A splitting theorem for multipeak path algebras

by

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Abstract. Let F be a division ring, $\mathcal{Q}$ a locally finite quiver, Ω an admissible ideal [4] in the path algebra $F\mathcal{Q}$ of $\mathcal{Q}$ and suppose that the bound quiver algebra $R = F(\mathcal{Q}, \Omega) := F\mathcal{Q}/\Omega$ of $(\mathcal{Q}, \Omega)$ [5] is a right multipeak algebra in the sense that $\operatorname{soc}(R_R)$ is an essential and projective submodule of R_R [22]. Let $\operatorname{mod}_{sp}(R)$ be the category of finitely generated socle projective right R-modules. Following an idea in [15, 11] we prove that if the bound quiver $(\mathcal{Q}, \Omega)$ admits a splitting decomposition (3.2) explained in Fig. 6 then there are two proper bound subquivers $(\mathcal{J}', \Omega')$, $(\mathcal{J}'', \Omega'')$ of $(\mathcal{Q}, \Omega)$ and full faithful exact embeddings

$$\operatorname{mod}_{sp} F(\mathcal{J}', \Omega') \xrightarrow{T} \operatorname{mod}_{sp}(R) \xleftarrow{L} \operatorname{mod}_{sp} F(\mathcal{J}'', \Omega'')$$

such that any indecomposable module in $\operatorname{mod}_{sp}(R)$ belongs either to $\operatorname{Im} \boldsymbol{T}$ or to $\operatorname{Im} \boldsymbol{L}$. The functors $\boldsymbol{T}$ and $\boldsymbol{L}$ carry Auslander–Reiten sequences to Auslander–Reiten sequences and induce a splitting of the Auslander–Reiten translation quiver $\Gamma_{sp}(R)$ of $\operatorname{mod}_{sp}(R)$ (see 3.10, 3.11 and Fig. 7).

1. Introduction. Multipeak rings and socle projective modules play an important role in the study of matrix problems and vector space categories [21–24] as well as in the classification of indecomposable modules over finite-dimensional algebras [17–19] and lattices over orders [20]. One can prove that if $R = F(\mathcal{Q}, \Omega)$ is a simply connected [0] right peak algebra [21] and for any vertex $i \in \mathcal{Q}_0$ having no oriented path $w\colon i \to i$ in $(\mathcal{Q}, \Omega)$ there is a unique path (up to a scalar) from i to the unique sink in $(\mathcal{Q}, \Omega)$ then $\operatorname{mod}_{sp}(R)$ is equivalent to the category I-sp of I-spaces, where I is a poset, and therefore its representation type can be determined by the criteria of Kleiner [10] and Nazarova [14]. In case R is not simply connected there exists a universal Galois covering $(\tilde{\mathcal{Q}}, \tilde{\Omega})$ of $(\mathcal{Q}, \Omega)$ [5, 8, 13] such that $\tilde{R} = F(\tilde{\mathcal{Q}}, \tilde{\Omega})$ is a multipeak algebra and by [22; Theorem 1.10] the push-down functor [4] reduces the study of $\operatorname{ind}_{sp}(R)$ to the study of $\operatorname{ind}_{sp}(\tilde{R})$, where $\operatorname{ind}_{sp}(R)$ is the full subcategory of $\operatorname{mod}_{sp}(R)$ consisting of pairwise nonisomorphic representatives of indecomposable modules (see 4.2 and 4.3). In order to apply the technique developed in [4, 7, 8] one needs a description of supports of modules in $\operatorname{ind}_{sp}(\tilde{R})$. This is one of the motivations for studying splitting decompositions of $(\tilde{\mathcal{Q}}, \tilde{\Omega})$ because in many situations our splitting theorems 3.10 and 3.11 allow us to determine

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the supports of modules in $\operatorname{ind}_{sp}(\tilde{R})$ as well as the Auslander–Reiten quiver $\Gamma_{sp}(\tilde{R})$. This technique is applied in [24, 26, 27] where among other things we determine the sp-representation type of a class of right peak algebras R, which are incidence algebras of bipartite posets (comp. [16]). We also determine $\Gamma_{sp}(R)$ for any such algebra R which is sp-representation-finite. One of our tools applied in [26] is the splitting theorem. Another application of our splitting theorem is given in Section 4, where we determine the sp-representation type of $R = F(\mathcal{Q}, \Omega)$ in case $(\mathcal{Q}, \Omega)$ is a multiserial tree. The splitting theorem reduces the problem to a corresponding problem for I-spaces (comp. [5]).

In Section 2 we collect basic definitions and notation. In particular a reflection duality $D^{\bullet}\colon \operatorname{mod}_{sp}(R) \to \operatorname{mod}_{sp}(R^{\bullet})^{op}$ is defined. Moreover, given $(\mathcal{Q}, \Omega)$ we define a bound quiver $(\mathcal{Q}^{\bullet}, \Omega^{\bullet})$ such that $(F(\mathcal{Q}, \Omega))^{\bullet} \cong F(\mathcal{Q}^{\bullet}, \Omega^{\bullet})$ provided $\dim_F e_i F(\mathcal{Q}, \Omega) e_p \leqslant 1$ for all $i \in \mathcal{Q}_0$ and all sinks $p \in \mathcal{Q}_0$ (Corollary 2.22).

In Section 3 we present our main results (Theorem 3.10 and Corollary 3.11). As a consequence we get in Corollary 3.13 a generalization of an edge reduction given by Ringel and Roggenkamp [19]. Section 4 contains various examples illustrating our main results and their applications.

Throughout this paper $\operatorname{mod}(R)$ denotes the category of finitely generated right R-modules. The projective cover and the injective envelope of a module X in $\operatorname{mod}(R)$ will be denoted by $P(X)$ and $E_R(X)$, respectively. We call R sp-*representation-finite* (resp. *-tame*) if $\operatorname{ind}_{sp}(R)$ is finite (resp. $\operatorname{mod}_{sp}(R)$ is tame in the sense of [6]).

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2. Preliminaries. We collect here the results and notation we need for our splitting theorem. Throughout we suppose that

$$R = \bigoplus_{i \in I_R} e_i R \tag{2.0}$$

is a basic semiperfect F-algebra (in general without identity) and $\{e_i\}_{i \in I_R}$ is a fixed set of primitive orthogonal idempotents of R. We suppose that R is *locally bounded* (i.e. $\sum_j \dim_F e_i R e_j$ and $\sum_j \dim_F e_j R e_i$ are finite for every $i \in I_R$ [4]) and R is a *right multipeak algebra* [22] with the set $\{e_p\}_{p \in \boldsymbol{p}(I_R)}$, $\boldsymbol{p}(I_R) \subseteq I_R$, of *peak idempotents* (i.e. $e_p R$ is simple for any $p \in \boldsymbol{p}(I_R)$, $\operatorname{soc}(e_j R)$ is essential in $e_j R$ and isomorphic to a finite direct sum of modules $e_p R$, $p \in \boldsymbol{p}(I_R)$, for all $j \in I_R$). It follows that

$$R = \begin{pmatrix} A & {}_A N_B \\ 0 & B \end{pmatrix} \tag{2.1}$$

where $A = eRe$, $B = e_* R e_*$, $N = eRe_*$ and

$$e_* = \sum_{p \in \boldsymbol{p}(I_R)} e_p, \qquad e = \sum_{j \in I_R - \boldsymbol{p}(I_R)} e_j. \tag{2.2}$$

Here the sums are formal if the index set is infinite. By $(\sum_{j \in J} e_j) R$ we shall mean $\sum_{j \in J} e_j R$. Note that B is a direct sum of division rings $B_p = e_p R e_p$, ${}_A N$ is A-faithful and $\dim_F e_j N$ is finite for $j \in I_R$.

In the case $|\boldsymbol{p}(I_R)| = 1$, e_* is primitive and we call R a *right peak algebra with a peak* $e_* R$ [21].

Following [21; 2.6], [3; p. 906], [9] we associate with R two *reflection forms*

$$R^{\blacktriangledown} = \begin{pmatrix} B & D(Re_*) \\ 0 & R \end{pmatrix} \cong \begin{pmatrix} B & \tilde{N} & B \\ 0 & A & N \\ 0 & 0 & B \end{pmatrix}, \qquad R^{\triangledown} = \begin{pmatrix} B & \tilde{N} \\ 0 & A \end{pmatrix} \cong R^{\blacktriangledown}/R^{\blacktriangledown} e_* R^{\blacktriangledown}, \tag{2.3}$$

where $D(-) = \operatorname{Hom}_F(-, F)$, ${}_B\tilde{N}_A = \operatorname{Hom}_B({}_A N_B, B) \cong D({}_A N_B)$ and multiplication in $R^{\blacktriangledown}$ is given by the evaluation map ${}_B\tilde{N} \otimes {}_A N_B \to B$. It is clear that $R^{\blacktriangledown}$ is a left multipeak algebra with left peak idempotents

$$e_p^- = \begin{pmatrix} e_p & 0 \\ 0 & 0 \end{pmatrix},$$

where $p \in \boldsymbol{p}(I_R)$ and e_p is considered here as an element of B. We put

$$e_*^- = \sum_{p \in \boldsymbol{p}(I_R)} e_p^-.$$

Following [21, 23] we define a pair of *reflection functors*

$$\operatorname{mod}_{sp}(R) \underset{\nabla_+}{\overset{\nabla_-}{\rightleftarrows}} \operatorname{mod}_{ti}(R^{\triangledown}) \tag{2.4}$$

where $\operatorname{mod}_{ti}(R^{\triangledown})$ is the category of finitely generated *top injective right R-modules*. Given $X_R = (X'_A, X''_B, \varphi\colon X' \otimes_A N_B \to X''_B)$, $Y_R = (Y'_B, Y''_A, \psi\colon Y' \otimes_B N_A \to Y''_A)$ we put

$$\nabla_-(X_R) = (X''_B, \operatorname{Coker}\varphi', \tilde{\varphi}), \qquad \nabla_+(Y) = (\operatorname{Ker}\psi', Y'_B, \tilde{\psi}),$$

where φ' and ψ' are the composed maps $X'_A \xrightarrow{\bar{\varphi}} \operatorname{Hom}_B({}_A N_B, X''_B) \cong X'' \otimes_B \tilde{N}_A$ and $\operatorname{Hom}_B({}_A N_B, Y'_B) \cong Y' \otimes_B \tilde{N}_A \xrightarrow{\psi} Y''_A$ respectively, $\bar{\varphi}$ is adjoint to φ, $\tilde{\varphi}$ is the cokernel map and $\tilde{\psi}$ is adjoint to the natural embedding $\operatorname{Ker}\psi' \hookrightarrow \operatorname{Hom}_B({}_A N_B, Y'_B)$. The functors are defined on maps in a natural way.

PROPOSITION 2.5. (a) *$R^{\blacktriangledown}$ is a left multipeak algebra with left peak idempotents e_p^-, $p \in \boldsymbol{p}(I_R)$, and a right multipeak algebra with right peak idempotents e_p, $p \in \boldsymbol{p}(I_R)$. Moreover, $e_p^- R^{\blacktriangledown} \cong E(e_p R^{\blacktriangledown})$ for $p \in \boldsymbol{p}(I_R)$ and $E(R^{\blacktriangledown})$ is projective. If X is in $\operatorname{ind}_{sp}(R^{\blacktriangledown})$, Y is in $\operatorname{ind}_{ti}(R^{\blacktriangledown})$ and $Xe_*^- \neq 0$, $Ye_* \neq 0$ then $X \cong E(e_p R^{\blacktriangledown})$, $Y \cong E(e_q R^{\blacktriangledown})$ for some $p, q \in \boldsymbol{p}(I_R)$.*

(b) *∇_- and ∇_+ are equivalences of categories preserving exactness and inverse to each other. X_R is* sp*-injective if and only if $\nabla_-(X_R)$ is injective in* $\operatorname{mod}(R^{\triangledown})$ (see below for definition).

(c) *Given X_R in $\operatorname{mod}_{sp}(R)$ and Y in $\operatorname{mod}_{ti}(R^{\triangledown})$ we have*

$$\nabla_-(X_R) \cong \operatorname{Coker}(X \hookrightarrow E_{R^{\blacktriangledown}}(X)), \qquad \nabla_+(Y) \cong \operatorname{Ker}(P_{R^{\blacktriangledown}}(Y) \to Y)$$

where X and Y are considered as $R^{\blacktriangledown}$-modules via the epimorphisms $R \leftarrow R^{\blacktriangledown} \to R^{\triangledown}$.

Proof. (a) and (b) can be proved by applying arguments in the proof of Propositions 2.6 and 2.8 in [21] and in [3; Proposition 1.6].

(c) By (a) indecomposable summands of $E_{R^{\blacktriangledown}}(X_R)$ are summands of $e_*^- R^{\blacktriangledown}$ which is the row ideal $(B, \tilde{N}, B)$ in the form (2.3). It follows that up to isomorphism the embedding $X \hookrightarrow E_{R^\blacktriangledown}(X)$ is given by the natural monomorphism $(0, \varphi', 0)$: $(0, X'_A, X''_B) \to (X''_B, X' \otimes_A \tilde{N}_B, X''_B)$, where φ' is the map in the formula defining $\nabla_-(X_R)$. Hence the first isomorphism in (c) follows. The second one can be established in a similar way.

We recall that X in $\mathrm{mod}_{\mathrm{sp}}(R)$ is sp-*injective* if X is injective with respect to monomorphisms $f\colon Z \to Z'$ in $\mathrm{mod}_{\mathrm{sp}}(R)$ such that $\mathrm{Coker}\, f$ is in $\mathrm{mod}_{\mathrm{sp}}(R)$ [21]. An indecomposable sp-injective module X in $\mathrm{mod}_{\mathrm{sp}}(R)$ is said to be *hereditary* if every module X' in $\mathrm{ind}_{\mathrm{sp}}(R)$ such that $\mathrm{Hom}_R(X, X') \neq 0$ is sp-injective.

We shall call $R^\bullet = (R^\nabla)^{\mathrm{op}}$ the *reflection dual algebra* to R and the functor

$$D_R^\bullet := D\nabla_-\colon \mathrm{mod}_{\mathrm{sp}}(R) \to \mathrm{mod}_{\mathrm{sp}}(R^\bullet) \tag{2.6}$$

will be called the *reflection duality*. It follows from Proposition 2.5 that $D_R^\bullet$ is a duality, there is an algebra isomorphism $R^{\bullet\bullet} \cong R$, $D_{R^\bullet}^\bullet \cong \nabla_+ D \cong (D_R^\bullet)^{-1}$ and X_R is hereditary sp-injective if and only if $D_R^\bullet(X_R)$ is *hereditary projective* [28], i.e. every submodule of $D_R^\bullet(X_R)$ is projective. The modules

$$Q^{(p)} = D_{R^\bullet}^\bullet(e_p^- R^\bullet), \quad p \in \boldsymbol{p}(I_R), \qquad Q^{(j)} = D_{R^\bullet}^\bullet(e_j R^\bullet), \quad j \in I_R - \boldsymbol{p}(I_R) \tag{2.7}$$

form a complete set of nonisomorphic indecomposable sp-injective modules in $\mathrm{mod}_{\mathrm{sp}}(R)$.

It is known that there are Auslander–Reiten sequences in $\mathrm{mod}(R)$ [4]. Then by [2, 21, 23] we get

PROPOSITION 2.8. *If R is an F-algebra as above then for every nonprojective module X in $\mathrm{ind}_{\mathrm{sp}}(R)$ and for every non-sp-injective module Y in $\mathrm{ind}_{\mathrm{sp}}(R)$ there are Auslander–Reiten sequences in $\mathrm{mod}_{\mathrm{sp}}(R)$*

$$0 \to X \to Z \to \Delta^-(X) \to 0, \qquad 0 \to \Delta(Y) \to U \to Y \to 0 \tag{2.9}$$

which are unique up to isomorphism and $\Delta^-(X)$, $\Delta(Y)$ are indecomposable.

Now suppose $J \subseteq I_R$ is such that $\boldsymbol{p}(J) \subseteq \boldsymbol{p}(I_R)$ and $R_J = vRv$ with $v = \sum_{j \in J} e_j$ is a right multipeak algebra with peak idempotents e_q, $q \in \boldsymbol{p}(J)$. Following [1], [22; 1.14], [23] we consider three functors

$$\mathrm{mod}_{\mathrm{sp}}(vRv) \underset{\boldsymbol{r}_v}{\overset{\boldsymbol{T}_v, \boldsymbol{L}_v}{\rightleftarrows}} \mathrm{mod}_{\mathrm{sp}}(R) \tag{2.10}$$

where $\boldsymbol{r}_v(X) = Xv$, $\boldsymbol{T}_v(Y) = Y \otimes_{vRv} vR$ and $\boldsymbol{L}_v(Y) = \mathrm{Hom}_{vRv}(Rv, Y)$.

In view of [1] and [22; Corollary 1.16] we get

PROPOSITION 2.11. *If $J \subseteq I_R$ and v are as above then*

(a) *$\boldsymbol{T}_v$ and $\boldsymbol{L}_v$ are fully faithful embeddings, $\boldsymbol{r}_v$ is exact, $\boldsymbol{T}_v$ is left adjoint to $\boldsymbol{r}_v$, $\boldsymbol{L}_v$ is right adjoint to $\boldsymbol{r}_v$ and $\boldsymbol{r}_v \boldsymbol{T}_v \cong \mathbf{id} \cong \boldsymbol{r}_v \boldsymbol{L}_v$.*

(b) *$\boldsymbol{T}_v(e_j Rv) \cong e_j R$ for $j \in J$, $\boldsymbol{T}_v$ preserves projective covers and $\mathrm{Im}\, \boldsymbol{T}_v$ is the full subcategory of $\mathrm{mod}_{\mathrm{sp}}(R)$ consisting of modules X such that $P(X) \cong \bigoplus_{j \in J} (e_j R)^{s_j}$.*

To any X in $\mathrm{ind}_{\mathrm{sp}}(R)$ we associate two integral vectors [21; Section 3]

$$\mathbf{dim}(X) = (x_i)_{i \in I_R}, \qquad \mathbf{cdn}(X) = (s_i)_{i \in I_R}, \tag{2.12}$$

where $x_i = \dim_F X e_i$, $s_p = \dim_F X e_p$ for $p \in \boldsymbol{p}(I_R)$ and given $j \in I_R - \boldsymbol{p}(I_R)$, s_j is such that

$$P(X) \cong \bigoplus_{i \in I_R} (e_i R)^{s_i}.$$

We call $\mathbf{dim}(X)$ and $\mathbf{cdn}(X)$ the *dimension vector* of X and the *coordinate vector* of X, respectively. The sets

$$\mathbf{supp}(X) = \{i \in I_R;\ x_i \neq 0\}, \qquad \mathbf{csup}(X) = \{i \in I_R;\ s_i \neq 0\} \tag{2.13}$$

are called the *support* and the *coordinate support* of X, respectively. Note that $\mathbf{csup}(X)$ is finite. X in $\mathrm{ind}_{\mathrm{sp}}(R)$ is called sp-*sincere* [11] if $\mathbf{csup}(X) = I_R$. R is said to be sp-*sincere* if there is an sp-sincere module in $\mathrm{ind}_{\mathrm{sp}}(R)$.

It is clear that if X is in $\mathrm{ind}_{\mathrm{sp}}(R)$ and $v = \sum_{j \in \mathbf{csup}(X)} e_j$ then vRv is a right multipeak algebra, $\boldsymbol{r}_v(X)$ is an sp-sincere vRv-module and $\boldsymbol{T}_v \boldsymbol{r}_v(X) \cong X$. It follows that up to the equivalence $\mathrm{mod}_{\mathrm{sp}}(vRv) \simeq \mathrm{Im}\, \boldsymbol{T}_v$ any X in $\mathrm{ind}_{\mathrm{sp}}(R)$ can be considered as an sp-sincere module over $vRv \subseteq R$.

Note also that if $J \subseteq I_R$ and v are as in Proposition 2.11 then we have

$$\mathbf{cdn}(\boldsymbol{T}_v(Y))|_J = \mathbf{cdn}(Y), \qquad Y \in \mathrm{ind}_{\mathrm{sp}}(vRv). \tag{2.14}$$

In describing a bound quiver of $R^\blacktriangledown$ in terms of the algebra R the following simple lemma will be useful:

LEMMA 2.15. *Let*

$$R = \begin{pmatrix} A & N \\ 0 & B \end{pmatrix}$$

be a locally bounded right multipeak F-algebra (2.1) *and let $R^\blacktriangledown$, R^∇ be the reflection forms* (2.3) *of R. In the notation above we have*

(a)
$$\begin{aligned} e_i R^\blacktriangledown e_j &= e_i R^\nabla e_j = e_i R e_j && \textit{for } i \in I_R - \boldsymbol{p}(I_R),\ j \in I_R, \\ e_p^- R^\blacktriangledown e_q &= 0 && \textit{for } p \neq q,\ p, q \in \boldsymbol{p}(I_R), \\ &= e_p R e_p && \textit{for } p = q. \end{aligned}$$

(b) *There is a bimodule isomorphism $\sigma\colon {}_B\tilde{N}_A \to D({}_A N_B)$ which induces F-linear isomorphisms*

$$\sigma_{pj}\colon e_p^- \tilde{N} e_j = e_p^- R^\blacktriangledown e_j = e_p^- R^\nabla e_j \simeq D(e_j N e_p) = D(e_j R e_p)$$

for all $p \in \boldsymbol{p}(I_R)$, $j \in I_R - \boldsymbol{p}(I_R)$, such that given $i \in I_R - \boldsymbol{p}(I_R)$ the diagram

$$\begin{array}{ccc} e_p^- \tilde{N} e_j \otimes e_j R e_i & \xrightarrow{\sigma_{pj} \otimes 1} & D(e_j N e_p) \otimes e_j R e_i \\ \downarrow \mu_{pji} & & \downarrow \tilde{c}_{pji} \\ e_p^- \tilde{N} e_i & \xrightarrow{\sigma_{pi} \otimes 1} & D(e_i N e_p) \end{array}$$

is commutative, where $D(-) = \mathrm{Hom}_F(-, F)$, $\mu_{pji}(g \otimes y) = gy$, $\tilde{c}_{pji}(f \otimes y)(n) = f(yn)$ *for* $n \in e_i N e_p$, $y \in e_j R e_i$, $f \in D(e_j N e_p)$, $g \in e_p^- \tilde{N} e_j$.

The proof is easy and is left to the reader.

Now suppose that $R = F(\mathcal{Q}, \Omega)$ is a bound quiver algebra. In this case we take $I_R = \mathcal{Q}_0$ (the set of vertices of $\mathcal{Q}$) and we take for $\{e_i\}_{i \in \mathcal{Q}_0}$ the standard set of idempotents which are the trivial paths 1_i: $i \to i$ in $\mathcal{Q}$. Then $\boldsymbol{p}(I_R)$ is the set $\boldsymbol{p}(\mathcal{Q})$ of all sinks in $\mathcal{Q}$ and R is a right multipeak algebra if and only if for any $k_1, \ldots, k_t \in F$ and paths $w_1, \ldots, w_t$: $i \to j$ such that $w = k_1 w_1 + \ldots + k_t w_t \notin \Omega$ there exist $p \in \boldsymbol{p}(\mathcal{Q})$ and a path $u: j \to p$ such that $wu \notin \Omega$. If $(\mathcal{Q}, \Omega)$ has the above property we call it a *right multipeak bound quiver.*

DEFINITION 2.16. Suppose that $(\mathcal{Q}, \Omega)$ is a right multipeak bound quiver, Ω is an admissible ideal in $F\mathcal{Q}$, $R = F(\mathcal{Q}, \Omega)$ and suppose that R is *peak* $\tilde{A}_1$*-free,* i.e. $\dim_F e_i R e_p \leqslant 1$ for all $i \in \mathcal{Q}_0$ and $p \in \boldsymbol{p}(\mathcal{Q})$ (in the notation above). We define two *reflection forms* $(\mathcal{Q}^{\blacktriangledown}, \Omega^{\blacktriangledown})$ and $(\mathcal{Q}^{\triangledown}, \Omega^{\triangledown})$ of $(\mathcal{Q}, \Omega)$ as follows. Let

$$\mathcal{Q}_0^{\blacktriangledown} = \boldsymbol{p}(\mathcal{Q})^- \,\dot{\cup}\, \mathcal{Q}_0, \qquad \mathcal{Q}_0^{\triangledown} = \mathcal{Q}_0^{\blacktriangledown} - \boldsymbol{p}(\mathcal{Q})$$

where $\boldsymbol{p}(\mathcal{Q})^-$ consists of vertices p^- with $p \in \boldsymbol{p}(\mathcal{Q})$ and $\dot{\cup}$ means disjoint union. The sets of edges in $\mathcal{Q}^{\blacktriangledown}$ and in $\mathcal{Q}^{\triangledown}$ between the vertices in $\mathcal{Q}_0$ remain the same as in $\mathcal{Q}$. For any $i \in \mathcal{Q}_0 - \boldsymbol{p}(\mathcal{Q})$ and $p \in \boldsymbol{p}(\mathcal{Q})$ such that $e_i R e_p \neq 0$ we fix a path u_{ip}: $i \to p$ in $\mathcal{Q}$ which does not belong to Ω. We define a unique edge

$$u_{ip}^-: p^- \to i$$

in $\mathcal{Q}^{\blacktriangledown}$ (and in $\mathcal{Q}^{\triangledown}$) if u_{ip} is maximal modulo Ω in the sense that $uu_{ip} \in \Omega$ for every edge u in $\mathcal{Q}$ ending at i (see Fig. 0).

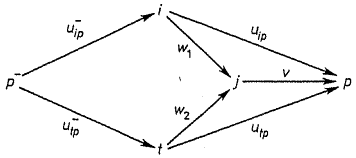

Fig. 0

We define $\Omega^{\blacktriangledown}$ as the two-sided ideal in $F\mathcal{Q}^{\blacktriangledown}$ generated by Ω and the following elements:

(i) $u_{ip}^- w$, where $w: i \to j$ runs through all paths in $\mathcal{Q}$ such that $wv \in \Omega$ for any path v: $j \to p$, $p \in \boldsymbol{p}(\mathcal{Q})$, $j \in \mathcal{Q}_0$.

(ii) $\lambda_1 u_{ip}^- w_1 - \lambda_2 u_{tp}^- w_2$, where $p \in \boldsymbol{p}(\mathcal{Q})$, $i, t, j \in \mathcal{Q}_0$, $\lambda_1, \lambda_2 \in F$ and w_1: $i \to j$, w_2: $t \to j$ are paths in $\mathcal{Q}$ such that

$$u_{ip} - \lambda_1 w_1 v \in \Omega \quad \text{and} \quad u_{tp} - \lambda_2 w_2 v \in \Omega$$

for some nonzero path v: $j \to p$ in $\mathcal{Q}$ (see Fig. 0). In particular $u_{ip}^- u_{ip} - u_{tp}^- u_{tp} \in \Omega$. The ideal $\Omega^{\triangledown}$ is generated by $\Omega^{\blacktriangledown} \cap F\mathcal{Q}^{\triangledown}$.

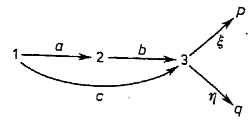

Fig. 1

Let us illustrate the definition by two examples.

EXAMPLE 2.17. Let $\mathcal{Q}$ be the quiver of Fig. 1 and $\Omega = (c\eta, ab\xi - c\xi)$. Then $R = F(\mathcal{Q}, \Omega)$ is a right two-peak algebra which is peak $\tilde{A}_1$-free, and $\mathcal{Q}^{\blacktriangledown}$ is shown in Fig. 2 and $\Omega^{\blacktriangledown} = (c\eta, ab\xi - c\xi, fc, eab - ec)$, where $e = (c\xi)^-$ and $f = (ab\eta)^-$. We note that $R \cong F(\mathcal{Q}^{\blacktriangledown}, \Omega^{\blacktriangledown})$.

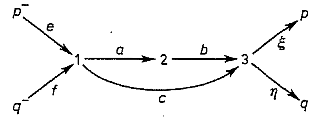

Fig. 2

EXAMPLE 2.18. Let $(\mathcal{Q}, \Omega)$ be the bound quiver obtained from $(\mathcal{Q}', \Omega')$ presented in Fig. 3, where

$$\Omega' = \big(hac,\ ab,\ ba, fe - hd,\ (ge)^- g - (fe)^- f,\ (fe)^- h - (bd)^- b,\ u^- g,\ (ge)^- u,$$
$$(ac)^- d,\ (bd)^- c,\ (ac)^- ac - (bd)^- bd\big),$$

by removing the vertices p^-, q^- and the edges starting from p^- and q^-. Note that $R = F(\mathcal{Q}, \Omega)$ is a right two-peak algebra, R is not peak $\tilde{A}_1$-free, $(\mathcal{Q}^{\blacktriangledown}, \Omega^{\blacktriangledown}) = (\mathcal{Q}', \Omega')$ and $R^{\blacktriangledown} \cong F(\mathcal{Q}', \Omega')$.

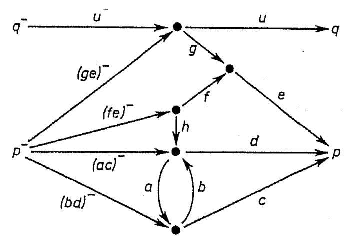

Fig. 3

PROPOSITION 2.19. *Let* $R = F(\mathcal{Q}, \Omega)$ *be a locally bounded right multipeak algebra which is peak* $\tilde{A}_1$*-free and nonsemisimple indecomposable. Then*

(a) $\mathcal{Q}^{\blacktriangledown}$ *and* $\mathcal{Q}^{\triangledown}$ *are isomorphic to the quivers of* $R^{\blacktriangledown}$ *and* $R^{\triangledown}$, *respectively.*

(b) *There are F-algebra isomorphisms*

$$\zeta\colon F(\mathcal{Q}^{\blacktriangledown}, \Omega^{\blacktriangledown}) \to R^{\blacktriangledown}, \qquad \zeta'\colon F(\mathcal{Q}^{\triangledown}, \Omega^{\triangledown}) \to R^{\triangledown},$$

where $(\mathcal{Q}^{\blacktriangledown}, \Omega^{\blacktriangledown})$ *and* $(\mathcal{Q}^{\triangledown}, \Omega^{\triangledown})$ *are the reflection forms of* $(\mathcal{Q}, \Omega)$.

Proof. Let us define an algebra homomorphism $f\colon F\mathcal{Q}^{\blacktriangledown} \to R$ by taking for $f|_{F\mathcal{Q}}$ the natural epimorphism $F\mathcal{Q} \to R$ and by putting

$$f(e_{p^-}) = e_p^- \quad \text{if } p \in p(\mathcal{Q}),$$

$$f(u_{ip}^-) = \bar{u}_{ip}^* \quad \text{if } u_{ip} \text{ is the fixed edge maximal modulo } \Omega,$$

where $\bar{u}_{ip}^* \in e_p^- R^{\blacktriangledown} e_i = D(e_i R e_p)$ is given by $\bar{u}_{ip}^*(u_{ip}) = 1$.

Since R is locally bounded, nonsemisimple and indecomposable as an algebra, given $p \in p(\mathcal{Q})$ there is $i \in \mathcal{Q}_0 - p(\mathcal{Q})$ such that $e_i R e_p \neq 0$ and $e_i R e_p$ contains the coset $\bar{u}_{ip}$ of the maximal path u_{ip} modulo Ω. Since R is peak $\tilde{A}_1$-free, according to Lemma 2.15 the map

$$\mu_{pip}\colon e_p^- R^{\blacktriangledown} e_i \otimes e_i R e_p \to e_p^- R^{\blacktriangledown} e_p$$

is bijective and therefore $e_p^- R^{\blacktriangledown} e_p = \bar{u}_{ip}^* R^{\blacktriangledown} \subseteq \operatorname{Im} f$. In order to prove that f is surjective it is sufficient to show that $e_p^- R^{\blacktriangledown} e_i \subseteq \operatorname{Im} f$ for all $i \in \mathcal{Q}_0$. For this purpose suppose that $e_p^- R^{\blacktriangledown} e_i = D(e_i R e_p) \neq 0$. Hence $e_i R e_p \neq 0$ and by our assumption there exists $j \in \mathcal{Q}_0$ such that u_{jp} is maximal modulo Ω and $u_{jp} - \lambda v u_{ip} \in \Omega$ for some $\lambda \in F$ and a path $v\colon j \to i$. It follows from Lemma 2.15 that $\bar{u}_{jp}^* \bar{v} \in e_p^- R^{\blacktriangledown} e_i$ is nonzero since $\bar{c}_{pji}(\bar{u}_{jp}^* \otimes v)(\lambda \bar{u}_{ip}) = \bar{u}_{jp}^*(\bar{u}_{jp}) = 1$. Hence $e_p^- R^{\blacktriangledown} e_i = \bar{u}_{jp}^* R^{\blacktriangledown} \subseteq \operatorname{Im} f$ and therefore f is surjective. Moreover, the considerations above show that

$$e_p^- R^{\blacktriangledown} e_j / e_p^- J(R^{\blacktriangledown})^2 e_j = \bar{u}_{jp}^* F \quad \text{if } u_{jp} \text{ is maximal modulo } \Omega,$$

$$= 0 \qquad \text{otherwise.}$$

It follows that the quiver of $R^{\blacktriangledown}$ is isomorphic to $\mathcal{Q}^{\blacktriangledown}$, which proves (a).

Applying Lemma 2.15(b) we easily check that $\Omega^{\blacktriangledown} \subseteq \operatorname{Ker} f$ and therefore f induces an algebra epimorphism $\zeta\colon F(\mathcal{Q}^{\blacktriangledown}, \Omega^{\blacktriangledown}) \to R^{\blacktriangledown}$. In order to show that ζ is injective it is sufficient to check that the induced surjections

$$\zeta_{pi}\colon e_{p^-} F(\mathcal{Q}^{\blacktriangledown}, \Omega^{\blacktriangledown}) e_i \to e_p^- R^{\blacktriangledown} e_i$$

are injective for all $p \in p(\mathcal{Q})$ and $i \in \mathcal{Q}_0$. Since a simple analysis shows that $\dim_F(e_{p^-} F(\mathcal{Q}^{\blacktriangledown}, \Omega^{\blacktriangledown}) e_i) \leqslant 1$ it remains to show that if $e_p^- R^{\blacktriangledown} e_i = 0$ then $e_{p^-} F(\mathcal{Q}^{\blacktriangledown}, \Omega^{\blacktriangledown}) e_i = 0$. Indeed, let $u\colon p^- \to i$ be a nonzero path in $\mathcal{Q}^{\blacktriangledown}$. Then $u = u_{ip}^- w$ for some path $w\colon i \to j$ in $\mathcal{Q}$. Since $e_p^- R^{\blacktriangledown} e_i \cong D(e_i R e_p) = 0$ we get $e_i R e_p = 0$ and therefore $wv \in \Omega$ for every path $v\colon j \to p$ in $\mathcal{Q}$. It follows from Definition 2.16(i) that $u = u_{ip}^- w \in \Omega^{\blacktriangledown}$. This shows that $e_{p^-} F(\mathcal{Q}^{\blacktriangledown}, \Omega^{\blacktriangledown}) e_i = 0$ and proves that ζ is an isomorphism. Since the isomorphism ζ' is the restriction of ζ to $F(\mathcal{Q}^{\triangledown}, \Omega^{\triangledown})$ the proof is complete.

It would be interesting to describe the bound quiver of $F(\mathcal{Q}, \Omega)^{\blacktriangledown}$ and of $F(\mathcal{Q}, \Omega)^{\triangledown}$ in terms of $(\mathcal{Q}, \Omega)$ in the general situation.

Note that the algebra R in Example 2.18 is not peak $\tilde{A}_1$-free while $R^{\blacktriangledown} \cong F(\mathcal{Q}^{\blacktriangledown}, \Omega^{\blacktriangledown})$. However, this fact does not hold in general because of the following example.

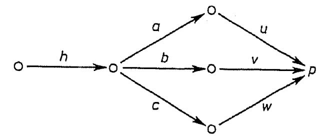

Fig. 4

EXAMPLE 2.20. Let $R = F(\mathcal{Q}, \Omega)$, where $\mathcal{Q}$ is the quiver of Fig. 4 and $\Omega = (hb, au + cw - bv)$. Applying the same type of argument as in the proof of Proposition 2.19 one can show that $R^{\blacktriangledown} \cong F(\mathcal{Q}', \Omega')$, where $\mathcal{Q}'$ is shown in Fig. 5 and $\Omega' = (\Omega, (cw)^* a, (cw)^* c + (hau)^* hc)$. Note that $\mathcal{Q}^{\blacktriangledown}$ is not isomorphic to $\mathcal{Q}'$.

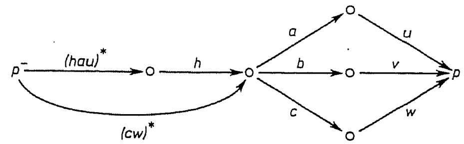

Fig. 5

DEFINITION 2.21. Let $(\mathcal{Q}, \Omega)$ be a right multipeak bound quiver. The *reflection dual bound quiver of* $(\mathcal{Q}, \Omega)$ is the bound quiver $(\mathcal{Q}^{\bullet}, \Omega^{\bullet}) = (\mathcal{Q}^{\triangledown}, \Omega^{\triangledown})^{\mathrm{op}}$.

As an immediate consequence of Proposition 2.19 we get

COROLLARY 2.22. *If* $(\mathcal{Q}, \Omega)$ *is a right multipeak bound quiver such that* $F(\mathcal{Q}, \Omega)$ *is* $\tilde{A}_1$-*free then* $(\mathcal{Q}^{\bullet}, \Omega^{\bullet})$ *is a right multipeak bound quiver and there is an F-algebra isomorphism* $(F(\mathcal{Q}, \Omega))^{\bullet} \cong F(\mathcal{Q}^{\bullet}, \Omega^{\bullet})$.

3. Main results. Throughout this section $(\mathcal{Q}, \Omega)$ denotes a connected right multipeak bound quiver and

$$R = F(\mathcal{Q}, \Omega) = \bigoplus_{i \in \mathcal{Q}_0} e_i R, \tag{3.1}$$

where $e_i\colon i \to i$ is the trivial path in $\mathcal{Q}$. Moreover, we suppose that $\mathcal{Q}$ is *directed*, i.e. the relation

$$i \prec j \Leftrightarrow \text{there is a nonzero path } i \to j \text{ in } \mathcal{Q}$$

is a partial order in $\mathcal{Q}_0$ [17]. Given $t \in \mathcal{Q}_0$ we denote by $t^{\triangledown}$ (resp. by $t^{\vartriangle}$) the full bound subquiver of $(\mathcal{Q}, \Omega)$ consisting of vertices $s \in \mathcal{Q}_0$ such that there is a path $s \to t$ (resp. $t \to s$) in $\mathcal{Q}$ which does not belong to Ω. The idea of the splittting decomposition of posets [15] and of right peak rings [11] is extended as follows.

Following [11; Section 4] we say that subposets $\mathcal{Q}_0'$, $\mathcal{C}_0$ and $\mathcal{Q}_0''$ of $\mathcal{Q}_0$ form a *triangular decomposition* of the bound quiver $(\mathcal{Q}, \Omega)$ if

(3.2) $$\mathcal{Q}_0 = \mathcal{Q}'_0 + \mathcal{C}_0 + \mathcal{Q}''_0$$

is a disjoint union of subposets, $\mathcal{Q}'_0$ and $\mathcal{Q}''_0$ are nonempty and

(i) there are no relations $j'' \prec j'$, $j'' \prec c$, $c \prec j'$ with $j' \in \mathcal{Q}'_0$, $c \in \mathcal{C}_0$ and $j'' \in \mathcal{Q}''_0$,

(ii) $\mathcal{C}_0 \subseteq \mathcal{Q}_0 - \boldsymbol{p}(\mathcal{Q})$ and $(\mathcal{C} + \mathcal{Q}'', \Omega|)$ is a right multipeak bound quiver with the set of peaks $\boldsymbol{p}(\mathcal{C} + \mathcal{Q}'') = \mathcal{Q}''_0 \cap \boldsymbol{p}(\mathcal{Q})$, where $\Omega| = \Omega|_{\mathcal{C}+\mathcal{Q}''}$.

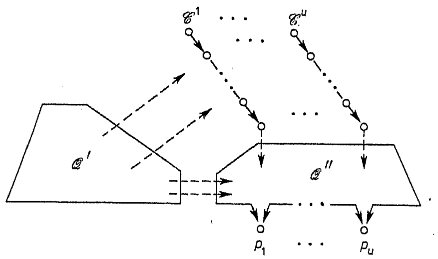

Fig. 6

DEFINITION 3.3. The triangular decomposition (3.2) is called a *splitting decomposition of* $(\mathcal{Q}, \Omega)$ if there exist a nonempty set

$$\mathcal{P} = \{p_1, \ldots, p_u\} \subseteq \boldsymbol{p}(\mathcal{Q}) \cap \mathcal{Q}''_0, \quad p_j \neq p_i \quad \text{for } j \neq i,$$

and a disjoint union poset decomposition (see Fig. 6)

$$\mathcal{C}_0 = \mathcal{C}_0^1 + \ldots + \mathcal{C}_0^u$$

where $\mathcal{C}_0^1, \ldots, \mathcal{C}_0^u$ are pairwise incomparable chains such that

(a) $(\mathcal{Q}' + \mathcal{C} + \mathcal{P}, \Omega|)$ is a connected right multipeak bound quiver satisfying $\boldsymbol{p}(\mathcal{Q}' + \mathcal{C} + \mathcal{P}) = \boldsymbol{p}(\mathcal{Q}) \cap \mathcal{Q}'_0 \cup \mathcal{P}$.

(b) The full bound subquiver $\hat{\mathcal{C}} = \mathcal{C} \cup \mathcal{P}$ of $(\mathcal{Q}, \Omega)$ consisting of vertices $\mathcal{C}_0 \cup \mathcal{P}$ is a poset disjoint union

$$\hat{\mathcal{C}} = \mathcal{C} \cup \mathcal{P} = \hat{\mathcal{C}}^1 + \ldots + \hat{\mathcal{C}}^u$$

of pairwise unrelated chains

$$\hat{\mathcal{C}}^i\colon\ c_1^i \to c_2^i \to \ldots \to c_{t_i}^i \to c_{t_i+1}^i = p_i$$

where $t_i \geqslant 0$. The restriction of Ω to $\hat{\mathcal{C}}$ is empty.

(c) For any $i \in \mathcal{Q}'_0$ and $j \in \mathcal{Q}''_0$ we have

$$d(i, j) = \sum_{p \in \mathcal{P}} d(i, p) d(j, p)$$

where $d(s, t) = \dim_F(e_s R e_t)$. Moreover, $d(c, q) = 0$ for all $c \in \mathcal{C}$ and $q \in \boldsymbol{p}(\mathcal{Q}) - \mathcal{P}$.

It follows that $(\mathcal{Q}, \Omega)$ has the form of Fig. 6.

Note also that $d(i, q) = 0$ for all $i \in \mathcal{Q}'_0$, $q \in \boldsymbol{p}(\mathcal{Q}'') - \mathcal{P}$, for any $p \in \mathcal{P}$ there is $i \in \mathcal{Q}'_0$ such that $d(i, p) \neq 0$, and $d(i, j) \neq 0$ iff there is $p \in \mathcal{P}$ such that $d(i, p) \neq 0$ and $d(j, p) \neq 0$. In general the sets $\mathcal{Q}'_0 - \boldsymbol{p}(\mathcal{Q})$ and $\boldsymbol{p}(\mathcal{Q}'') - \mathcal{P}$ are not empty.

Our main motivation for the definition above is Theorem 3.10 below which shows how the category $\mathrm{mod}_{\mathrm{sp}}(R)$ can be obtained by glueing $\mathrm{mod}_{\mathrm{sp}} F(\mathcal{Q}' + \hat{\mathcal{C}}, \Omega|)$ and $\mathrm{mod}_{\mathrm{sp}} F(\mathcal{C} + \mathcal{Q}'', \Omega|)$ if the decomposition (3.2) is splitting. This result is of "recollements" character.

The reader is referred to Section 4 for typical examples of splitting decomposition and its applications.

Remark 3.3′. Suppose that $R = F(\mathcal{Q}, \Omega)$. It is easy to check that $R^\bullet \cong F(\mathcal{Q}^\bullet, \Omega^\bullet)$ and

$$\mathcal{Q}_0^\bullet = (\mathcal{Q}_0 - \boldsymbol{p}(\mathcal{Q})) \cup \boldsymbol{p}(\mathcal{Q})^-, \quad \mathcal{Q}^\bullet - \boldsymbol{p}(\mathcal{Q})^- \cong (\mathcal{Q} - \boldsymbol{p}(\mathcal{Q}))^{\mathrm{op}}$$

where $\boldsymbol{p}(\mathcal{Q})^- = \{p^-, p \in \boldsymbol{p}(\mathcal{Q})\}$. One can check that if (3.2) is a splitting decomposition of $(\mathcal{Q}, \Omega)$ then $(\mathcal{Q}^\bullet, \Omega^\bullet)$ admits a splitting decomposition given by the set $\mathcal{P}^-$ of peaks and by

$$\mathcal{Q}_0^\bullet = {}^\bullet\mathcal{Q}''_0 + \mathcal{C}_0^{\mathrm{op}} + {}^\bullet\mathcal{Q}'_0$$

where ${}^\bullet\mathcal{Q}''_0 = (\mathcal{Q}''_0 - \boldsymbol{p}(\mathcal{Q}'')) \cup (\boldsymbol{p}(\mathcal{Q}'') - \mathcal{P})^-$, ${}^\bullet\mathcal{Q}'_0 = (\mathcal{Q}'_0 - \boldsymbol{p}(\mathcal{Q}')) \cup \boldsymbol{p}(\mathcal{Q}')^- \cup \mathcal{P}^-$. We note that

$$\dim_F(e_j R^\bullet e_i) = d(i, j) \quad \text{for } i \in \mathcal{Q}'_0,\ j \notin \boldsymbol{p}(\mathcal{Q}),$$

$$\dim(e_i R^\bullet e_p^-) = d(i, p) \quad \text{for } p \in \boldsymbol{p}(\mathcal{Q}).$$

Given a splitting decomposition (3.2) we consider

(3.4) $$\hat{S} = \hat{\eta} R \hat{\eta} = F(\mathcal{Q}' + \hat{\mathcal{C}}, \Omega|) \quad \text{and} \quad R = \begin{pmatrix} S & {}_S M_T \\ 0 & T \end{pmatrix}$$

where $S = \eta R \eta = F(\mathcal{Q}', \Omega|)$, $M = \eta R \xi$, $T = \xi R \xi = F(\mathcal{C} + \mathcal{Q}'', \Omega|)$ and

(3.5) $$\eta = \sum_{i \in \mathcal{Q}'_0} e_i, \quad \xi = \sum_{j \in \mathcal{Q}_0 - \mathcal{Q}'_0} e_j, \quad \hat{\eta} = e_{p_1} + \ldots + e_{p_u} + \sum_{j \in \mathcal{Q}'_0 \cup \mathcal{C}} e_j.$$

Our splitting theorem will be formulated in terms of the functors (see (2.10))

(3.6) $$\mathrm{mod}_{\mathrm{sp}}(\hat{S}) \underset{r_{\hat{\eta}}}{\overset{L_{\hat{\eta}}, T_{\hat{\eta}}}{\rightleftarrows}} \mathrm{mod}_{\mathrm{sp}}(R) \underset{r_\xi}{\overset{T_\xi}{\rightleftarrows}} \mathrm{mod}_{\mathrm{sp}}(T).$$

Let us start with three technical lemmas.

LEMMA 3.7. *Let* $A = F(\mathcal{Q}'', \Omega|) = fRf$, *where* $f = \sum_{j \in \mathcal{Q}''} e_j$. *A module* X *in* $\mathrm{mod}_{\mathrm{sp}}(R)$ *belongs to* $\mathrm{Im}\, \boldsymbol{L}_{\hat{\eta}}$ *if and only if* $Xf \cong \bigoplus_{p \in \mathcal{P}} E_A(e_p A)^{r_p}$ *for some* $r_p \geqslant 0$. *In this case* $X \cong \boldsymbol{L}_{\hat{\eta}} r_{\hat{\eta}}(X) \cong \boldsymbol{T}_{\hat{\eta}} r_{\hat{\eta}}(X)$.

Proof. Suppose that $X \cong L_{\hat{\eta}}(Y)$ with Y in $\mathrm{mod_{sp}}(\hat{S})$ and let $g = e_{p_1} + \ldots + e_{p_u}$. Since $G = gAg$ is a product of division rings and the functor L_g: $\mathrm{mod}(G) \to \mathrm{mod}(A)$ is right adjoint to the restriction functor r_g, $L_g(V)$ is injective and $\mathrm{soc} L_g(V) \cong L_g(V)g \cong V$ for any V in $\mathrm{mod}(G)$. Since obviously $L_{\hat{\eta}}(Y)f = L_g(Yg)$, Xf is A-injective as required.

In order to prove the converse we note that since $L_{\hat{\eta}}$ is right adjoint to $r_{\hat{\eta}}$ there is a natural R-homomorphism h: $X \to L_{\hat{\eta}} r_{\hat{\eta}}(X)$ such that $r_{\hat{\eta}}(h) = \mathrm{id}$ and the induced map $\mathrm{soc}(X) \to \mathrm{soc}(L_{\hat{\eta}} r_{\hat{\eta}}(X))$ is bijective. By the "if" part of the lemma $L_{\hat{\eta}} r_{\hat{\eta}}(X) f$ is A-injective and therefore h restricted to Xf is an isomorphism. It follows that h is bijective. Since analogously the natural map $T_{\hat{\eta}} r_{\hat{\eta}}(X) \to X$ is also bijective the proof is complete.

LEMMA 3.8. (a) *Given $i = 1, \ldots, u$ the T-modules*

$$H_0^i := E_T(e_{p_i}T) = D_T^{\bullet}(e_{p_i}^{-} T^{\bullet}) \quad \text{and} \quad H_j^i := D_T^{\bullet}(e_{c_j^i} T^{\bullet}) \quad \text{for } j = 1, \ldots, t_i$$

(see (3.3)) *are indecomposable hereditary* sp-*injective in* $\mathrm{mod_{sp}}(T)$ *such that* $H_j^i f \cong E_A(e_{p_i}A)$, $H_j^i|_{\mathscr{C}^t} = 0$ *for* $t \neq i$ *and*

$$H_j^i|_{\mathscr{C}^i} = (0 \to 0 \to \ldots \to 0 \to F \to F \to \ldots \to F) \qquad (j \text{ zeros})$$

for all $i = 1, \ldots, u$, $j \leqslant t_i + 1$. *The natural embeddings*

$$H_{t_i}^i \subseteq \ldots \subseteq H_1^i \subseteq H_0^i = E_T(e_{p_i}T)$$

are irreducible in $\mathrm{mod_{sp}}(T)$. *If* $\mathrm{Hom}_T(H_j^i, Z) \neq 0$ *and Z is in* $\mathrm{ind_{sp}}(T)$ *then* $Z \cong H_r^i$ *for some* $0 \leqslant r \leqslant j$. *Moreover,* $\mathrm{Hom}_T(H_r^i, H_s^j) = 0$ *for all* r, s *and* $i \neq j$.

(b) *The $\hat{S}$-modules* $P_j^i = e_{c_j^i}\hat{S}$, $j = 1, \ldots, t_i$, $t_i + 1$, *are hereditary projective and the natural embeddings*

$$e_{p_i}\hat{S} = P_{t_i+1}^i \subseteq P_{t_i}^i \subseteq \ldots \subseteq P_1^i$$

are irreducible in $\mathrm{mod_{sp}}(\hat{S})$.

(c) $\quad L_{\hat{\eta}}(P_j^i) \cong T_{\xi}(H_{j-1}^i) \quad$ *for* $j = 1, \ldots, t_i + 1$, $i = 1, \ldots, u$.

Proof. Applying the same type of argument as in the proof of Proposition 2.19 we show that the support of the module $X = e_{c_j^i} T^{\bullet}$ is contained in $\mathscr{S}$: $p_i^- \leftarrow c_1^i \leftarrow \ldots \leftarrow c_{t_i}^i$ and we have

$$X|_{\mathscr{S}} = (F^{\mathrm{op}} \leftarrow \ldots \leftarrow \underset{j}{F^{\mathrm{op}}} \leftarrow \underset{j+1}{0} \leftarrow \ldots \leftarrow 0).$$

Then X is hereditary projective and Proposition 2.5(c) together with the reflection duality $D_T^{\bullet}$ yields (a).

(b) Since obviously $\mathbf{supp}(P_j^i) \subseteq \mathscr{C}^i$, (b) follows.

(c) It follows from the definitions that $\mathbf{supp}(L_{\hat{\eta}}(P_j^i))$, $\mathbf{supp}(T_{\xi}(H_{j-1}^i)) \subseteq \mathscr{C}^i \cup \mathscr{Q}''$, $T_{\xi}(H_{j-1}^i)|_{\mathscr{C}^i} \cong H_{j-1}^i|_{\mathscr{C}^i} \cong P_j^i|_{\mathscr{C}^i}$ and $T_{\xi}(H_{j-1}^i)f \cong E_A(e_{p_i}A)$ (in the notation of Lemma 3.7). Then Lemma 3.7 yields $T_{\xi}(H_{j-1}^i) \cong L_{\hat{\eta}} r_{\hat{\eta}} T_{\xi}(H_{j-1}^i) \cong L_{\hat{\eta}}(P_j^i)$ and (c) follows.

LEMMA 3.9. (a) *Given $i \in \mathscr{Q}_0'$ the restriction of e_iR to $\mathscr{C} + \mathscr{Q}''$ is either zero or is isomorphic to a direct sum of copies of H_j^i for some indices i and j. The module M_T in* (3.4) *is hereditary* sp-*injective and M_T is isomorphic to a direct sum of copies of* H_j^i, $i = 1, \ldots, u$, $j = 0, \ldots, t_i$ (comp. [28]).

(b) *The functors* T_{ξ}, $L_{\hat{\eta}}$ *are exact and*

$$T_{\xi}(e_iT) \cong e_iR \quad \text{for } i \in \mathscr{Q}_0 - \mathscr{Q}_0', \qquad L_{\hat{\eta}}(e_j\hat{S}) \cong e_jR \quad \text{for } j \in \mathscr{Q}_0',$$
$$T_{\xi}(Q_T^{(j)}) \cong Q^{(j)} \quad \text{for } j \in \mathscr{Q}_0'', \qquad L_{\hat{\eta}}(Q_S^{(j)}) \cong Q^{(j)} \quad \text{for } j \in \mathscr{Q}_0' + \hat{\mathscr{C}}.$$

Proof. (a) Let $i \in \mathscr{Q}_0'$. Since e_iR is socle projective, Definition 3.3(c) yields

$$\mathrm{soc}(e_iRf) = \bigoplus_{p \in \mathscr{P}} (e_pA)^{d(i,p)}$$

where f and A are as in Lemma 3.7. It follows that there is an A-monomorphism

$$h\colon\ e_iRf \to \bigoplus_{p \in \mathscr{P}} E_A(e_pA)^{d(i,p)}.$$

Since $\dim_F E_A(e_pA)e_j = d(j, p)$, the splitting assumption (c) yields

$$\dim_F(e_iRfe_j) = d(i, j) = \sum_{p \in \mathscr{P}} d(i, p) d(j, p)$$
$$= \dim_F\Big(\bigoplus_{p \in \mathscr{P}} E_A(e_pA)^{d(i,p)}\Big)e_j$$

for any $j \in \mathscr{Q}_0''$ and therefore h is bijective.

In order to finish the proof we consider the functors

$$\mathrm{mod_{sp}}(\Lambda) \underset{r_w}{\overset{L_w}{\rightleftarrows}} \mathrm{mod_{sp}}(T)$$

where $\Lambda = wTw$ and $w = \sum_{j \in \mathscr{C}} e_j$. By our assumptions $\Lambda \cong F\mathscr{C}^1 \times \ldots \times F\mathscr{C}^u$ and $\mathrm{mod_{sp}}(F\mathscr{C}^j) \cong \mathscr{C}^j$-sp, where $\mathscr{C}^1, \ldots, \mathscr{C}^u$ are the chains in Definition 3.3. Hence if X is in $\mathrm{ind_{sp}}(\Lambda)$ then $\mathbf{supp}(X) \subseteq \hat{C}^r$ for some $r \leqslant u$ and

$$X|_{\mathscr{C}^r} = (0 \to 0 \to \ldots \to 0 \to F \to F \to \ldots \to F) \qquad (j \text{ zeros}).$$

Since we know from Lemma 3.7 that $L_w(X)f \cong L_{\hat{\eta}}(X)f \cong E_A(e_{p_r}A)$, Lemma 3.8(a) yields $L_w(X) \cong H_j^r$ and therefore in order to prove that the restriction Y of e_iR to $\mathscr{C} + \mathscr{Q}''$ is a direct sum of copies of H_s^i it is sufficient to show that Y is in $\mathrm{Im} L_w$. For this purpose we note that since $Yf \cong e_iRf$, $Y\eta = 0$ and the map h is an A-isomorphism, according to Lemma 3.7 we have $Y \cong L_{\hat{\eta}} r_{\hat{\eta}}(Y) \cong L_w r_w(Y)$ as required.

Since

$$M_T = \eta R \xi = \sum_{i \in \mathscr{Q}_0'} \sum_{j \in \mathscr{Q}_0 - \mathscr{Q}_0'} e_iRe_j = \bigoplus_{i \in \mathscr{Q}_0'} r_{\xi}(e_iR)$$

and since it follows from the fact proved above that $r_{\xi}(e_iR)$ is either zero or a direct sum of copies of H_j^r, (a) follows in view of Lemma 3.8.

(b) First we note that T_{ξ} is the embedding of $\mathrm{mod_{sp}}(T)$ in $\mathrm{mod_{sp}}(R)$ induced by the natural epimorphism $R \to T$ derived from the triangular form (3.4) of R. The functor

$L_{\hat\eta}$ is left exact by Proposition 2.11(a). Therefore from Lemma 3.7 and from $r_{\hat\eta}L_{\hat\eta}\cong\mathrm{id}$ we easily conclude that $L_{\hat\eta}$ is exact.

The first isomorphism in (b) is obvious. In order to prove the second one we fix $j\in\mathcal{Q}''_0-\boldsymbol{p}(\mathcal{Q})$ and recall from (2.7) that $Q^{(j)}\cong V_+D(e_jR^\bullet)\cong DV_-(e_jR^\bullet)$. By Proposition 2.5(c) there is an exact sequence

$$(*)\qquad 0\to e_jR^\bullet\xrightarrow{h}E_{\hat R^\bullet}(e_jR^\bullet)\to V_-(e_jR^\bullet)\to 0$$

in $\mathrm{mod}(\hat R^\bullet)$, where $\hat R^\bullet=(R^\blacktriangledown)^{\mathrm{op}}$ and $\mathrm{mod}_{\mathrm{sp}}(R^\bullet)$ is considered as a full subcategory of $\mathrm{mod}(\hat R^\bullet)$ via the natural epimorphism $\hat R^\bullet\to R^\bullet$. Consider the idempotents

$$\eta_1=\sum_{i\in{}^\bullet\mathcal{Q}'_0}e_i,\qquad \xi_1=\sum_{j\in{}^\bullet\mathcal{Q}''_0+\mathcal{C}+\mathcal{P}^-}e_j$$

in $R^\bullet$ and the idempotent

$$\xi_2=\xi_1+\sum_{p\in\boldsymbol{p}(\mathcal{Q}'')}e_p$$

in $R^\bullet$, where we put $e_{p^-}=e_p^-$ for $p\in\boldsymbol{p}(\mathcal{Q})$ and ${}^\bullet\mathcal{Q}'$, ${}^\bullet\mathcal{Q}''$ are as in Remark 3.3′. It is easy to see that $T^\bullet\cong\xi_1\hat R^\bullet\xi_1$ and $\hat T^\bullet\cong\xi_2\hat R^\bullet\xi_2$. A simple analysis shows that $E_{\hat T^\bullet}(e_jT^\bullet)=E_{\hat R^\bullet}(e_jR^\bullet)\xi_2$ (the restriction of $E_{\hat R^\bullet}(e_jR^\bullet)$ to $(\mathcal{P}^-+\mathcal{C}+\mathcal{Q}'')^{\mathrm{op}}$). By Remark 3.3′ the decomposition $\mathcal{Q}^\bullet={}^\bullet\mathcal{Q}'+\mathcal{C}^{\mathrm{op}}+{}^\bullet\mathcal{Q}''$ induces a splitting decomposition of $(\mathcal{Q}^\bullet,\Omega^\bullet)$. Then applying (a) to $R^\bullet$ we conclude that the restriction $e_jR^\bullet\eta_1$ of $e_jR^\bullet$ to ${}^\bullet\mathcal{Q}'_0$ is an injective module and therefore the restriction $e_jR^\bullet\eta_1\to E_{\hat R^\bullet}(e_jR^\bullet)\eta_1$ of the map h to $e_jR^\bullet\eta_1$ is an isomorphism. It follows that Coker$h$ is a $T^\bullet$-module and according to Proposition 2.5(c) and remarks above the sequence $(*)$ induces a commutative diagram

$$\begin{array}{ccccccccc}0\to & e_jR^\bullet\xi_2 & \to & E_{\hat R^\bullet}(e_jR^\bullet)\xi_2 & \to & V_-(e_jR^\bullet) & \to 0\\ & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & \\ 0\to & e_jT^\bullet & \to & E_{\hat T^\bullet}(e_jT^\bullet) & \to & V_-(e_jT^\bullet) & \to 0\end{array}$$

of $\hat T^\bullet$-modules. Since T_ξ is a natural embedding we have

$$T_\xi(Q_T^{(j)})=Q_T^{(j)}\cong DV_-(e_jT^\bullet)\cong DV_-(e_jR^\bullet)\cong Q^{(j)}$$

as required. For $j\in\boldsymbol{p}(\mathcal{Q}'')$ the proof is similar.

In order to establish the right hand isomorphisms in (b) we note that according to (a) and Lemma 3.8 the restriction of e_jR to $\mathcal{Q}''$ is isomorphic to a direct sum of copies of the A-modules $E_A(e_pA)$, $p\in\mathcal{P}$. Then by Lemma 3.7 we get

$$e_jR\cong L_{\hat\eta}r_{\hat\eta}(e_jR)\cong L_{\hat\eta}(e_jR\hat\eta)\cong L_{\hat\eta}(e_j\hat S).$$

Since the remaining isomorphism in (b) can be established in a similar way the proof is complete.

Now we are able to prove our multipeak splitting theorem.

THEOREM 3.10. *Suppose that $R=F(\mathcal{Q},\Omega)$ is a right multipeak algebra* (3.1) *where F is a division ring and $(\mathcal{Q},\Omega)$ is a bound quiver with a splitting decomposition* (3.2). *In the notation of* (3.6) *we have*:

(a) *If X is in $\mathrm{ind}_{\mathrm{sp}}(R)$ then either $X\in\mathrm{Im}\,T_\xi$ or $X\in\mathrm{Im}\,L_{\hat\eta}$. Moreover, $X\in\mathrm{Im}\,T_\xi\cap\mathrm{Im}\,L_{\hat\eta}$ if and only if $X\cong L_{\hat\eta}(P_j^i)\cong T_\xi(H_{j-1}^i)$ for some $i=1,\dots,u$, $j=1,\dots,t_i,t_{i+1}$* (see 3.8). *If Y is in $\mathrm{ind}_{\mathrm{sp}}(\hat S)$, Z is in $\mathrm{ind}_{\mathrm{sp}}(T)$ and $Y\eta\neq 0$ then $\mathrm{Hom}_R(L_{\hat\eta}(Y),T_\xi(Z))=0$.*

(b) *The functors $L_{\hat\eta}$ and T_ξ are full faithful exact and carry Auslander–Reiten sequences to Auslander–Reiten sequences. If $\mathfrak{X}$ is an Auslander–Reiten sequence in $\mathrm{mod}_{\mathrm{sp}}(R)$ and the starting term of $\mathfrak{X}$ is indecomposable then either $\mathfrak{X}\cong L_{\hat\eta}(\mathfrak{Y})$ or $\mathfrak{X}\cong T_\xi(\mathfrak{Z})$, where $\mathfrak{Y}$ and $\mathfrak{Z}$ are Auslander–Reiten sequences in $\mathrm{mod}_{\mathrm{sp}}(\hat S)$ and $\mathrm{mod}_{\mathrm{sp}}(T)$, respectively.*

Proof. Let $X=(X'_S,X''_T,\varphi\colon X'\otimes_SM_T\to X''_T)$ be a module in $\mathrm{ind}_{\mathrm{sp}}(R)$. It is clear that $X''_T=X\xi$ is in $\mathrm{mod}_{\mathrm{sp}}(T)$. Assume $X'_T=X\eta\neq 0$. If $\varphi\neq 0$, Y is an indecomposable summand of X''_T and $p\colon X''\to Y$ is the natural projection then $p\varphi\neq 0$ and therefore $\mathrm{Hom}_T(M_T,Y)\neq 0$. It follows from Lemma 3.8(a) that $Y\cong H_j^r$ for some r and j. Consequently, either $\varphi=0$ and $X''=0$, or X''_T is a direct sum of copies of H_j^r. Therefore $Xf=X''f$ is injective because $H_j^rf\cong E_A(e_{p_r}A)$ in the notation of Lemma 3.7. It follows from Lemma 3.7 that $X\cong L_{\hat\eta}r_{\hat\eta}(X)\cong T_\xi r_\xi(X)\in\mathrm{Im}\,L_{\hat\eta}$ and $r_{\hat\eta}(X)$ is indecomposable. If $X'_S=0$ then obviously $X\cong T_\xi r_\xi(X)\in\mathrm{Im}\,T_\xi$ and the first statement in (a) follows.

If $X\cong L_{\hat\eta}(Y)\cong T_\xi(Z)$ then $X\eta=Y\eta=0$ and therefore $\mathbf{supp}(Y)\subseteq\hat{\mathcal{C}}^r$ for some $r=1,\dots,u$ because of (3.3). Hence $Y\cong P_j^r$ for some j and in view of Lemma 3.8(c) the second statement in (a) follows.

Now take Y in $\mathrm{ind}_{\mathrm{sp}}(\hat S)$ such that $Y\eta\neq 0$, put $X\cong L_{\hat\eta}(Y)$ and suppose that there is a nonzero $h\in\mathrm{Hom}_R(X,T_\xi Z)$. It follows from the first part of the proof that $X\xi$ is a direct sum of copies of H_l^j, $j=1,\dots,u$, $i=0,\dots,t_j$. Now since $h\neq 0$ and $(T_\xi Z)\eta=0$, $r_\xi(h)\colon X\xi\to(T_\xi Z)\xi\cong Z$ is nonzero and we conclude from Lemma 3.8 that $Z\cong H_{j-1}^i$ for some i and j, and $T_\xi Z\cong T_\xi H_{j-1}^i\cong L_{\hat\eta}P_j^i$. Hence $\mathrm{Hom}_R(X,T_\xi Z)\cong\mathrm{Hom}_R(L_{\hat\eta}Y,L_{\hat\eta}P_j^i)\cong\mathrm{Hom}_{\hat S}(Y,P_j^i)=0$ because otherwise $Y\cong P_l^i$ for some $l\geqslant j$ (Lemma 3.8(b)) and therefore $Y\eta=0$; a contradiction. This finishes the proof of (a).

(b) Suppose that $\mathfrak{Y}\colon 0\to Y\xrightarrow{u}Y'\to Y''\to 0$ is an Auslander–Reiten sequence in $\mathrm{mod}_{\mathrm{sp}}(\hat S)$ and Y, Y'' are indecomposable. In order to prove that $L_{\hat\eta}\mathfrak{Y}$ is an Auslander–Reiten sequence it is sufficient to show that $L_{\hat\eta}(u)$ is a left almost split map. Let $g\colon L_{\hat\eta}Y\to X$ be a nonzero nonisomorphism in $\mathrm{ind}_{\mathrm{sp}}(R)$. It follows from (a) that $X\cong L_{\hat\eta}U$ for some U in $\mathrm{ind}_{\mathrm{sp}}(\hat S)$ and we get a factorization of g through $L_{\hat\eta}(u)$ because $L_{\hat\eta}$ is fully faithful and u is left almost split. The proof for the functor T_ξ is similar.

Now suppose that $\mathfrak{X}\colon 0\to X\to X'\to X''\to 0$ is an Auslander–Reiten sequence in $\mathrm{mod}_{\mathrm{sp}}(R)$ and X is non-sp-injective indecomposable. If $X\cong L_{\hat\eta}Y$ then by Lemma 3.9(b), Y is non-sp-injective indecomposable and therefore there exists an Auslander–Reiten sequence $\mathfrak{Y}$ in $\mathrm{mod}_{\mathrm{sp}}(\hat S)$ starting with Y. Since we know that $L_{\hat\eta}\mathfrak{Y}$ is an Auslander–Reiten sequence we have $\mathfrak{X}\cong L_{\hat\eta}\mathfrak{Y}$.

Next suppose that $X\notin\mathrm{Im}\,L_{\hat\eta}$. It follows from (a) that $X\cong T_\xi Z$ for some Z and X is not isomorphic to a module of the form $T_\xi H_{j-1}^i$. Hence in view of Lemma 3.9(b), Z is not sp-injective and therefore there exists an Auslander–Reiten sequence $\mathfrak{Z}$ in $\mathrm{mod}_{\mathrm{sp}}(T)$ starting with Z. Since we know that $T_\xi\mathfrak{Z}$ is an Auslander–Reiten sequence, $\mathfrak{X}\cong T_\xi\mathfrak{Z}$ and the proof is complete.

Following Ringel [17] we call $\mathrm{Irr}(X,X')=\mathrm{Hom}_R(X,X')/\mathrm{rad}^2(X,X')$ the $E(X')$-$E(X)$-bimodule of irreducible maps from X to X', where $E(X)=\mathrm{End}(X)/J\mathrm{End}(X)$.

As a consequence of the results above we get

COROLLARY 3.11. *Let R be as in Theorem* 3.10. *Then*

(a) *R is not* sp-*sincere if one of $\mathcal{Q}'$, $\mathcal{Q}''$ is not empty.*

(b) *The natural bimodule epimorphisms $\tilde{L}_{\hat{\eta}}$:* $\mathrm{Irr}(Y, Y') \to \mathrm{Irr}(L_{\hat{\eta}} Y, L_{\hat{\eta}} Y')$, $\tilde{T}_{\xi}$: $\mathrm{Irr}(Z, Z') \to \mathrm{Irr}(T_{\xi} Z, T_{\xi} Z')$ *are isomorphisms for Y, Y' in* $\mathrm{ind}_{\mathrm{sp}}(\hat{S})$ *and Z, Z' in* $\mathrm{ind}_{\mathrm{sp}}(T)$.

(c) $\Gamma_{\mathrm{sp}}(R) = L_{\hat{\eta}} \Gamma_{\mathrm{sp}}(\hat{S}) \cup T_{\xi} \Gamma_{\mathrm{sp}}(T)$ *is obtained from* $\Gamma_{\mathrm{sp}}(\hat{S})$ *and* $\Gamma_{\mathrm{sp}}(T)$ *by the identification of the final hereditary* sp-*injective section* $H^i_{t_i} \to \ldots \to H^i_0$ *in* $\Gamma_{\mathrm{sp}}(T)$ *with the starting hereditary projective section* $P^i_{t_i+1} \to \ldots \to P^i_1$ *in* $\Gamma_{\mathrm{sp}}(\hat{S})$ *for* $i = 1, \ldots, u$ (see 3.8 and Fig. 7, comp. [27]).

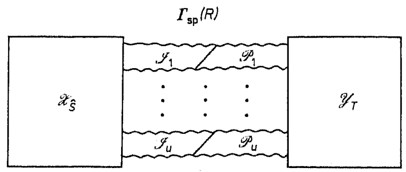

Fig. 7

Applying the same type of argument as in the proof of Theorem 3.10(a) we get

PROPOSITION 3.12. *Suppose that* $R = F(\mathcal{Q}, \Omega)$ *and* $(\mathcal{Q}, \Omega)$ *admits a triangular decomposition* (3.2) *and a set* $\mathcal{P} \subseteq \mathbf{p}(\mathcal{Q}'')$ *satisfying* 3.3(a) *and the following condition*:

(S) *If* $i \in \mathcal{Q}_0'$ *and Z is in* $\mathrm{ind}_{\mathrm{sp}}(T)$ (see (3.4)) *such that* $\mathrm{Hom}_T(r_{\xi}(e_i R), Z) \neq 0$ *then* $Z|_{\mathcal{Q}''} = Zf$ *is a direct sum of copies of* $E_A(e_p A)$, $p \in \mathcal{P}$ (see 3.7).

Then every X in $\mathrm{ind}_{\mathrm{sp}}(R)$ *belongs either to* $\mathrm{Im} L_{\hat{\eta}}$ *or to* $\mathrm{Im} T_{\xi}$.

Let us finish this section by a generalization of the Ringel–Roggenkamp [19] edge reduction.

COROLLARY 3.13. *Let* $R = F(\mathcal{Q}, \Omega)$ *be a right multipeak algebra such that* $(\mathcal{Q}, \Omega)$ *is a disjoint union of* $(\mathcal{Q}', \Omega')$ *and a quiver*

$$\mathscr{C}\colon\ p \leftarrow \ldots \leftarrow \circ \rightarrow \ldots \rightarrow \circ \leftarrow \ldots$$

of type A_n having $\geqslant 1$ *sinks. Suppose that there is no path* $c \to i \in \mathcal{Q}_0'$, $c \in \mathscr{C}_0$, *and there is an edge* $\gamma\colon s \to p$, *s a sink in* $\mathcal{Q}'$, *such that any nonzero path* $i \to c \in \mathscr{C}_0$, $i \in \mathcal{Q}_0'$, *in* $(\mathcal{Q}, \Omega)$ *has a factorization through* γ. *Then any X in* $\mathrm{ind}_{\mathrm{sp}}(R)$ *is in the image of one of the functors*

$$\mathrm{mod}_{\mathrm{sp}} F(\mathcal{Q} - \{s\}, \Omega) \xrightarrow{L^s} \mathrm{mod}_{\mathrm{sp}}(R) \xleftarrow{T^s} \mathrm{mod}_{\mathrm{sp}} F\mathscr{C}$$

where $T^s = T_g$, $g = \sum_{c \in \mathscr{C}} e_c$, *is the natural embedding and* $L^s = L_e$, $e = \sum_{i \neq s} e_i$.

Proof. Similarly to the proof of Theorem 3.10 one can show that if X is in $\mathrm{ind}_{\mathrm{sp}}(R)$ and $X(\gamma)$: $Xe_s \to Xe_p$ is nonzero then $X(\gamma)$ is bijective and according to Lemma 3.7, $X \in \mathrm{Im} L^s$. Hence the corollary follows.

We shall show that the splitting theorem remains true for a class of artinian rings under a suitable modification of Definition 3.3.

For this purpose we suppose that $R = \bigoplus_{i \in I_R} e_i R$ is a basic artinian right multipeak ring, we keep the notation of Section 2 and given $i \neq j$ we set

$$d_{ij} = \mathrm{length}(e_i R e_j)_{e_j R e_j}, \qquad d'_{ij} = \mathrm{length}_{e_i R e_i}(e_i R e_j).$$

DEFINITION 3.14. The ring R has a *splitting decomposition* if there exist a set $\mathcal{P} = \{p_1, \ldots, p_u\} \subseteq \mathbf{p}(I_R)$ and a disjoint union decomposition

$$I_R = I' \dot{\cup} C \dot{\cup} I''$$

such that $C = C^1 \dot{\cup} \ldots \dot{\cup} C^u \subseteq I_R - \mathbf{p}(I_R)$, $\mathcal{P} \subseteq I''$, I'' is not empty and the following conditions are satisfied:

(a) $\hat{S} = \hat{\eta} R \hat{\eta}$ and $T = \xi R \xi$ are right multipeak rings with peak idempotents e_p, $p \in \mathbf{p}(I_R) \cap I' \cup \mathcal{P}$, and e_q, $q \in \mathbf{p}(I) \cap I''$, respectively, where $\hat{\eta} = e_{p_1} + \ldots + e_{p_u} + \sum_{i \in I' \cup C} e_i$ and $\xi = \sum_{j \in I''} e_j$.

(b) $d_{ij} d'_{ij} = 1$ for $i, j \in \hat{C}^t := C^t \cup \{p_t\}$, $t = 1, \ldots, u$.

(c) $d_{ij} = 0$ if either $i \in C^t$, $j \in C^s$, $t \neq s$, or $i \in C \cup I''$, $j \in I'$, or $i \in I''$, $j \in C$, or $i \in C$, $j \in \mathbf{p}(I_R) - \mathcal{P}$, or $i \in I'$, $j = (\mathbf{p}(I_R) \cap I'') - \mathcal{P}$.

(d) $d_{ij} = \sum_{p \in \mathcal{P}} d_{ip} d'_{jp}$ for any $i \in I'$ and $j \in I''$.

In this case we have the induced functors (3.6) and by a modification of the arguments in the proof of our splitting results above and those in [11, 12] we easily prove

THEOREM 3.15. *Let R be a basic artinian right multipeak* PI-*ring* (2.0) *such that* $\mathrm{mod}_{\mathrm{sp}}(R)$ *has Auslander–Reiten sequences. If R admits a splitting decomposition induced by $\mathcal{P}$ and* $I_R = I' \dot{\cup} C \dot{\cup} I''$ *then in the notation above the statements* (a) *and* (b) *in Theorem* 3.10 *as well as* (a)–(c) *in Corollary* 3.11 *are true.*

4. Examples and concluding remarks.

4.0. *Multipeak posets.* Suppose that I is a finite poset and consider I as a bound quiver whose arrows are pairs (i, j), where $i \prec j$ and there is no $t \neq i, j$ such that $i \prec t \prec j$. The relations are the natural commutativity ones. Suppose that

$$I = I' + (C^1 + \ldots + C^u) + (p_1^{\nabla} + \ldots + p_u^{\nabla})$$

is a disjoint union triangular poset decomposition, where $p_1 \ldots, p_u \in \mathbf{p}(I)$ and $C^1, \ldots, C^u$ are linearly ordered subposets of $I - \mathbf{p}(I)$ (see (3.2)). The decomposition is splitting in the sense of 3.3 if and only if all elements $u \in C^i + p_i^{\nabla}$, $v \in C^j + p_j^{\nabla}$ are unrelated for $i \neq j$, and $s \prec p_i$, $s \in I'$, implies $s \prec r$ for all $r \prec p_i$, where $i = 1, \ldots, u$.

If I is a one-peak poset, then $u = 1$, p_1 is the unique maximal element in I and the definition above is the usual one given by Nazarova and Roiter in [15]. For I arbitrary our definition is a natural extension of that given in [15]. The situation is more complicated for multipeak posets with zero relations. As a simple example consider the two-peak poset I of Fig. 8 with zero relations $\alpha\beta = \gamma\delta = \varepsilon\xi = 0$. The decomposition

$$I = I' + C^1 + C^2 + I''$$

where $I' = \{a, b \to d\}$, $C^1 = \{c_1 \to c_2\}$, $C^2 = \{c_3 \to c_4\}$, $I'' = I - (C^1 + C^2 + I')$, is obviously splitting.

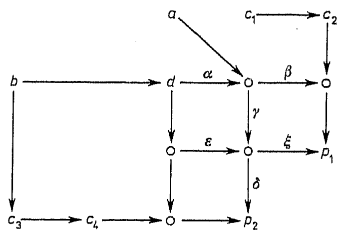

Fig. 8

4.1. The following two examples are not of the poset type. They are representative for the Galois covering reduction of nonschurian right peak algebras mentioned in the Introduction.

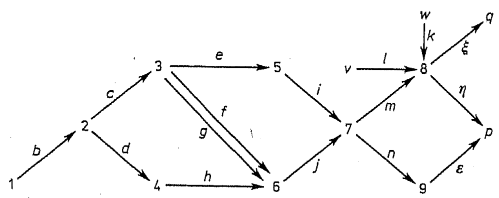

Fig. 9

(a) Consider a right two-peak algebra $R = F(\mathcal{Q}, \Omega)$, where $\mathcal{Q}$ is shown in Fig. 9 and $\Omega = (cg - dh,\ ei - fj,\ m\eta - n\varepsilon,\ hjm\xi,\ gjm\xi,\ in,\ k\xi,\ l\eta)$. If we set $\mathcal{Q}_0' = \{1, 2, 3, 4\}$, $\mathscr{C}_0^1 = \{u\}$, $\mathscr{C}_0^2 = \{v\}$, $\mathcal{Q}_0'' = \{5, 6, 7, 8, 9, p, q\}$ then

$$\mathcal{Q}_0 = \mathcal{Q}_0' + \mathscr{C}_0^1 + \mathscr{C}_0^2 + \mathcal{Q}_0''$$

defines a splitting decomposition of $(\mathcal{Q}, \Omega)$ with $u = 2$, $p_1 = p$ and $p_2 = q$. Note that the algebras (3.4)

$$\hat{S} = F(\mathcal{Q}_0' + \hat{\mathscr{C}}^1 + \hat{\mathscr{C}}^2, \Omega|), \qquad T = F(\mathscr{C}^1 + \mathscr{C}^2 + \mathcal{Q}'', \Omega|)$$

are the bound quiver algebras of the quivers of Fig. 10 with all commutativity relations and the following zero relations: $5 \dashrightarrow 9$, $v \dashrightarrow p$, $w \dashrightarrow q$ in the right hand quiver. It follows that $\hat{S}$ and T have the separation property for radicals of indecomposable projective

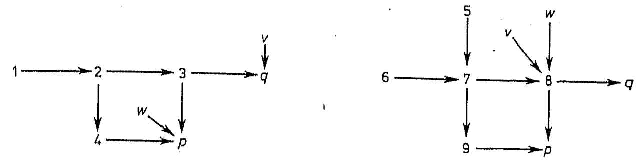

Fig. 10

right modules and $\Gamma_{sp}(\hat{S})$, $\Gamma_{sp}(T)$ have a preprojective and a preinjective component (see [26; Proposition 4.7]). Modifying the arguments used in [12, 24] one can show that

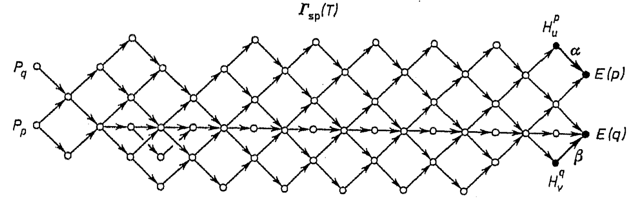

Fig. 11

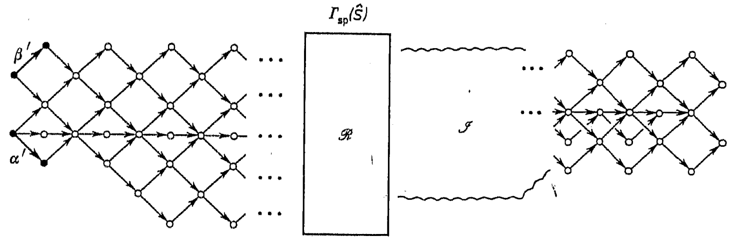

Fig. 12

$\Gamma_{sp}(T)$ and $\Gamma_{sp}(\hat{S})$ have the forms presented in Figures 11 and 12, respectively. By [25; Theorem 3.4 and Corollary 3.13] there is an equivalence of categories

$$\mathrm{mod}_{sp}(\hat{S})/\mathscr{L} \cong J\text{-sp}/[e_1FJ^*,\ e_2FJ^*,\ e_3FJ^*]$$

where $\mathscr{L}$ is the ideal in $\mathrm{mod}_{sp}(\hat{S})$ generated by $e_j\hat{S}$, $j = v, q, p, 4, w$ and by the submodule $e_4\hat{S} + e_w\hat{S}$ of $E(e_p\hat{S})$, and J^* is the poset of Fig. 13. Hence if $F = \bar{F}$, the category $\mathrm{mod}_{sp}(\hat{S})$

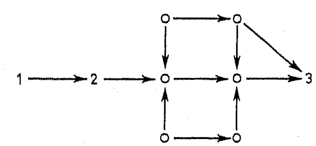

Fig. 13

is of tame type [6] because J^* is of tame type by the criterion of Nazarova [14]. It follows from Corollary 3.11 that

$$\Gamma_{sp}(R) = \Gamma_{sp}(T) \cup \Gamma_{sp}(\hat{S})/(\alpha \equiv \alpha', \beta \equiv \beta').$$

The part $\mathscr{R}$ in Figure 12 consists of tubes [17]. The category $\mathrm{mod}_{sp}(R)$ is of tame type. Note also that the algebras $\hat{S}$ and T are simply connected [0], whereas R is not.

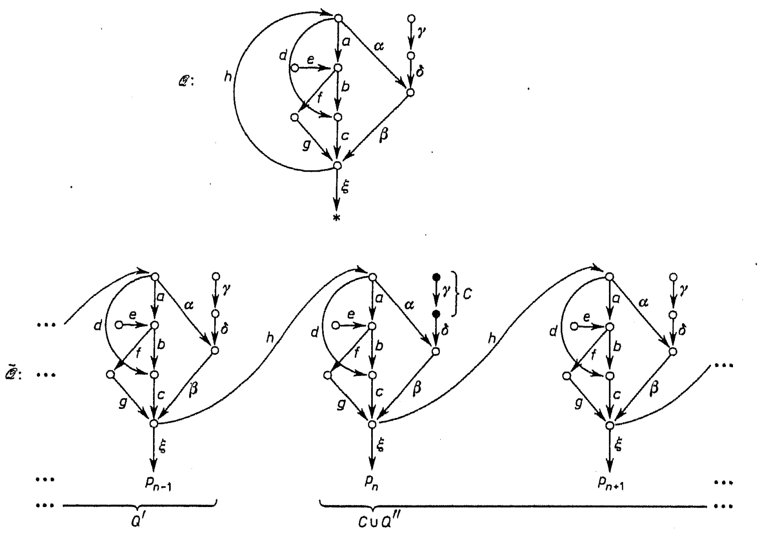

Fig. 14

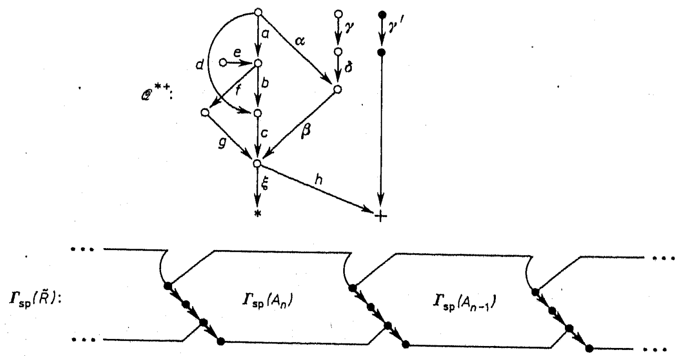

Fig. 15

(b) Consider the right peak algebra $R = F(\mathcal{Q}, \Omega)$ and the right multipeak algebra $\tilde{R} = F(\tilde{\mathcal{Q}}, \tilde{\Omega})$, where $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$ are the quivers in Fig. 13 and

$$\Omega = \tilde{\Omega} = (dc - afg,\ abc - dc - \alpha\beta,\ abc\xi - \alpha\beta\xi,$$
$$efg - ebc,\ h\alpha\beta - habc,\ bch - fgh,\ \beta h,\ hd,\ haf,\ g\xi).$$

Note that $(\tilde{\mathcal{Q}}, \tilde{\Omega})$ is the universal cover of $(\mathcal{Q}, \Omega)$ and $\mathcal{Q}'$, $\mathscr{C} \cup \mathcal{Q}''$, $\mathscr{C} = \{\bullet \xrightarrow{\gamma} \bullet\}$ marked in Fig. 14 induce a splitting decomposition of $(\tilde{\mathcal{Q}}, \tilde{\Omega})$ for any fixed n. By repeating the splitting procedure we easily conclude from Theorem 3.10 that the coordinate support of any X in $\mathrm{ind}_{sp}(\tilde{R})$ is a subquiver of the quiver $(\mathcal{Q}^{*+}, \tilde{\Omega}|)$ in Fig. 15. The two-peak algebra $A = F(\mathcal{Q}^{*+}, \tilde{\Omega}|)$ is sp-representation-infinite according to the criterion of Weichert [29]. It follows that $\tilde{R}$ is locally sp-representation-infinite and by [22; Theorem 1.10] R is sp-representation-infinite. Moreover, we conclude from Corollary 3.11 that $\Gamma_{sp}(\tilde{R})$ is obtained from $\Gamma_{sp}(A_n)$, $A_n = A$, $n = 0, \pm 1, \pm 2, \ldots$ by successive glueing presented in Fig. 15, whereas $\Gamma_{sp}(R)$ is obtained from $\Gamma_{sp}(A)$ by simple glueing of two linear three-vertex sections like in [24; 5.15]. Since A obviously has the separation property [26], $\Gamma_{sp}(A)$ has a preprojective component $\mathscr{P}_{sp}(A)$ which can be easily computed by the well-known preprojective component construction like in [12, 24].

4.2. *Multiserial multipeak trees.* A multipeak bound quiver $(\mathcal{Q}, \Omega)$ will be called a *multiserial tree* if $\mathcal{Q}$ is a tree, Ω consists of zero relations and the following conditions are satisfied (comp. [7; 5.2] and [5]):

(i) For every edge a in $\mathcal{Q}_1$ there exists at most one edge b and at most one edge c such that ac, ba are not in Ω.

(ii) For any $p \in \mathbf{p}(\mathcal{Q})$ there are chains $\mathscr{C}'_p$, $\mathscr{C}''_p$ such that $p^\nabla = \mathscr{C}'_p \cup \mathscr{C}''_p$, $p \in \mathscr{C}'_p$, $\mathscr{C}'_p \cap \mathscr{C}''_p = \varnothing$ and $\mathscr{C}''_p \cap q^\nabla$ is empty if $p \neq q \in \mathbf{p}(\mathcal{Q})$. We put $\hat{\mathscr{C}}''_p = \mathscr{C}''_p \cup \{p\}$.

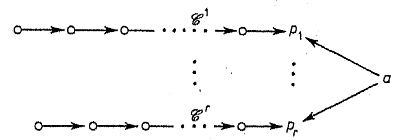

Fig. 16

If $\mathscr{C}^1, \ldots, \mathscr{C}^r$ are pairwise disjoint chains and $|\mathscr{C}^1_0| = m_1, \ldots, |\mathscr{C}^r_0| = m_r$ then we denote by $T_a(\mathscr{C}^1, \ldots, \mathscr{C}^r) = T(m_1, \ldots, m_r)$ the tree of Fig. 16. We say that $T(m_1, \ldots, m_r)$ is a *peak subtree* of $(\mathcal{Q}, \Omega)$ if there is a bound quiver embedding of $T(m_1, \ldots, m_r)$ in $(\mathcal{Q}, \Omega)$ which carries peak vertices to peak ones and oriented edges to nonzero oriented paths in $(\mathcal{Q}, \Omega)$.

PROPOSITION. *Let $R = F(\mathcal{Q}, \Omega)$, where F is a division ring and $(\mathcal{Q}, \Omega)$ is a locally finite multiserial multipeak tree.*

(a) *The radical of any indecomposable projective right R-module is a direct sum of uniserial modules. If X is in $\mathrm{ind}_{sp}(R)$ then* $\mathbf{csup}(X) = T_a(\mathscr{C}_{p_1}, \ldots, \mathscr{C}_{p_r})$ *for some $a \in \mathcal{Q}_0$, $p_1, \ldots, p_r \in \mathbf{p}(\mathcal{Q})$ and $\mathscr{C}_{p_1} \subseteq \hat{\mathscr{C}}''_{p_1}, \ldots, \mathscr{C}_{p_r} \subseteq \hat{\mathscr{C}}''_{p_r}$ with $p_i \in \mathscr{C}_{p_i}$.*

(b) *R is locally* sp-*representation-finite if and only if* $(\mathcal{Q}, \Omega)$ *does not contain peak subtrees of the form* $\boldsymbol{T}(1, 1, 1, 1)$, $\boldsymbol{T}(2, 2, 2)$, $\boldsymbol{T}(1, 3, 3)$, $\boldsymbol{T}(1, 2, 5)$.

(c) *R is locally* sp-*representation-tame if and only if* $(\mathcal{Q}, \Omega)$ *does not contain peak subtrees of the form* $\boldsymbol{T}(1, 1, 1, 1, 1)$, $\boldsymbol{T}(1, 1, 1, 2)$, $\boldsymbol{T}(2, 2, 3)$, $\boldsymbol{T}(1, 3, 4)$ *and* $\boldsymbol{T}(1, 2, 6)$.

Proof. (a) The first part follows immediately from the definition. In order to prove the second part one can suppose without loss of generality that $(\mathcal{Q}, \Omega)$ is finite sp-sincere and $\mathbf{csup}(X) = (\mathcal{Q}, \Omega)$. Let $\boldsymbol{p}(\mathcal{Q}) = \{p_1, \ldots, p_r\}$ and let a be a minimal vertex in $\mathscr{C}'_{p_1} \cup \ldots \cup \mathscr{C}'_{p_r}$ (see (ii)). Suppose that $p_1, \ldots, p_s$, $s \leqslant r$, are all $p \in \boldsymbol{p}(\mathcal{Q})$ such that there is a nonzero path $\gamma\colon a \to p$ in $(\mathcal{Q}, \Omega)$. We shall prove (a) by showing that $s = r$, and $\boldsymbol{T}_a(\hat{\mathscr{C}}''_{p_1}, \ldots, \hat{\mathscr{C}}''_{p_r}) = (\mathcal{Q}, \Omega)$. For this purpose consider the decomposition 3.3 with $u = s$, $\mathcal{Q}'_0 = \{a\}$, $\mathscr{C}^j = \mathscr{C}''_{p_j}$, $j = 1, \ldots, s$, and $\mathcal{Q}'' = \mathcal{Q} - (p_1^\nabla \cup \ldots \cup p_s^\nabla)$. Since a is minimal and $(\mathcal{Q}, \Omega)$ is a multiserial tree, the above decomposition is splitting, and since R is sp-sincere, by Corollary 3.11(a) the set $\mathcal{Q}''_0$ is empty. Hence $s = r$, $(\mathcal{Q}, \Omega) = \boldsymbol{T}_a(\hat{\mathscr{C}}''_{p_1}, \ldots, \hat{\mathscr{C}}''_{p_r})$ and (a) follows.

It follows from (a) that R is locally sp-representation-finite (resp. -tame) if and only if for every peak subtree $\boldsymbol{T}_a(\hat{\mathscr{C}}''_{p_1}, \ldots, \hat{\mathscr{C}}''_{p_r})$ of $(\mathcal{Q}, \Omega)$ the right multipeak hereditary algebra $R_a(m_1, \ldots, m_r) = F\boldsymbol{T}_a(\hat{\mathscr{C}}''_{p_1}, \ldots, \hat{\mathscr{C}}''_{p_r})$ is sp-representation-finite (resp. -tame), where m_j is the number of vertices in $\hat{\mathscr{C}}''_{p_j}$. Consider the poset $\boldsymbol{I}(m_1, \ldots, m_r) = \boldsymbol{T}_a(\hat{\mathscr{C}}''_{p_1}, \ldots, \hat{\mathscr{C}}''_{p_r}) - \{a\}$. It follows from [25; Corollary 3.13] applied to $I = \boldsymbol{T}_a(m_1, \ldots, m_r)$, $c = a$, that $\xi_c I = \boldsymbol{I}(m_1, \ldots, m_r)$ and there is a full dense functor

$$\mathbf{ad}\colon \operatorname{mod}_{\mathrm{sp}}(R_a(m_1, \ldots, m_r)) \to (\boldsymbol{I}(m_1, \ldots, m_r)\text{-sp})/[P_*]$$

preserving the representation type and such that $\operatorname{Ker}\mathbf{ad} = [\boldsymbol{I}(m_1)\text{-sp}, \ldots, \boldsymbol{I}(m_r)\text{-sp}]$, where I-sp means the category of I-spaces, P_* is the unique simple projective I-space and $\boldsymbol{I}(m_j)$-sp is considered as a full subcategory of $\operatorname{mod}_{\mathrm{sp}}(R_a(m_1, \ldots, m_r))$ in a natural way. Consequently, the problem reduces to the problem for $\boldsymbol{I}(m_1, \ldots, m_r)$-spaces and therefore (b) is a consequence of the criterion of Kleiner [10], whereas (c) is a consequence of the criterion of Nazarova [14]. The proof is complete.

For an illustration of the result above let us consider the right peak algebra

$$R = \begin{bmatrix} F & 0 & 0 & 0 & F \\ & F & F & F & F \\ & & F & F & F \\ & & & F & F \\ 0 & & & & F \end{bmatrix}$$

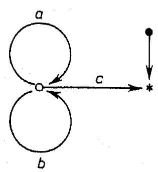

Fig. 17

where F is an algebraically closed field. Note that $R = F(\mathcal{Q}, \Omega)$, where $\mathcal{Q}$ is shown in Fig. 17 and $\Omega\colon a^2 = b^2 = ba = ac = 0$. The infinite tree $\tilde{\mathcal{Q}}$ in Fig. 18 with relations $\tilde{\Omega} = \Omega$ is a Galois cover of $(\mathcal{Q}, \Omega)$ with the free group $G = \boldsymbol{Z} * \boldsymbol{Z}$. Since $(\tilde{\mathcal{Q}}, \tilde{\Omega})$ is a multiserial

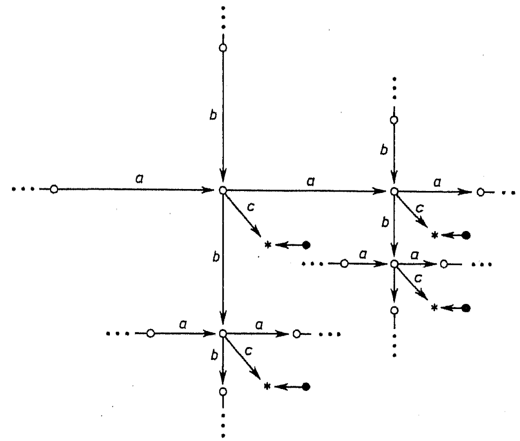

Fig. 18

multipeak tree we conclude from Proposition 4.2 that the coordinate support of every indecomposable in $\operatorname{mod}_{\mathrm{sp}} F(\tilde{\mathcal{Q}}, \tilde{\Omega})$ is a subquiver of $\Delta = \boldsymbol{T}(2, 2, 2)$ (see Fig. 19) and $F(\tilde{\mathcal{Q}}, \tilde{\Omega})$ is locally sp-representation-tame [6]. Hence by [22; Theorem 1.10], [24; Theorem 0.2] and the arguments applied in [6, 7] the category $\operatorname{mod}_{\mathrm{sp}}(R)$ is of tame type.

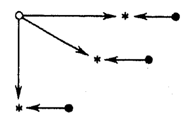

Fig. 19

Since $F\Delta$ is a hereditary algebra of Euclidean type $\tilde{E}_6$ it is tame and $\operatorname{mod}(F\Delta)$ as well as $\Gamma(F\Delta)$ are well known. Since every X in $\operatorname{ind}(F\Delta) - \operatorname{ind}_{\mathrm{sp}}(F\Delta)$ is simple injective the category $\operatorname{mod}_{\mathrm{sp}}(F\Delta)$ can be completely described and the restriction of the push-down functor to $\operatorname{mod}_{\mathrm{sp}}(F\Delta)$ yields a description of $\operatorname{mod}_{\mathrm{sp}}(R)$. By the arguments used in [26; Section 5] one can show that $\operatorname{mod}_{\mathrm{sp}}(R)$ is representation equivalent to a cofinite subcategory of $\operatorname{mod}_{\mathrm{sp}}(F\Delta)$.

Let us finish by an example which illustrates the generalized splitting theorems given in Proposition 3.12 and Theorem 3.15.

Example 4.3. For $t > 0$ consider the two-peak algebra $R_t = F(\mathcal{Q}, \Omega)$, where F is a division ring, $\mathcal{Q}$ is the quiver

$$\begin{array}{c}
\circ\to\circ\to\ldots\to\circ\to\circ\to p\\
\uparrow{\scriptstyle a_{t+1}}\\
1 \underset{b_2}{\overset{a_2}{\rightrightarrows}} 2 \underset{b_3}{\overset{a_3}{\rightrightarrows}} 3 \rightrightarrows \ldots \rightrightarrows t-1 \underset{b_t}{\overset{a_t}{\rightrightarrows}} t\\
\downarrow{\scriptstyle b_{t+1}}\\
\circ\to\circ\to\ldots\to\circ\to\circ\to q
\end{array}$$

and $\Omega = (a_2 b_3, b_2 a_3, a_3 b_4, b_3 a_4, \ldots, a_t b_{t+1}, b_t a_{t+1})$. Let $\mathcal{Q}'$ and $\mathcal{Q}''$ be the full subquivers consisting of points $1, \ldots, t-1$ and t, p, q respectively, and let $\mathcal{C}^p$ and $\mathcal{C}^q$ be the top chain and the bottom chain in $\mathcal{Q}$, respectively. Then $\mathcal{Q} = \mathcal{Q}' \cup \mathcal{C}^p \cup \mathcal{C}^q \cup \mathcal{Q}''$ satisfies the assumptions in Proposition 3.12 and in Theorem 3.15 with $u = 2$, $p_1 = p$, $p_2 = q$, there are F-algebra isomorphisms $\hat{S} \cong \hat{\eta} R_t \hat{\eta} \cong R_{t-1}$, $T \cong \xi R_t \xi \cong R_1$ and it follows by induction that $\mathrm{mod}_{sp}(R_t)$ is of finite type for all $t \geqslant 1$.

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