The prime spectrum of an infinite product of copies of $\mathbb{Z}$

by

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Abstract. G. Cherlin, using model-theoretic techniques, has characterized the prime and maximal ideals in the direct product of copies of the ring of integers. In this paper, we obtain a characterization of these ideals using less machinery. As a consequence, we are able to obtain more information about the structure and order type of the prime spectrum of the ring.

§ 0. Introduction. Let $I$ be an infinite index set and let $R = \prod_{i \in I} \mathbb{Z}$, the direct product of copies of $\mathbb{Z}$. The goals of this paper are to characterize the prime ideals of $R$ and to discuss the structure of $\Spec(R)$.

It is known that the ultrafilters $\mathcal{U}$ on $I$ are associated in a natural way with the minimal prime ideals of $R$ (denoted by $(\mathcal{U})$). $R/(\mathcal{U})$ is isomorphic to an ultrapower of $\mathbb{Z}$, and hence if $\mathcal{U}$ is non-principal, then $R/(\mathcal{U})$ is a so-called non-standard Peano ring (that is, a ring which is elementarily equivalent to $\mathbb{Z}$). G. Cherlin [C] studied the prime spectrum of such rings. His characterization is done in model-theoretic terms. Our construction has certain advantages. We use less machinery so our description is simpler. In addition, our elementwise description of the prime and maximal ideals enables us to obtain information about the order structure of the chain of prime ideals.

We use ultrafilters $\mathcal{F}$ on certain Boolean algebras of functions to describe the maximal ideals of $R$, denoted by $(\mathcal{F})$ (Theorems 3 & 4). If $\mathcal{U}$ is a principal ultrafilter on $I$, then rather trivially there are no non-maximal prime ideals strictly containing $(\mathcal{U})$. In any case the set of primes between a fixed maximal ideal and a prime ideal is linearly ordered. We show that over a fixed $(\mathcal{U})$ any two maximal chains of prime ideals are order isomorphic. Moreover, we prove that a maximal chain of prime ideals has cardinality either 2 or at least $2^\omega$. Finally, we prove that for $\omega_1$-complete ultrafilters $\mathcal{U}$ the set of prime ideals between $(\mathcal{U})$ and $(\mathcal{F})$ is essentially the Dedekind completion of the lexicographic product of an $\eta_1$-set with 2.

We will always work in at least ZFC, that is, Zermelo--Fraenkel set theory with the axiom of choice. We will, in certain cases, use additional axioms.

Following the classical definition, we say that $\mathcal{U}$ is a filter on $I$ if it is a subset of the power set of $I$ that satisfies the following conditions:
Let $J$ be the ideal generated by $\{\delta_p : p \text{ is a sequence of non-negative prime numbers}\}$. It is not difficult to see that $J$ does not contain $1$. Every proper ideal, in a ring with identity, is contained in a maximal ideal. If $M$ is any maximal ideal of $R$ containing $J$, then $M$ cannot be of the form $(\mathfrak{p}, p)$ because $p$ is relatively prime to $\delta_p$. Thus there must be non-principal maximal ideals.

We now investigate the non-principal maximal ideals of $R$. Let $\mathfrak{S}$ be the set of functions $\sigma : I \to \mathbb{F}$ where $\mathbb{F}$ is a finite set of positive prime integers. One example of such a $\sigma$ (with $I = \omega$) is given by defining $\sigma(n)$ to be $\{1, n \text{ prime integers}\}$. This function was used implicitly in the preceding example. Now let $\sigma$ and $\varphi$ be elements of $\mathfrak{S}$. We say that $\varphi \in [\sigma]$ is a subfunction of $\sigma$ if $\varphi(i) \subseteq \sigma(i)$ for each $i \in I$; in this case we define $\sigma(\varphi)$ to be the function given by $(\sigma(\varphi))(i) = \sigma(i)(\varphi)$ for each $i \in I$. If $\varphi_1$ and $\varphi_2$ are elements of $\mathfrak{S}$, we define $\varphi_1 \land \varphi_2$ (respectively $\varphi_1 \lor \varphi_2$) to be the function defined by $(\varphi_1 \land \varphi_2)(i) = \varphi_1(i) \land \varphi_2(i)$ (respectively $\varphi_1(i) \lor \varphi_2(i)$) for each $i \in I$. Finally, the blank function $\emptyset$ is defined by $\emptyset(i) = \emptyset$ for each $i \in I$. Let $\sigma$ be a fixed element of $\mathfrak{S}$. The set of subfunctions of $\sigma$ forms a Boolean algebra. Therefore it makes sense to talk about ultrafilters on the set of subfunctions of $\sigma$. In particular, a subset $\mathfrak{F}$ of this Boolean algebra is called an ultrafilter on $\sigma$ if:

1. $\varphi \notin \mathfrak{F}$ and $\sigma(\varphi) \in \mathfrak{F}$;
2. if $\varphi_1$ and $\varphi_2$ are in $\mathfrak{F}$, then so is $\varphi_1 \lor \varphi_2$;
3. if $\varphi \in \mathfrak{S}$, then either $\varphi \notin \mathfrak{F}$ or $\sigma(\varphi) \in \mathfrak{F}$.

We will use these ultrafilters to describe the maximal ideals of $R$. But first we give some definitions. Let $\sigma$ be a fixed element of $\mathfrak{S}$. For $a = (\alpha_n)_{n \in \mathbb{N}} \in R$, we define a subfunction $\sigma(a)$ of $\sigma$ via $\sigma(a)(i) = \{p \in \alpha(i) \subseteq \emptyset : p \text{ divides } \alpha_n\}$. For $\sigma(\varphi)$, we define $\prod_{\varphi \in \emptyset} \varphi = 1$ otherwise. Now define $\sigma(a) = (\alpha_n)_{n \in \mathbb{N}} \in R$, where $\alpha_n = \{\varphi \in \emptyset : \varphi \in \mathfrak{F}\}$, $\{\varphi \in \emptyset : \varphi \in \mathfrak{F}\}$, and $\{\varphi \in \emptyset : \varphi \in \mathfrak{F}\}$ divide $a$.

Theorem 3. Let $\mathfrak{S}$ be an ultrafilter on some $\sigma \in \mathfrak{S}(\emptyset)$. Then the set $\mathfrak{S} = \{a \in R : \sigma(a) \in \emptyset\}$ is a maximal ideal of $R$.

Proof. Let $a, b \in R$ be such that $\sigma(a), \sigma(b) \in \emptyset$. Then $\sigma(a \cdot b) \supseteq \sigma(a) \land \sigma(b)$. Also if $a, b \in R$, then $\sigma(a \cdot b) \supseteq \sigma(a) \land \sigma(b)$. Hence $\mathfrak{S}$ is an ideal. Finally, if $a \in R(\mathfrak{S})$, then $\sigma(a) \in \mathfrak{S}$, and hence $\sigma(a) \in \emptyset$. Therefore $\sigma(a) \in \mathfrak{S}$ (in $\emptyset$) and is relatively prime to $x$ (at each coordinate). Hence there exists $a \in R$ such that $x \notin 1$, $\sigma(a) \in \emptyset$. So $\sigma$ is a maximal ideal of $R$.

A function $\varphi \in \mathfrak{S}$ will be called principal if $\varphi(i)$ is either $\emptyset$ or a singleton for each $i \in I$. If $\mathfrak{S}$ contains a principal function $\varphi$, then $\mathfrak{S} = (\mathfrak{S}, p)$, for $\mathfrak{S} = (\emptyset, \emptyset)$ where $U_1 = \{i \in I : \varphi(i) \notin \emptyset\}$ and $p = (p_i)_{i \in I}$, where $p_i = \prod e \in \emptyset$. Conversely, it is clear that every maximal ideal of the form $(\mathfrak{S}, p)$ can be described as an $\emptyset$ for some suitable choice of $\emptyset$. Thus the maximal ideals described in Theorem 3 include the principal maximal ideals. The next result shows that these are all the maximal ideals.

Theorem 4. There is a surjection, given by $\mathfrak{S} \rightarrow (\emptyset)$, between the set of ultrafilters on elements of $\mathfrak{S}$ and the maximal ideals of $R$. 

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Proof. First we introduce some notation. For \( x = (x_i)_{i \in \mathbb{N}} \in R \), let \( \sqrt{x} = (y_i)_{i \in \mathbb{N}} \) where \( y_i \) is the product of all the distinct prime integers dividing \( x_i \), if \( x_i \neq 0 \) or 1. If \( x_i = 0 \) or 1, then \( y_i = x_i \).

Now let \( M \) be any maximal ideal of \( R \). If \( x \in M \), then so is \( \sqrt{x} \); for otherwise, \( \sqrt{x} \) is relatively prime to an element \( z \in M \) at each coordinate, and hence \( x \) is relatively prime to \( z \), which is a contradiction.

Let \( x \in M \) be such that \( \sqrt{x} = x \). Define \( \sigma \in \mathcal{F} \) such that for all \( i \in I \), \( \sigma(i) = \{ \text{positive prime integers that divide } x_i \} \). Let \( \mathcal{F} = \{ \sigma \in \mathcal{F} : a \in M \} \).

CLAIM 1. \( \mathcal{F} \) is an ultrafilter on \( \sigma \).

Clearly \( \emptyset \in \mathcal{F} \) (since \( 1 \in M \) and \( \sigma \in \mathcal{F} \) (since \( a \in \sqrt{x} = x \)). Also, if \( \sigma_1 \) and \( \sigma_2 \) are elements of \( \mathcal{F} \) (i.e., \( a_1 \) and \( a_2 \) are in \( M \)), then \( a_1 \cap a_2 \) is also in \( M \), since \( a_1 \cap a_2 \) is the greatest common divisor of \( a_1 \) and \( a_2 \). Finally, let \( \sigma \in \mathcal{F} \). Then \( a_\sigma \in a_\sigma \in \sqrt{x} = x \in M \) and hence \( \sigma \in \mathcal{F} \) or \( \sigma \in \mathcal{F} \).

CLAIM 2. \( \mathcal{F} \subseteq M \).

Let \( a \in \mathcal{F} \). Then \( a \in \mathcal{F} \) and thus \( a \in M \). But \( a \) divides \( a \), so \( a \in M \) proving the claim. By Theorem 3, \( \mathcal{F} \) is a maximal ideal and hence \( \mathcal{F} = M \).

§ 2. The prime ideals of \( R \). From now on, we fix \( \sigma \in \mathcal{F}, \mathcal{F}, \) an ultrafilter on \( \sigma \).

As pointed out previously, every prime ideal contains a minimal prime ideal. So in particular \( \mathcal{F} \) contains a prime ideal of the form \( \mathcal{F} \) for some ultrafilter \( \mathcal{F} \) on \( 

We now turn our attention to the prime ideals of \( R \). Let \( g \in \mathcal{F} \) and let \( x = (x_i)_{i \in \mathbb{N}} \in R \). We define \( T_g(x) \) to be the greatest non-negative integer \( k \) such that \( \prod_{i \in \mathbb{N}} x_i^k \) divides \( x \), if \( x \neq 0 \); we define \( T_g(0) \) to be \( 0 \). If \( h : I \to R \) is any function, we say that \( h \) is bounded on \( \mathcal{F} \) if for some \( U \in \mathcal{F} \), the set \( \{ h(i) : i \in \mathcal{F} \} \) is a bounded subset of \( R \). Otherwise \( h \) is unbounded on \( \mathcal{F} \). Now let \( g : I \to [1, \infty) \) be any function. We define \( \langle \mathcal{F} \rangle_g = \{ x \in \mathcal{F} : \text{there exists a } g \in \mathcal{F} \text{ such that the function } T_g(x)/g \text{ is bounded on } \mathcal{F} \} \), where \( T_g(x)/g \) is defined by \( (T_g(x)/g)(i) = T_g(x)/g \).

THEOREM 5. For any function \( g : I \to [1, \infty) \), the set \( \langle \mathcal{F} \rangle_g \) is a prime ideal of \( R \) satisfying \( (\mathcal{F})_g \subset \langle \mathcal{F} \rangle_g \subset \mathcal{F} \).

Proof. Let \( x = (x_i)_{i \in \mathbb{N}} \), \( y = (y_i)_{i \in \mathbb{N}} \in \langle \mathcal{F} \rangle_g \). Then, since \( \sigma \) is a filter, there exists \( g \in \mathcal{F} \) such that \( T_g(x)/g \text{ and } T_g(y)/g \text{ are bounded on } \mathcal{F} \). Now \( T_g(x+y)/g = \min(T_g(x), T_g(y)) \). Fix \( U \in \mathcal{F} \), then we let \( W = \{ i \in U : T_g(x_i+y_i) \geq T_g(y_i) \} \) and let \( V = U \setminus W \) (so for \( i \in V \), \( T_g(x_i+y_i) \geq T_g(y_i) \)). Either \( V \in \mathcal{F} \) or \( V \in \mathcal{F} \). In the first case, since \( T_g(x)/g \text{ is unbounded on } W \), so is \( T_g(x+y)/g \). Hence \( T_g(x+y)/g \text{ is unbounded on } U \). On the other hand, if \( V \in \mathcal{F} \), the same argument applies to \( T_g(y) \). Hence \( x+y \in \langle \mathcal{F} \rangle_g \). Also, if \( r \in \mathcal{F} \) is \( R \) then \( T_g(rx) \geq T_g(x) \), so that \( T_g(rx)/g \text{ is unbounded on } \mathcal{F} \). Hence \( rx \in \langle \mathcal{F} \rangle_g \) and \( \langle \mathcal{F} \rangle_g \) is a prime ideal of \( R \).
Theorem 8. The set $(\mathcal{F})'$ is a prime ideal of $R$ which contains $(\mathcal{F})$. Furthermore, $(\mathcal{F})'$ is the smallest prime ideal below $(\mathcal{F})$ containing the element $t = \alpha\beta$.

Proof. The proof of $(\mathcal{F})'$ being an ideal is analogous to the proof that $(\mathcal{F})$ is an ideal as given in Theorem 5. To show that the ideal is prime let $x, y$ be elements of $R$ such that $xy \in (\mathcal{F})'$. Then there exists $g \in \mathcal{F}$ and $u \in \mathcal{U}$ such that $g/T(x)y$ is bounded on $U$. As in Theorem 5, there exists $x \leq \mathcal{U}$ such that $T(x,y) \leq T(x,k)$ for all $i$. Clearly then $g/T(x)$ is bounded on $U$. If $x \in (\mathcal{F})'$, then $x \in (\mathcal{F})$. Otherwise $g \in \mathcal{F}$, which implies that $y$ is in $(\mathcal{F})$. So $(\mathcal{F})'$ is prime.

Finally, let $Q$ be a prime ideal of $R$ between $(\mathcal{F})$ and $(\mathcal{F})$ containing the element $t$. Pick $x \in (\mathcal{F})'$. Then there exists a positive integer $n$, an element $q \in \mathcal{Q}$ and $u \in \mathcal{U}$ such that $g(n) \leq T(x,n)$ for all $i$. Now let $\delta = \sigma_{\mathcal{Q}}q$ and let $z = a_q$. It follows that $t \in (\mathcal{F})$. Thus either $x \in \mathcal{Q}$ or $z \in (\mathcal{F})$. Therefore $x \in \mathcal{Q}$, proving that $(\mathcal{F})' \subseteq \mathcal{Q}$.

Note. If $g$ is constant or even bounded on some $U \in \mathcal{U}$, then $(\mathcal{F})' = (\mathcal{F})$.

If $g \in \mathcal{F}$, then $(\mathcal{F})$ defines an ultrafilter $\mathcal{F}$ on $\mathcal{Q}$ in a natural fashion: $x \in \mathcal{Q}$ if $g$ is in $\mathcal{F}$ if and only if $x \in \mathcal{F}$. It is not difficult to see that these two ultrafilters determine the same maximal ideal, namely $(\mathcal{F}) = (\mathcal{F})'$. In particular, if $x = a_q^1$ for some function $h: I \rightarrow \omega$, then $x \in (\mathcal{F})$ is the smallest prime ideal in $(\mathcal{F})$ which contains $x$.

Proposition 9. $(\mathcal{F})_b$ is the largest prime ideal of $R$ below $(\mathcal{F})$ not containing the element $t = \alpha\beta$.

Proof. In Theorem 8, we showed that $y \notin (\mathcal{F})_b$. Now let $Q$ be a prime ideal of $R$ below $(\mathcal{F})$ and $(\mathcal{F})$ not containing $y$. Assume that $Q$ is not contained in $(\mathcal{F})_b$, so there exists $x \in \mathcal{Q} \setminus (\mathcal{F})_b$. Since $x \in (\mathcal{F})$, we can, as in the proof of Theorem 8, assume that $x = a_q^1$ for some function $h: I \rightarrow \omega$ and some $g \in \mathcal{Q}$. Since $x$ is not in $(\mathcal{F})_b$, $h(g)(y) < \infty$ on $U$ for some positive integer $N$. Hence $x \in (\mathcal{F})$. Since $(\mathcal{F})_b$ is the smallest prime ideal containing $x$, $(\mathcal{F})_b \subseteq \mathcal{Q}$. Thus $y \notin \mathcal{Q}$, which is a contradiction.

The following corollaries are now immediate using Theorem 8 and Proposition 9.

Corollary 10. There are no prime ideals between $(\mathcal{F})_b$ and $(\mathcal{F})'$.

Note that we then have $(\mathcal{F})_b = (\mathcal{F})'$ if and only if $(\mathcal{F})_b \subseteq (\mathcal{F})'$. Furthermore, $(\mathcal{F})' = (\mathcal{F})'$ if and only if $(\mathcal{F})_b = (\mathcal{F})$. We will show in the next section that not every prime ideal is of the form $(\mathcal{F})_b$ or $(\mathcal{F})'$. However, all the prime ideals can be described in terms of these primes.

Corollary 11. Every prime ideal below $(\mathcal{F})$ can be written as the intersection of a family of $(\mathcal{F})_b$.

Corollary 12. Every prime ideal below $(\mathcal{F})$ can be written as the union of a family of $(\mathcal{F})'$.

Observe that the prime ideals between a given $(\mathcal{F})$ and $(\mathcal{F})$ are completely determined by functions $g$ from $I$ to $[1, \infty)$ modulo $\mathcal{U}$. In particular, if $(\mathcal{F})$ and $(\mathcal{F})'$ are two maximal ideals containing $\mathcal{U}$, then the chain of primes contained in $(\mathcal{F})$ is order isomorphic to the chain of primes contained in $(\mathcal{F})'$.

In view of the note after Theorem 8, the question arises as to whether functions $g$ from $I$ to $\omega$, which are unbounded on every $U \in \mathcal{U}$, exist. The next few paragraphs deal with this question.

For $f$ and $g$ functions from $I$ to $\omega$, we say that $f \prec g$ (modulo $\mathcal{U}$) if $f | g$ is bounded away from $0$ and $\infty$ on some $U \in \mathcal{U}$. Note that $\prec$ is an equivalence relation. The equivalence classes are linearly ordered and in order reversing bijection with the set of prime ideals of the form $(\mathcal{F})$ or $(\mathcal{F})'$ for a fixed $\mathcal{F}$.

An ultrafilter $\mathcal{U}$ on $I$ is called $\alpha_1$-complete if every countable partition of $I$ contains an element of $\mathcal{U}$ (see [CN]). An ultrafilter is called $\alpha_1$-incomplete if it is not $\alpha_1$-complete.

Let $\mathcal{U}$ be an $\alpha_1$-complete ultrafilter on $I$, and let $g$ be any function from $I$ to $\omega$. Then $\{g^{-1}(n): n \in \omega\}$ forms a countable partition of $I$, and hence $g$ is bounded on a filter element. Therefore, there are no prime ideals of $R$ strictly contained between $(\mathcal{F})$ and $(\mathcal{F})'$ by Corollary 11. Since every principal ultrafilter on $I$ is $\alpha_1$-complete, there are no prime ideals between $(\mathcal{F})$ and $(\mathcal{F})'$ as noted after Proposition 1.

Let $\mathcal{U}$ be an $\alpha_1$-incomplete ultrafilter on $I$. Then there exists a countable partition \{\langle $A_1$, $n \in \omega$ \}\} of $I$ such that $\mathcal{A} \notin \mathcal{U}$ for every $n \in \omega$. Define $g_1: I \rightarrow \{1, \infty\}$ via $g_1(A_1) = \mathcal{A}$; then $g_1$ is unbounded on every $U \in \mathcal{U}$. Now for $r \in [1, \infty]$, let $g_r: I \rightarrow \{1, \infty\}$ be defined by $g_r(y) = g_1(y)$. Then $g_r$ is unbounded on every $U \in \mathcal{U}$, and if $r < s$, then $g_r/g_s$ is unbounded on every $U \in \mathcal{U}$. Thus there are at least $\mathcal{c} = |\mathcal{R}|$ equivalence classes (modulo $\mathcal{U}$) of functions $g$ from $I$ to $[1, \infty)$ (or $\omega$).

If $I$ has non-measurable cardinal (e.g., if $I$ is countable), then every non-principal ultrafilter on $I$ is $\alpha_1$-incomplete. A set with a measurable cardinal (if such a set exists) has a non-principal $\alpha_1$-complete ultrafilter on it. (See [CN]).

The following example, which is due to John Kulesza, shows that, assuming Continuum Hypothesis (CH), there exists an ultrafilter $\mathcal{U}$ on $I$ of cardinality $\mathcal{c}$ for which there are more than $\mathcal{c}$ many equivalence classes (modulo $\mathcal{U}$) of functions $g$ from $I$ to $\omega$.

Example. Let $\mathcal{V}$ be a non-principal ultrafilter on $\omega$ and let $\mathcal{W}$ be a uniform ultrafilter on $R$, that is, an ultrafilter which each element of which has cardinality $\mathcal{c}$. Define an ultrafilter $\mathcal{U}$ on $I = \omega \times R$ by $U \in \mathcal{U}$ if and only if

\[ \{x: \{y: (x, y) \in U\} \in \mathcal{W}\} \in \mathcal{V}. \]

Assuming CH we well order $R$ as $\{\alpha: \alpha < \omega_1\}$. Let $\{f_\alpha: \alpha < \omega_1\}$ be a collection of $\mathcal{c}$ many functions from $I$ to $\omega$. For $\alpha < \omega_1$, let $g_\alpha: \omega \times \{y\} \rightarrow \omega$ be such that for all $\lambda < \omega$,

\[ \{\alpha < \omega_1: g_\alpha(n, y) < f_\alpha(n, y)\} \text{ is finite.} \]

Define $g: I \rightarrow \omega$ by $g | \alpha_{\mathcal{U}} = \mathcal{V}$.

Claim. For each $\lambda < \omega_1$, \{(x, y): f_\lambda(x, y) > g(x, y)\} \in \mathcal{V}$.

Assume to the contrary, that \{(x, y): f_\lambda(x, y) > g(x, y)\} \in \mathcal{U}. Since $\mathcal{W}$ is uniform, there is a $y = y_\lambda$ such that $\lambda > \mathcal{c}$ and \{(x, y): f_\lambda(x, y) > g(x, y)\} \in \mathcal{V}$. But $f_\lambda(x, y) \leq \mathcal{V} \Rightarrow g(x, y) = g_{\lambda_{\mathcal{U}}(y)}$. This proves the claim.
Now let \( h(x, y) = g(x, y) (1 + x + y) \). Then for each \( U \in \mathfrak{U} \), \( h|_U \) is unbounded on \( U \), so \( h > f_\lambda \) for each \( \lambda < \omega_1 \). Therefore there are more than continuum many equivalence classes (modulo \( U \)) of functions \( f \) from \( I \) to \( \omega \).

\section{The structure of prime ideals}

We want to investigate the order type of the chain of prime ideals between \((\mathfrak{U})\) and \((\mathfrak{F})\). As we have seen, if \( \mathfrak{U} \) is \( \omega_1 \)-incomplete, then there are at least \( c \) such prime ideals. Now if \( |I| = \omega_1 \), then the ring \( \mathfrak{R} \) has \( 2^\omega \) elements and hence the chain of prime ideals between \((\mathfrak{U})\) and \((\mathfrak{F})\) contains at most \( 2^{2^\omega} \) elements.

We can actually get more information on the structure of the order of the prime ideals of the form \((\mathfrak{F})_I\) in the case where \( I = \omega_1 \). A linearly ordered set \( S \) is called an \( \eta_1 \)-set if given any countable subsets \( F \) and \( G \) of \( S \) such that every element of \( F \) is less than every element of \( G \), then there exists an element \( h \in S \) such that \( f < h < g \) for every \( f \in F \) and every \( g \in G \).

**Theorem 13.** Let \( I = \omega \) and let \( \mathfrak{F} \) be an ultrafilter on some set \( \mathfrak{E} \). Assume that the associated ultrafilter \( \mathfrak{U} \) on \( \omega \) is non-principal. Then the set of prime ideals of the form \((\mathfrak{F})_I\) between \((\mathfrak{U})\) and \((\mathfrak{F})\), where \( g \) is unbounded on \( \mathfrak{U} \), is an \( \eta_1 \)-set.

**Proof.** Recall that the prime ideals of the form \((\mathfrak{F})_I\) between \((\mathfrak{U})\) and \((\mathfrak{F})\) are in order reversing bijection with the equivalence classes of functions \( f \) from \( I \) to \( \omega_1 \) where \( f \sim g \) if \( fg \) is bounded away from \( 0 \) and \( \omega_1 \) on some \( U \in \mathfrak{U} \). The order is given by \( f < g \) if \( fg \) is unbounded on \( \mathfrak{U} \). It is therefore sufficient to show that the set of equivalence classes with this order is an \( \eta_1 \)-set.

Let \( F = \{ f_0, f_1, f_2, \ldots \} \) and \( G = \{ g_0, g_1, g_2, \ldots \} \) be countable sets of equivalence classes of functions \( f_n \) and \( g_n \) from \( I \) to \( \omega_1 \) such that \( f_n < g_n \) for all \( n \) and \( m \). Clearly, we can assume that

\[
f_0 < f_1 < f_2 < \ldots \leq g_1 < g_2 < \ldots \]

We will assume that \( F \) and \( G \) have infinitely many equivalence classes. The other cases are similar and easier. Therefore, we can assume that the above inequalities are strict.

**Claim.** We can assume that \( f_i < f_{i+1} \) and \( g_i < g_{i+1} \) for all \( i \leq I \).

Let \( f_0 \) and \( g_0 \) be defined via \( f_0 = \max_{k \in I} f_k(0) \) and \( g_0 = \min_{k \in I} g_k(0) \). Then \( f_0 < f_1 \) and \( g_0 > g_1 \). So we have the claim.

It is not too difficult to see that it is possible to define a countable collection \( \{ U_k : k = 0, 1, 2, \ldots \} \) of elements of \( \mathfrak{U} \) satisfying the following:

1. \( U_0 = I = \omega \);
2. \( U_i \supseteq U_{i+1} \) for all \( k \);
3. \( \bigcap U_i = \emptyset \) (e.g., delete \( k \) from \( U_k \) for each \( k \));
4. \( g_k(0)(x) > 0 \) for all \( i \in U_k \) and for all \( k \).

To accomplish 4) we use the fact that, since \( g_k(0) \) is unbounded on \( \mathfrak{U} \), the set \( \{ k : g_k(0)(x) > 0 \} \) is in \( \mathfrak{U} \).

Let \( h \) be the function from \( I \) to \( \omega_1 \) defined via \( h(i) = \sqrt{g_i(0)}(0) \) for \( i \in U_k \setminus U_{k+1} \).

Note that conditions 1), 2) and 3) assure that \( h \) is well defined.

**Claim.** \( h > f_\lambda \) for all \( \lambda \).

Suppose to the contrary that for a fixed \( k \), \( h(0)f_\lambda(0) < 0 \) for some \( U \in \mathfrak{U} \) and some \( B \in \omega \). Let \( m \) be a positive integer larger than \( 2^\omega \) and \( k \), and let \( t \) be \( U \cap U_m \). Then \( h(0) = \sqrt{f_\lambda(0)g_0(0)} \) for some \( r \) \( \geq m \). Hence

\[
\frac{h(0)}{f_\lambda(0)} = \frac{\sqrt{f_\lambda(0)g_0(0)}}{f_\lambda(0)} > \frac{\sqrt{f_\lambda(0)g_0(0)}}{f_\lambda(0)} = \frac{g_0(0)}{f_\lambda(0)}
\]

By condition 4) and the assumptions we have on \( r \) and \( m \), we also know that

\[
\sqrt{g_0(0)f_\lambda(0)} > \sqrt{r} \geq \sqrt{m} \geq B.
\]

The two inequalities combined give a contradiction, so the claim is proved.

The proof that \( h > g_\lambda \) for all \( \lambda \) is done in a similar fashion.

**Remarks.** (i) The proof of Theorem 13 can be generalized to the case where \( \mathfrak{U} \) is an \( \omega_2 \)-incomplete ultrafilter on an arbitrary index set \( I \). The only potential difficulty is in condition 3) on the sets \( U_k \). So let \( U_{k+1} \supseteq U_k \). Then we can replace \( U_k \) with \( U_k \setminus U_{k+1} \) if needed, which satisfies condition 3).

(ii) It follows from Theorem 13 that, if \( \mathfrak{U} \) is \( \omega_1 \)-incomplete, then the set of prime ideals of the form \((\mathfrak{F})_I\) and \((\mathfrak{F})_j\), where \( g \) is unbounded, is the lexicographic product of an \( \eta_1 \)-set with \( \{ 0, 1 \} \).

(iii) It follows from (ii) and Corollaries 11 and 12 that, if \( \mathfrak{U} \) is \( \omega_1 \)-incomplete, then the primes between \((\mathfrak{F})_I\) and \((\mathfrak{F})_j\) can be obtained from the Dedekind completion of an \( \eta_1 \)-set by splitting each element of the \( \eta_1 \)-set into two consecutive elements and then adjoining a first element, a last element and last element (respectively) of \((\mathfrak{F})_I\) (where \( g \) is any constant). Furthermore, since any Dedekind complete space containing an \( \eta_1 \)-set has cardinality at least \( 2^\omega \) (see [GJ]), there are at least \( 2^\omega \) primes between \((\mathfrak{U})\) and \((\mathfrak{F})\). In particular, if \( I = \omega_1 \), then, assuming \( \text{CH} \), there are exactly \( 2^\omega \) prime ideals between \((\mathfrak{U})\) and \((\mathfrak{F})\).

**Corollary 14.** Let \( \mathfrak{U} \) be an \( \omega_1 \)-incomplete ultrafilter on \( I \). Then no prime between \((\mathfrak{U})\) and \((\mathfrak{F})\) is the union of countably many \((\mathfrak{F})_I\)'s and the intersection of countably many \((\mathfrak{F})_I\)'s.

**Proof.** In the Dedekind completion of an \( \eta_1 \)-set, no singleton is the intersection of countably many non-degenerate intervals.

**Corollary 15.** Let \( I = \omega \) and assume that \( 2^\omega < 2^\omega \). (This set-theoretic assumption will hold if, for example, the continuum hypothesis is assumed.) Then there exist prime ideals between \((\mathfrak{U})\) and \((\mathfrak{F})\) which are neither a union of countably many \((\mathfrak{F})_I\)'s nor an intersection of countably many \((\mathfrak{F})_I\)'s.

**Proof.** There are only continuum many ideals of the form \((\mathfrak{F})_I \) or \((\mathfrak{F})_j \) because there are only continuum many choices for the functions \( g \). Hence, there are only \( 2^{\omega_2} \) intersections or unions of countably many sets of this form. But as noted in Remark (iii) above, there are \( 2^{\omega_1} \) prime ideals between \((\mathfrak{U})\) and \((\mathfrak{F})\).
We close with some questions:
1. Does Corollary 15 hold with no set-theoretic assumptions?
2. Does Corollary 15 hold under MA+$\neg$CH?
3. Is there a generalization of Corollary 15 to higher cardinals?
4. Given an index set $I$ and an ultrafilter $\mathcal{U}$ on $I$, is it possible to determine the cardinality of the set of prime ideals between $(\mathcal{U})$ and $(\mathcal{F})$ (perhaps assuming some additional set-theoretic hypothesis)?

References


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On some subclasses of Darboux functions

by

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Abstract. The maximal additive, multiplicative and lattice-like classes for some classes of real functions are computed.

1. Introduction. We shall consider mainly real functions of a real variable, however, some of the considered functions will be defined on different sets, and sometimes the range sets will be different. Let us settle some of the notations to be used in the article.

$\mathcal{C}$ — the class of constant functions,
$\mathcal{D}$ — the class of connected functions,
$\mathcal{E}$ — the class of continuous functions,
$\mathcal{A}$ — the class of almost continuous functions,
$\mathcal{B}$ — the class of Darboux functions,
$\mathcal{B}_1$ — the class of Darboux functions of the first class of Baire,
$\mathcal{F}$ — the class of functionally connected functions ([5]),
$\mathcal{Lc}$ ($\mathcal{Us}$) — the class of lower (upper) semicontinuous functions,
$\mathcal{M}$ — the class of Darboux functions $f$ with the following property: if $x_0$ is a right-hand (left-hand) point of discontinuity of $f$, then $f(x_0) = 0$ and there is a sequence $(x_n)$ converging to $x_0$ such that $x_n > x_0$ ($x_n < x_0$) and $f(x_n) = 0$,
$\mathcal{D}_0$ — the class of all functions $f$ such that for each $x$ from the domain $f(x) \in L^-(f, x) \cap L^+(f, x)$ and the sets $L^-(f, x)$, $L^+(f, x)$ are closed intervals.

The symbols $L^-(f, x)$, $L^+(f, x)$ denote the cluster sets from the left and right, respectively, of the function $f$ at the point $x$.

Notice that if $f \in \mathcal{D}$, then the set $E$ of all points of discontinuity of $f$ is nowhere dense and $f(x) = 0$ for each $x$ in $E$. Consequently, $f$ is a function of the first class of Baire, hence $\mathcal{M} \subseteq \mathcal{D} \subseteq \mathcal{F}$, Since $\mathcal{D} \cap \mathcal{A}_1 = \mathcal{A} \cap \mathcal{B}_1 = \mathcal{C} \cap \mathcal{D} \cap \mathcal{B}_1 ([1])$, we have $\mathcal{M} \subseteq \mathcal{A}_1$. Thus for the classes of real functions defined on an interval we have $\mathcal{C} \subseteq \mathcal{M} \subseteq \mathcal{A}_1 \subseteq \mathcal{D} \subseteq \mathcal{F} \subseteq \mathcal{B}_0$.

Let $\mathcal{F}$ be a class of real functions. The maximal additive (multiplicative, lattice-like, respectively) class for $\mathcal{F}$ is defined to be the class of all $f \in \mathcal{F}$ for which $f + g \in \mathcal{F}$ ($fg \in \mathcal{F}$, $\max(f, g)$ and $\min(f, g) \in \mathcal{F}$, respectively) whenever $g \in \mathcal{F}$. The respective classes are denoted by $\mathcal{M}_a(\mathcal{F})$, $\mathcal{M}_m(\mathcal{F})$, $\mathcal{M}(\mathcal{F})$. 