A}) = \mathcal{C}$ and $\mathcal{M}_m(\mathcal{A}) = \mathcal{M}$.*

Proof. By Corollary 3.1 we have

$$\mathcal{C} \subseteq \mathcal{M}_a(\mathcal{A}) \cap \mathcal{M}_m(\mathcal{A}) \cap \mathcal{M}_l(\mathcal{A}).$$

Notice that the class $\mathcal{A}$ fulfils the conditions (2.1)–(2.5) from Lemma 2.2. Hence

$$\mathcal{M}_a(\mathcal{A}) \subseteq \mathcal{C}, \quad \mathcal{M}_{\min}(\mathcal{A}) \subseteq lsc, \quad \mathcal{M}_{\max}(\mathcal{A}) \subseteq usc,$$

therefore

$$\mathcal{M}_l(\mathcal{A}) \subseteq \mathcal{C} \quad \text{and} \quad \mathcal{M}_m(\mathcal{A}) \subseteq \mathcal{M}.$$

It now suffices to prove that $\mathcal{M} \subseteq \mathcal{M}_m(\mathcal{A})$.

Let $f \in \mathcal{M}$. In view of Lemma 3.5 one can assume f to be defined on a closed interval $I = [a, b]$, where $f(a) \neq 0 \neq f(b)$. Let $g\colon I \to \boldsymbol{R}$ be an almost continuous function and G an open set in $I \times \boldsymbol{R}$ such that $fg \subseteq G$. Put $E = \{x \in I;\ f(x) = 0\}$. This set is closed and contains the set of all points of discontinuity of f. For each $x \in E$ we take intervals $U_x \subseteq I$ and $V_x \subseteq \boldsymbol{R}$ such that

$$(x, 0) \in U_x \times V_x \subseteq \bar{U}_x \times \bar{V}_x \subseteq G.$$

There is a finite sequence of intervals $U_{x_1}, \ldots, U_{x_m}$ such that $E \subseteq \bigcup_{i=1}^m U_{x_i}$. Now let $V = \bigcap_{i=1}^m V_{x_i}$. Let $W_1, \ldots, W_n$ be the components of the set $\bigcup_{i=1}^m \bar{U}_{x_i}$ such that if $x \in W_i$, $y \in W_j$ for $i < j$ then $x < y$. Notice that $E \subseteq W_1 \cup \ldots \cup W_n$ and $\bar{W}_i \cap \bar{W}_j = \emptyset$ for $i \neq j$. Moreover, the end-points of the intervals W_i do not belong to $\bar{E}$.

For each $i = 1, \ldots, n$ choose a closed interval $[a_{2i}, a_{2i+1}] \subseteq W_i$ such that $E \cap W_i = E \cap [a_{2i}, a_{2i+1}]$ and $a_{2i}, a_{2i+1} \in E$. Let us remark that it can happen that $a_{2i} = a_{2i+1}$, and $f|[a_{2i-1}, a_{2i}]$ is continuous. Since $g|[a_{2i-1}, a_{2i}]$ is almost continuous, the product $fg|[a_{2i-1}, a_{2i}]$ is almost continuous too. Since $fg|[a_{2i-1}, a_{2i}] \subseteq G$ there exist continuous functions $h_i\colon [a_{2i-1}, a_{2i}] \to \boldsymbol{R}$ such that $h_i \subseteq G$, $h_i(a_{2i}) = h_i(a_{2i-1}) = 0$. Therefore the function $h\colon I \to \boldsymbol{R}$ defined by

$$h = \bigcup_{i=1}^{n+1} h_i \cup [(I \setminus \bigcup_{i=1}^{n} [a_{2i-1}, a_{2i}]) \times \{0\}]$$

is contained in G and continuous, which completes the proof.

PROBLEM. Is it true that $\mathcal{M}_{\max}(\mathcal{A}) = \mathcal{D} \cap usc$ and $\mathcal{M}_{\min}(\mathcal{A}) = \mathcal{D} \cap lsc$?

4. Connectedness. If X, Y are two topological spaces, then a function $f\colon X \to Y$ is called *connected* provided f is a connected subset of $X \times Y$. The next two lemmas follow from the fact that the continuous image of a connected set is connected.

LEMMA 4.1. *Let X, Y, Z be topological spaces. If $f\colon X \to Y$ is a connected function and $g\colon Y \to Z$ is continuous, then $g \circ f$ is connected.*

LEMMA 4.2. *If $f\colon X \to Y$ is a connected function, $g\colon X \to Z$ is a continuous function, then $h = (f, g)\colon X \to Y \times Z$ is connected.*

COROLLARY 4.1. *If $F\colon X \times Y \to Z$ is a continuous function, $g\colon S \to Y$ is connected, $f\colon S \to X$ is continuous, then $F \circ (f, g)\colon S \to Z$ is a connected function.*

In particular, the following inclusions hold:

$$\mathcal{C} \subseteq \mathcal{M}_a(\mathcal{C}on) \cap \mathcal{M}_m(\mathcal{C}on) \cap \mathcal{M}_{\max}(\mathcal{C}on) \cap \mathcal{M}_{\min}(\mathcal{C}on).$$

LEMMA 4.3. *If $h\colon (a, b) \to \mathbf{R}$ is a connected function, $y \in L^+(h, a)$, $z \in L^-(h, b)$, then the functions h_1, h_2, h_3 are connected, where $h_1 = h \cup \{(a, y)\}$, $h_2 = h \cup \{(b, z)\}$, $h_3 = h_1 \cup h_2$.*

LEMMA 4.4. *If $I \subseteq \mathbf{R}$ is an interval and I is the union of a finite or countable sequence (I_n) of closed intervals such that $\bigcup_{n \leqslant k} I_n$ is connected for every k, and a function $f|I_n$ is connected for every n, then $f|I$ is connected.*

THEOREM 4.1. $\mathcal{M}_a(\mathcal{C}on) = \mathcal{M}_l(\mathcal{C}on) = \mathcal{C}$, $\quad \mathcal{M}_m(\mathcal{C}on) = \mathcal{M}$.

Proof. First, notice that the class $\mathcal{C}on$ fulfils the conditions (2.1)–(2.5) of Lemma 2.2. Thus

$$\mathcal{M}_a(\mathcal{C}on) \subseteq \mathcal{C}, \quad \mathcal{M}_m(\mathcal{C}on) \subseteq \mathcal{M}, \quad \mathcal{M}_l(\mathcal{C}on) \subseteq \mathcal{C}.$$

According to Lemma 4.1 we infer that

$$\mathcal{M}_a(\mathcal{C}on) = \mathcal{C}, \quad \mathcal{M}_l(\mathcal{C}on) = \mathcal{C}$$

(compare [5]). Only the inclusion $\mathcal{M} \subseteq \mathcal{M}_m(\mathcal{C}on)$ has to be proved. By Lemma 4.4 it suffices to consider functions $f \in \mathcal{M}$ defined on a closed interval $I = [a, b]$ with $a, b \notin \bar{D}$, where D denotes the set of all points of discontinuity of f.

Suppose that there exists a connected function $g\colon I \to \mathbf{R}$ such that fg is not connected. Let (A, B) be a partition of fg and suppose that $(a, f(a)g(a)) \in A$, $(b, f(b)g(b)) \in B$. Let now

$$A_1 = \{x \in I;\ (x, f(x)g(x)) \in A\}, \quad B_1 = \{x \in I;\ (x, f(x)g(x)) \in B\},$$

and let $c = \sup A_1$. Of course, $c < b$. Notice that if J is a component of $I \setminus \bar{D}$, then either $\bar{J} \subseteq A_1$, or $\bar{J} \subseteq B_1$, for if not then $fg|\bar{J}$ would not be connected. Thus $c \in \bar{D}$. Moreover, $(c, f(c)g(c)) \in \bar{A} \cap \bar{B}$. Indeed, either c is an end-point of a component of the set $I \setminus \bar{D}$ that is contained in A_1, and hence $c \in A_1$ or there exists a sequence (J_n) of components of the set $I \setminus \bar{D}$ that are contained in A_1 and such that $\bar{J}_n \subseteq A_1$. Therefore there exists a sequence (x_n) of points $x_n \in A_1$ such that $f(x_n) = 0$ and $x_n \to c$. Then $(x_n, 0) \in A$, for $n = 1, 2, \ldots$, and $(c, 0) \in \bar{A}$. Similarly one can prove that $(c, 0) \in \bar{B}$. Since $(c, 0) \in A$ or $(c, 0) \in B$ we infer that $(\bar{A} \cap B) \cup (A \cap \bar{B}) \neq \emptyset$, hence a contradiction. This completes the proof.

PROBLEM. Is it true that $\mathcal{M}_{\min}(\mathcal{C}on) = \mathcal{D} \cap lsc$ and $\mathcal{M}_{\max}(\mathcal{C}on) = \mathcal{D} \cap usc$?

5. Functional connectedness. Assume now that I is an interval. A function $f\colon I \to \mathbf{R}$ is called *functionally connected* provided for each $[a, b] \subseteq I$ and each continuous function $g\colon [a, b] \to \mathbf{R}$, if $(f(a) - g(a))(f(b) - g(b)) < 0$, then there exists a point $c \in [a, b]$ for which $f(c) = g(c)$.

LEMMA 5.1. *If $f\colon I \to \mathbf{R}$ is functionally connected and $x \in I$, then $f|I \cap (-\infty, x]$ and $f|I \cap [x, \infty)$ are functionally connected.*

LEMMA 5.2. *If I is a finite or countable union of a sequence (I_n) of closed intervals such that $\bigcup_{n \leqslant k} I_n$ is connected for every k, and a function $f|I_n$ is functionally connected, then $f|I$ is functionally connected.*

LEMMA 5.3. *If $h\colon (a, b) \to \mathbf{R}$ is functionally connected, $y \in L^+(h, a)$, $z \in L^-(h, b)$ then the functions h_1, h_2, h_3 are functionally connected, where $h_1 = h \cup \{(a, y)\}$, $h_2 = h \cup \{(b, z)\}$, $h_3 = h_1 \cup h_2$.*

LEMMA 5.4. (a) *If $f\colon I \to \mathbf{R}$ is functionally connected and both $g\colon I \to (0, +\infty)$ and $h\colon I \to (-\infty, 0)$ are continuous, then fh, fg are functionally connected.*

(b) *If $f\colon I \to (0, \infty)$ is functionally connected, then so is $1/f$.*

LEMMA 5.5. *Let C be a nowhere dense subset of an interval I. Let $(I_n)_n$ be the sequence of all components of the set $I \setminus \bar{C}$.*

If $f\colon I \to \mathbf{R}$ fulfils the conditions

(i) *$f|I_n$ is functionally connected for each $n \in \mathbf{N}$,*

(ii) *$f(x) = 0 \in L^-(f, x) \cap L^+(f, x)$ for x in $\bar{C}$,*

then f is functionally connected.

Proof. Let $a, b \in I$ and let $h\colon I \to \mathbf{R}$ be a continuous function such that

$$h(a) < f(a) \quad \text{and} \quad f(b) < h(b).$$

By Lemma 5.2, with no loss of generality we can assume that $a, b \in C$. Hence

$$h(a) < f(a) = 0 = f(b) < h(b).$$

Notice that $\{x \in (a, b);\ h(x) = 0\}$ is a non-empty compact set.

If $h(x) = 0$ for some $x \in \bar{C} \cap (a, b)$, then we are done. Assume now that $h(x) \neq 0$ for all $x \in \bar{C} \cap (a, b)$. Then

$$\{x \in (a, b);\ h(x) = 0\} \subseteq \bigcup_n I_n$$

and $\{n;\ I_n \cap \{x \in (a, b);\ h(x) = 0\} \neq \emptyset\}$ is a finite set. Let $I_{n_1}, \ldots, I_{n_k}$ be the sequence of all components which meet the set $\{x \in (a, b);\ h(x) = 0\}$ and for which

$$\sup I_{n_i} \leqslant \inf I_{n_j} \quad \text{if } i < j.$$

Let $I_{n_m} = (a_{n_m}, b_{n_m})$ for $m \leqslant k$ and $n_0 = \min\{m \leqslant k;\ h(b_{n_m}) > 0\}$. Notice that $h(b_{n_0}) > 0$ and $h(a_{n_0}) < 0$. Since $f|I_{n_0}$ is functionally connected, so is $f|\bar{I}_{n_0}$ and consequently $h(z) = f(z)$ for some $z \in I_{n_0}$, which completes the proof.

THEOREM 5.1. $\mathscr{M}_a(\mathscr{F}) = \mathscr{C} = \mathscr{M}_l(\mathscr{F})$, $\mathscr{M}_m(\mathscr{F}) = \mathscr{M}$, $\mathscr{M}_{\min}(\mathscr{F}) = \mathscr{D} \cap lsc$, $\mathscr{M}_{\max}(\mathscr{F}) = \mathscr{D} \cap usc$.

Proof. For the equality $\mathscr{M}_a(\mathscr{F}) = \mathscr{C}$ see [5]. To prove the other equalities notice that $\mathscr{F}$ fulfils the conditions (2.1)–(2.5) of Lemma 2.2. Thus

$$\mathscr{M}_m(\mathscr{F}) \subseteq \mathscr{M}, \quad \mathscr{M}_{\min}(\mathscr{F}) \subseteq \mathscr{D} \cap lsc, \quad \mathscr{M}_{\max}(\mathscr{F}) \subseteq \mathscr{D} \cap usc.$$

First we shall show that $\mathscr{M}_{\min}(\mathscr{F}) = \mathscr{D} \cap lsc$ and $\mathscr{M}_{\max}(\mathscr{F}) = \mathscr{D} \cap usc$. Let f, g, h be real functions defined on an interval I such that $f \in \mathscr{F}$, $g \in \mathscr{D} \cap usc$, $h \in \mathscr{C}$ and

$$(\max(f, g)(a) - h(a))(\max(f, g)(b) - h(b)) < 0,$$

for some $a, b \in I$. Since $\max(f, g) - h = \max(f-h, g-h)$ and $g-h \in \mathscr{D} \cap usc$, the function $k = \max(f, g) - h$ has the Darboux property (see [5]). Since $k(a)k(b) < 0$, there exists $c \in [a, b]$ such that $k(c) = 0$. Hence $\max(f, g)(c) = h(c)$ and, consequently, $\max(f, g)$ is functionally connected.

In the analogous way we prove that $\mathscr{D} \cap lsc \subseteq \mathscr{M}_{\min}(\mathscr{F})$. Hence

$$\mathscr{M}_{\min}(\mathscr{F}) = \mathscr{D} \cap lsc, \quad \mathscr{M}_{\max}(\mathscr{F}) = \mathscr{D} \cap usc, \quad \mathscr{M}_l(\mathscr{F}) = \mathscr{C}.$$

Now we shall prove that $\mathscr{M} \subseteq \mathscr{M}_m(\mathscr{F})$. Let f, g be real functions defined on I such that $f \in \mathscr{F}$ and $g \in \mathscr{M}$. Then the set E of all points of discontinuity of g is nowhere dense and the set

$$B = \{x \in I:\ g(x) = 0\}$$

is closed. Let the sequence (I_n) be the union of the set of all components of $\mathrm{Int}(B)$ and the set of all components of $I \setminus B$; moreover, let $C = I \setminus \bigcup_n I_n$. Thus C is a closed, nowhere dense set contained in B and all the assumptions of Lemma 5.6 for the function fg are satisfied. Consequently, fg is functionally connected.

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