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Strong cellularity and global asymptotic stability

by

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Abstract. For (semi)dynamical systems on infinite-dimensional Banach spaces, a topological characterization of nonempty compact invariant globally asymptotically stable sets is given. The proofs are based on a paper by McCoy [11] and on other results of infinite-dimensional topology.

I. Introduction and the finite-dimensional case. Let $(X, \|\cdot\|)$ be a Banach space. The origin of X is denoted by 0_X. The closed ball and sphere of radius r centered at 0_X are denoted by $B(r)$ and $\partial B(r)$, respectively. In general, ∂ denotes the boundary of sets in X. The distance between a point $x \in X$ and a nonempty set $Y \subset X$ is defined as $d(x, Y) = \inf\{\|x-y\| \mid y \in Y\}$.

A closed subset C of X is called a *cell* in X if there exists a homeomorphism from the pair $(B(1), \partial B(1))$ onto the pair $(C, \partial C)$. A subset A of X is called *cellular* if there is a cellular sequence for A, i.e. a sequence $\{C_n\}$ of cells in X such that $\bigcap\{C_n \mid n \in N\} = A$ and $C_{n+1} \subset \operatorname{int}(C_n)$ for each $n \in N$. A subset A of X is called *strongly cellular* if there is a strongly cellular sequence for A, i.e. a cellular sequence $\{C_n\}$ with the additional property that for each open set U in X containing A, there is an integer n such that $C_n \subset U$.

A subset A of X is called *point-like* if $X \setminus A$ is homeomorphic to $X \setminus \{0_X\}$. A compact connected subset A of X is cellular if and only if it is point-like. Strongly cellular subsets are compact and connected. Compact subsets of infinite-dimensional Banach spaces are point-like and cellular. In the finite-dimensional case, cellularity is equivalent to strong cellularity. For these and other properties of cellularity resp. strong cellularity, see [11], [10].

The continuous mapping π: $\mathbf{R} \times X \to X$ (π: $\mathbf{R}^+ \times X \to X$) is called a *dynamical (semidynamical) system* if $\pi(0, x) = x$ for all $x \in X$ and $\pi(t+\tau, x) = \pi(t, \pi(\tau, x))$ for all t, $\tau \in \mathbf{R}$, $x \in X$ (for all $t, \tau \in \mathbf{R}^+$, $x \in X$). In most applications, dynamical systems are induced (both on finite- and infinite-dimensional Banach spaces) by ordinary differential equations. Similarly, in most applications, semidynamical systems are induced (on infinite-dimensional Banach spaces) by retarded or partial differential equations.

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The following definitions make sense for dynamical as well as for semidynamical systems. A subset Y of X is said to be *invariant* if $\{\pi(t, y) \mid y \in Y\} = Y$ for all $t \in \boldsymbol{R}^+$. Let M be a nonempty compact invariant set. Its *region of attraction* is defined by $A(M) = \{x \in X \mid d(\pi(t, x), M) \to 0 \text{ as } t \to \infty\}$. The set M is said to be *asymptotically stable* if $A(M) \supset \{y \in X \mid d(y, M) \leqslant \eta\}$ for some $\eta > 0$ and if, given $\varepsilon > 0$ arbitrarily, there exists a $\delta > 0$ such that $d(\pi(t, x), M) < \varepsilon$ whenever $d(x, M) < \delta$, $t \geqslant 0$. If, in addition, $A(M) = X$, then M is called *globally asymptotically stable*. The study of (semi)dynamical systems with globally asymptotically stable compact invariant sets is one of the major topics of infinite-dimensional topological dynamics [6].

The following result is essentially known [2].

THEOREM 1.1. *Let X be a finite-dimensional Banach space and let π: $\boldsymbol{R} \times X \to X$ be a dynamical system on X. Moreover, let M be a nonempty compact invariant asymptotically stable subset of X. Then the following statements are equivalent:*

(i) *M is strongly cellular.*
(ii) *$A(M)$ is homeomorphic to X.*
(iii) *There exists a neighborhood U of M in $A(M)$ which is homeomorphic to X.*

Proof. In case $M = \{0_X\}$, the implication (i)$\Rightarrow$(ii) was proved in [2, Thm. V.3.4]. The general case follows from the very same argument. In fact, assume that M is strongly cellular. Then there is a cell C such that $M \subset \text{int}(C) \subset C \subset A(M)$. By a simple compactness argument, $C \subset \{\pi(-p, y) \mid y \in C\}$ for some fixed $p \in N \backslash \{0\}$. For $n \in N$, define $\tilde{C}_{-n} = \{\pi(-pn, y) \mid y \in C\}$. Hence $\text{int}(C) \subset \text{int}(\tilde{C}_{-1})$ and, by induction, $\text{int}(\tilde{C}_{-n}) \subset \text{int}(\tilde{C}_{-(n+1)})$ for each $n \in N$. Observe that $A(M) = \bigcup \{\text{int}(\tilde{C}_{-n}) \mid n \in N\}$. Since the union of an increasing sequence of open n-cells is an open n-cell [5], $A(M)$ is homeomorphic to X. The implication (ii)$\Rightarrow$(iii) is trivial. Suppose now that (iii) is satisfied. Then there is a cell C such that $M \subset \text{int}(C) \subset C \subset U$. By a simple compactness argument, $\{\pi(q, y) \mid y \in C\} \subset \text{int}(C)$ for some fixed $q \in N \backslash \{0\}$. For $n \in N$, define $\hat{C}_n = \{\pi(qn, y) \mid y \in C\}$. It is easy to see that $\{\hat{C}_n\}$ is a strongly cellular sequence for M. (We remark here that the implication (ii)$\Rightarrow$ "M is point-like" was proved earlier [2, Thm. V.3.6]. (Since M is obviously connected, this is equivalent to strong cellularity.))

By a simple compactness argument, each cellular sequence is, in finite-dimensional spaces, strongly cellular. Consequently, in stating Theorem 1.1, it is possible (and would have probably been more natural) to replace (i) by

(i)′ *M is cellular.*

However (and this is why we prefer (i)), Theorem 2.1, the infinite-dimensional version of Theorem 1.1, does not remain valid if (i) is replaced by (i)′. For completeness, we give a simple counterexample: Let X be an arbitrary infinite-dimensional Banach space and let $M = \{x_1, x_2\}$ be an arbitrary two-point subset of X. It is not hard to define a dynamical system on X for which x_i $(i = 1, 2)$ is an equilibrium and M is asymptotically stable. A simple connectedness argument implies that $A(M)$ is not connected. Hence no neighborhood of M in $A(M)$ is homeomorphic to X. But M is cellular [11].

Remarkably, Theorem 2.1 works for semidynamical systems and not only for dynamical ones as Theorem 1.1. (See also Remark 2.5.) We state and prove Theorem 2.1 for semidynamical systems. This is of some importance since functional differential equations, large classes of partial differential equations etc. give rise [6] only to semidynamical systems and not to dynamical ones.

It is well-known that all the previous dynamical concepts can be defined for discrete dynamical/semidynamical systems as well. (In all the previous definitions, $\boldsymbol{R}$ and $\boldsymbol{R}^+$ should be replaced by $\boldsymbol{Z}$ and $\boldsymbol{N}$, respectively.) It is easy to see that Theorem 1.1 remains valid for discrete dynamical systems. No alterations in the proof are needed. We do not know whether Theorem 1.1 remains true for semidynamical systems. (See also Remark 2.5.) On the other hand, Theorem 1.1 is false for discrete semidynamical systems. This is shown by the following two simple examples:

EXAMPLE 1.2. For $x \in \boldsymbol{R}$, define

$$f(x) = \begin{cases} 2|x|/(1+|x|) & \text{if } |x| < 1, \\ 1 & \text{if } |x| \geqslant 1. \end{cases}$$

Observe that f: $\boldsymbol{R} \to \boldsymbol{R}$ is continuous and consider the induced discrete semidynamical system. It is easy to check that $|f(x) - 1| \leqslant |x - 1|$ for all $x \in \boldsymbol{R}$, $f(1) = 1$, $f(0) = 0$ and $f^n(x) \to 1$ as $n \to \infty$ whenever $x \neq 0$. Thus, $x_0 = 1$ is an asymptotically stable equilibrium point with $A(\{x_0\}) = \boldsymbol{R} \backslash \{0\}$. In particular, $M = \{x_0\}$ satisfies (i) and (iii) but not (ii).

EXAMPLE 1.3. Write

$$Q = \{(x, y) \in \boldsymbol{R}^2 \mid 1/4 \leqslant x^2 + y^2 \leqslant 1\}, \quad S = \{(x, y) \in \boldsymbol{R}^2 \mid x^2 + y^2 = 1\}.$$

Let g: $\boldsymbol{R}^2 \to Q$ be an arbitrary continuous function with $g(S) \supset S$. Consider the induced discrete semidynamical system. It is well-known (see e.g. [6, Thm. 2.4.2]) that $M = \bigcap \{g^n(Q) \mid n \in N\}$ is a nonempty compact invariant globally asymptotically stable subset of $\boldsymbol{R}^2$. In particular, M satisfies (ii). But (i)′ is violated: since $S \subset M \subset Q$, M is not cellular.

II. The infinite-dimensional case. The main result of this paper is the following infinite-dimensional generalization of Theorem 1.1.

THEOREM 2.1. *Let X be an infinite-dimensional Banach space and let π: $\boldsymbol{R}^+ \times X \to X$ be a semidynamical system on X. Moreover, let M be a nonempty compact invariant asymptotically stable subset of X. Then the following statements are equivalent:*

(i) *M is strongly cellular.*
(ii) *$A(M)$ is homeomorphic to X.*
(iii) *There exists a neighborhood U of M in $A(M)$ which is homeomorphic to X.*

The proof is based on three lemmas. The first is an elementary result on Lyapunov functions [2]. Lemma 2.3 collects several results from shape theory [4]. (We will make use only of the implication (c)$\Rightarrow$(d) (in the case of X being infinite-dimensional) but we have not been able to find any direct references. (For the definition of contractibility, see the proof of Theorem 2.1.)) The core of the whole proof is Lemma 2.4, a fundamental result of infinite-dimensional topology [1].

LEMMA 2.2. *Let X be a Banach space, π: $R^+ \times X \to X$ a semidynamical system on X and M a nonempty compact invariant asymptotically stable subset of X. Then $A(M)$ is open and there is a continuous function V: $A(M) \to R^+$ with the properties that $V(x) = 0$ whenever $x \in M$, and $V(x) \geqslant d(x, M)$, $V(\pi(t, x)) < V(x)$ whenever $x \in A(M) \setminus M$, $t > 0$.*

Proof. For $x \in A(M)$, define

$$V(x) = \sup\{(1+\arctan t)\cdot d(\pi(t, x), M) | \ t \geqslant 0\}$$

and repeat the proof of [2, Prop. V.4.15] and [2, Thm. V.4.18].

LEMMA 2.3. *Let X be a Banach space and let W be a nonempty compact subset of X. Then the following statements are equivalent:*

(a) *W has trivial shape in the sense of Borsuk.*

(b) *W is a fundamental absolute retract.*

(c) *W is contractible in its every neighborhood in X.*

Further, property (a) *is implied by*

(d) *W is strongly cellular.*

If X is infinite-dimensional, then (a) *is equivalent to* (d).

Proof. The equivalence of (a), (b) and (c) is a combination of [3, Thm. 7.1] ((a)⇔(b)) and [9, Thm. (b) and (i)] ((b)⇔(c)). The rest is [11, Thm. 3.2]. The implication (a)⇒(d) was proved in [11] under the condition that X is homeomorphic to $X \times l_2$ where l_2 is the usual infinite-dimensional separable Hilbert sequence space. Then it was conjectured that this latter property holds true for all infinite-dimensional Banach spaces. The verification of this conjecture was done several years later [13].

LEMMA 2.4. *Let X be an infinite-dimensional Banach space and let Q be a contractible open subset of X. Then Q is homeomorphic to X.*

Proof. This is a special case of [1, Thm. IX.7.3], which was proved in [1] under the "extra" assumption that X is homeomorphic to X^N. The conjecture that this "extra" assumption holds true for all infinite-dimensional Banach spaces was verified several years later: infinite-dimensional Banach spaces of the same density character are homeomorphic [13].

Proof of Theorem 2.1. (i)⇒(ii). Fix $p \in M$ arbitrarily. In virtue of Lemma 2.4, we have to prove that $A(M)$ is an open subset of X and there exists a continuous mapping H: $[0, 1] \times A(M) \to A(M)$ with $H(0, x) = x$ and $H(1, x) = p$ for all $x \in A(M)$. For some fixed $m \in N$, the strong cellularity of M implies that $C_n \subset \{y \in X | \ d(y, M) < \eta\} \subset A(M)$ whenever $n \geqslant m$. Since cells are contractible, there is a continuous mapping h: $[0, 1] \times C_m \to C_m$ such that $h(0, x) = x$ and $h(1, x) = p$ for all $x \in C_m$. By Lemma 2.2, $A(M)$ is open and if $c > 0$ is sufficiently small, then $V^{-1}([0, c]) \subset C_m$ and further, given $x \in V^{-1}([c, \infty))$ arbitrarily, there exists a unique $\tau_c(x) \in R^+$ such that $\pi(\tau_c(x), x) \in V^{-1}(\{c\})$. It is easy to see that the mapping τ_c: $V^{-1}([c, \infty)) \to R^+$ is continuous. The desired homotopy H: $[0, 1] \times A(M) \to A(M)$ can be defined by

$$H(t, x) = \begin{cases} \pi(2\tau_c(x)t, x) & \text{if } x \in V^{-1}([c, \infty)),\ 0 \leqslant t \leqslant 1/2, \\ x & \text{if } x \in V^{-1}([0, c]),\ 0 \leqslant t \leqslant 1/2, \\ h(2t-1, \pi(\tau_c(x), x)) & \text{if } x \in V^{-1}([c, \infty)),\ 1/2 \leqslant t \leqslant 1, \\ h(2t-1, x) & \text{if } x \in V^{-1}([0, c]),\ 1/2 \leqslant t \leqslant 1. \end{cases}$$

(ii)⇒(iii). This is trivial.

(iii)⇒(i). Fix $p \in M$ arbitrarily. In virtue of Lemma 2.3, we have to prove that given an open neighborhood D of M in X, there is a continuous mapping K: $[0, 1] \times M \to D$ with $K(0, x) = x$ and $K(1, x) = p$ for all $x \in M$. There is no loss of generality in assuming that $D \subset U$. Since U is homeomorphic to X, it is contractible. Consequently, there is a continuous mapping k: $[0, 1] \times U \to U$ such that $k(0, x) = x$ and $k(1, x) = p$ for all $x \in U$. Applying Lemma 2.2 again, there is a constant $c > 0$ such that $V^{-1}([0, c]) \subset D$ and that the function τ_c: $V^{-1}([c, \infty)) \to R^+$ defined by $\pi(\tau_c(x), x) \in V^{-1}(\{c\})$ is continuous. The desired homotopy K: $[0, 1] \times M \to D$ can be defined by

$$K(t, x) = \begin{cases} k(t, x) & \text{if } k(t, x) \in V^{-1}([0, c]), \\ \pi(\tau_c(k(t, x)), k(t, x)) & \text{if } k(t, x) \in V^{-1}([c, \infty)). \end{cases}$$

Remark 2.5. It is worth mentioning that Theorem 2.1 remains true if $X = R^2$ (or if $X = R$, the latter being trivial). No alterations in the proof are needed. In fact, the implication (c)⇒(*d*) holds true if $X = R^2$. This is an easy consequence of the topological characterization of plane fundamental absolute retracts: they are [4] exactly those nonempty continua which do not decompose the plane. Similarly, Lemma 2.4 is valid if $X = R^2$: this is the well-known topological characterization of simply-connected open plane subsets (see e.g. [12, Section VI.2]). Unfortunately, in general finite-dimensional spaces, the proof of Theorem 2.1 breaks down. As is indicated in [11], the implication (c)⇒(d) does not hold true if $X = R^n$, $n \geqslant 3$. Probably the simplest counterexample is a wild arc W in R^n, $n \geqslant 3$, whose complement is not simply-connected. (Observe that W is compact contractible but not point-like.) Similarly, Lemma 2.4 is false for $X = R^n$, $n \geqslant 3$. The first counterexample was constructed by J. H. C. Whitehead in 1935. For details, see [10, esp. p. 540]. So we do not know whether Theorem 2.1 is true or false when $X = R^n$, $n \geqslant 3$.

PROPOSITION 2.6. *Let X be an infinite-dimensional Banach space and let π: $Z \times X \to X$ be a discrete dynamical system on X. Moreover, let M be a nonempty compact invariant asymptotically stable subset of X. Assume that there is a cell C in X such that $M \subset \mathrm{int}(C) \subset C \subset A(M)$.*

(A) *M is connected.*

(B) *If $\{\pi(q, y) | \ y \in C\} \subset C$ for some $q \in N \setminus \{0\}$, then $A(M)$ is homeomorphic to X.*

(C) *If $\sup\{d(\pi(n, y), M) \in R^+ | \ y \in C\} \to 0$ as $n \to \infty$, then M is strongly cellular.*

Proof. (A) The connectedness of M follows from an easy application of [6, Lemma 2.4.1].

(B) For $n\in N$, define $\tilde{C}_{-n}=\{\pi(-qn, y)|\ y\in C\}$. It is easy to see that $C\subset\tilde{C}_{-1}$, $\mathrm{int}(C)\subset\mathrm{int}(\tilde{C}_{-1})$ and, by induction, $\mathrm{int}(\tilde{C}_{-n})\subset\mathrm{int}(\tilde{C}_{-(n+1)})$ for each $n\in N$. Observe that $A(M)=\bigcup\{\mathrm{int}(\tilde{C}_{-n})|\ n\in N\}$. By [11, Lemma 4.1], $A(M)$ is contractible. Hence, in virtue of Lemma 2.4, $A(M)$ is homeomorphic to X.

(C) Observe that $\{\pi(r, y)|\ y\in C\}\subset\mathrm{int}(C)$ for some $r\in N\backslash\{0\}$. For $n\in N$, define $\hat{C}_n=\{\pi(rn, y)|\ y\in C\}$. It is easy to see that $\{\hat{C}_n\}$ is a strongly cellular sequence for M.

We do not know whether Theorem 2.1 is true or false for discrete dynamical systems. This problem and the one discussed in Remark 2.5 are certainly worth further investigations. In spite of Theorem 2.1 and several results in [7], [8], the link between shape theory and stability/attraction properties of (discrete) (semi)dynamical systems is not fully clarified either. Asymptotic fixed point theory [6] is also related. To the best of our knowledge, for dynamical systems in infinite-dimensional Banach spaces, it is not known whether the existence of a nonempty compact invariant globally asymptotically stable set implies the existence of an equilibrium point.

The following result, combined with Theorem 2.1, gives a full topological characterization of nonempty compact invariant globally asymptotically stable sets for (semi)dynamical systems on infinite-dimensional Banach spaces:

THEOREM 2.7. *Let X be a (finite- or infinite-dimensional) Banach space and let M be a nonempty strongly cellular compact subset of X. Then there exists a dynamical system $\pi\colon \mathbf{R}\times X\to X$ on X such that M consists of equilibria and M is globally asymptotically stable.*

Proof. Let $\{C_n\}$ be a strongly cellular sequence for M. In virtue of [11, Thm. 2.1], there is no loss of generality in assuming that there is a homeomorphism h of $X\backslash\{0_X\}$ onto $X\backslash M$ such that for all $n\in N$, h maps the pair $(X\backslash B(2^{-n}), \partial B(2^{-n}))$ onto the pair $(X\backslash C_n, \partial C_n)$. Write $a_n=2^{-n}$, $n\in N$, and $\mathscr{S}=\partial B(1)$.

We claim that given $n\in N$ arbitrarily, there exists a continuous function $b_n\colon \mathscr{S}\to(0, a_n]$ such that $\|h(\lambda s)-h(\mu s)\|\leqslant a_n$ whenever $\lambda, \mu\in[a_{n-3}, a_n]$, $|\lambda-\mu|<b_n(s)$, $s\in\mathscr{S}$.

We point out first that, for each $z\in\mathscr{S}$, there exists an open neighborhood U_z of z in $\mathscr{S}$ and a positive number c_z such that $\|h(\lambda s)-h(\mu s)\|\leqslant a_n$ whenever $\lambda, \mu\in[a_{n-3}, a_n]$, $|\lambda-\mu|<c_z$, $s\in U_z$. If not, there exist sequences $\{s_m\}\subset\mathscr{S}$, $\{\lambda_m\}, \{\mu_m\}\subset[a_{n-3}, a_n]$ such that $s_m\to z$ and $|\lambda_m-\mu_m|\to 0$ as $m\to\infty$ but $\|h(\lambda_m s_m)-h(\mu_m s_m)\|>a_n$ for all $m\in N$. By passing to a subsequence, we may assume that $\lambda_m\to\lambda$, $\mu_m\to\mu$ as $m\to\infty$ for some $\lambda=\mu\in[a_{n+3}, a_n]$. It follows that $0=\|h(\lambda z)-h(\mu z)\|\geqslant a_n$, a contradiction.

By the standard paracompactness argument, there exists a continuous function $b_n\colon \mathscr{S}\to(0, a_n]$ such that $b_n(s)<\sup\{c_z|\ s\in U_z\}$ for all $s\in\mathscr{S}$. It is easy to check that b_n has the desired property.

Replacing $b_n(s)$ by $\min\{b_k(s)|\ k=0, 1, \ldots, n-1\}$, we may assume that $b_{n+1}(s)\leqslant b_n(s)$ for all $s\in\mathscr{S}$, $n\in N$. For each $s\in\mathscr{S}$, define $\tau_0^s=0$ and $\tau_{n+1}^s=\tau_n^s+1/b_n(s)$, $n\in N$.

By letting

$$\varphi_s(t)=\begin{cases}1-t & \text{if } t\leqslant 0,\\ a_n(1-(t-\tau_n^s)b_n(s)/2) & \text{if } \tau_n^s<t\leqslant\tau_{n+1}^s,\ n\in N,\end{cases}$$

we define a function $\varphi_s\colon \mathbf{R}\to\mathbf{R}$. Observe that φ_s is a strictly decreasing continuous function, $\varphi_s(t)\to 0$ as $t\to\infty$, $\varphi_s(0)=1$ and the mapping $\mathscr{S}\times\mathbf{R}\to(0, \infty)$, $(s, t)\to\varphi_s(t)$, is continuous. It is easy to see that there is a uniquely defined dynamical system $\varrho\colon \mathbf{R}\times(X\backslash\{0_X\})\to X\backslash\{0_X\}$ such that $\varrho(t, s)=\varphi_s(t)s$ for all $t\in\mathbf{R}$, $s\in\mathscr{S}$. By letting

$$\pi(t, x)=\begin{cases}h(\varrho(t, h^{-1}(x))) & \text{if } x\in X\backslash M,\ t\in\mathbf{R},\\ x & \text{if } x\in M,\ t\in\mathbf{R},\end{cases}$$

we define a function $\pi\colon \mathbf{R}\times X\to X$.

Observe that $\pi(0, x)=x$, $\pi(t, \pi(\tau, x))=\pi(t+\tau, x)$ for all $x\in X$, $t, \tau\in\mathbf{R}$. The continuity of π on $\mathbf{R}\times(X\backslash M)$ is obvious. By the construction, given $n\in N$ arbitrarily, $s\in\mathscr{S}$, $\tilde{t}, t\in\mathbf{R}$, $\varphi_s(t)\in[a_{n+2}, a_{n+1}]$, $|\tilde{t}-t|<1/a_n$ imply that $\varphi_s(\tilde{t})\in[a_{n+3}, a_n]$ and $|\varphi_s(\tilde{t})-\varphi_s(t)|<b_n(s)$. Assume that $x\in C_{n+1}\backslash C_{n+2}$. Then $h^{-1}(x)=\varphi_s(t)s$ for some $s\in\mathscr{S}$ and $\varphi_s(t)=\mu\in[a_{n+2}, a_{n+1}]$. Consequently, if $\tilde{t}\in\mathbf{R}$, $|\tilde{t}-t|<1/a_n$, then $\varrho(\tilde{t}, s)=\varphi_s(\tilde{t})s$ where $\varphi_s(\tilde{t})=\lambda\in[a_{n+3}, a_n]$ and $|\varphi_s(\tilde{t})-\varphi_s(t)|<b_n(s)$. Since

$$\varrho(\tilde{t}-t, h^{-1}(x))=\varrho(\tilde{t}-t, \varphi_s(t)s)=\varrho(\tilde{t}-t, \varrho(t, s))=\varrho(\tilde{t}, s)=\varphi_s(\tilde{t})s,$$

it follows from the claim that

$$\|\pi(\tilde{t}-t, x)-x\|=\|h(\varrho(\tilde{t}-t, h^{-1}(x)))-h(h^{-1}(x))\|\leqslant a_n$$

whenever $x\in C_{n+1}\backslash C_{n+2}$, $\tilde{t}, t\in\mathbf{R}$, $|\tilde{t}-t|<1/a_n$. Consequently, since $\{a_n\}$ is decreasing, we have

$$\|\pi(\tau, y)-m\|\leqslant\|\pi(\tau, y)-y\|+\|y-m\|\leqslant a_k+\|y-m\|$$

whenever $m\in M$, $y\in C_{k+1}$, $k\in N$, $\tau\in\mathbf{R}$ and $|\tau|<1/a_k$. For each $(t, m)\in\mathbf{R}\times M$, the continuity of π at (t, m) follows immediately. Hence π is a dynamical system on X.

By construction, for all $x\in X$, $n\in N$, we have $\pi(t, x)\in C_n$ for t sufficiently large. Since $\{C_n\}$ is a neighborhood basis for M and M is compact, it follows that $d(\pi(t, x), M)\to 0$ as $t\to\infty$. On the other hand, for each $n\in N$, $x\in C_n$ implies that $\pi(t, x)\in C_n$ whenever $t\geqslant 0$. Thus, M is asymptotically stable and $A(M)=X$.

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