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W R O C Ł A W S K A D R U K A R N I A N A U K O W A

Comparing $\prod_2^0$ sets of the Baire space by means of general recursive operators

by

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Abstract. By applying a notion of reducibility suggested in [DPH] to the domains of a recursion category introduced in [MS], we get many-one reducibility between $\prod_2^0$ sets of the arithmetical hierarchy of sets of functions by means of general recursive operators. We give a characterization of the complete domains in this reducibility. We also introduce an upper semilattice $\mathfrak{B}$ to which this reducibility gives rise in a standard way. Several facts about $\mathfrak{B}$ are proved: we characterize the finite ideals of $\mathfrak{B}$; the first order theory of $\mathfrak{B}$ is shown to be undecidable.

1. Introduction. Our basic references for recursion theory and recursion-theoretic hierarchies are [RO] and [HI], to which the reader is referred for any unexplained notation and terminology, used in the paper. Although a natural way to read this paper is to frame it in the context of recursion categories, no substantial knowledge of the theory of recursion categories (developed in [DPH]) is needed throughout the paper: despite same occasional use of terminology from [DPH], the exposition is, in this regard, completely self-contained.

Let $\mathscr{P}$ be the set of partial functions from ω into ω, where ω denotes the set of natural numbers. For every $\phi \in \mathscr{P}$, let $S_\phi = \{\psi \in \mathscr{P}: \phi \subseteq \psi\}$; the family $\{S_\phi: \text{domain}(\phi)$ is finite$\}$ is easily seen to be a basis for a topology on $\mathscr{P}$; the *Baire space* is the subspace ω^ω of $\mathscr{P}$, with the relative topology. A partial mapping $\Psi: \omega^\omega \to \omega^\omega$ is a *partial continuous operator* (see [MS]) if there exists a continuous mapping $F: \mathscr{P} \to \mathscr{P}$ such that $\text{domain}(\Psi) = \omega^\omega \cap F^{-1}(\omega^\omega)$, and, for every $f \in \text{domain}(\Psi)$, $\Psi(f) = F(f)$; we say in this case that Ψ *is defined through* F. It turns out ([MS]) that a subset of the Baire space is G_δ in the Borel hierarchy of the Baire space if and only if it is the domain of some partial continuous operator. Amongst the continuous mappings from $\mathscr{P}$ into $\mathscr{P}$ stand out the recursive operators: let $\lambda x, y \cdot \langle x, y\rangle$ be a fixed recursive encoding of ω^2 onto ω and let D_u denote the finite set with canonical index u; a mapping $\Omega: \mathscr{P} \to \mathscr{P}$ is a *recursive operator* if there exists a recursively enumerable set W which *determines* Ω, i.e., for every $\phi \in \mathscr{P}$, $\Omega(\phi) = \{(x, y): (\exists u)[\langle\langle x, y\rangle, u\rangle \in W \,\&\, D_u \subseteq \text{graph}(\phi)]\}$, where $\text{graph}(\phi) = \{\langle x, y\rangle: (x, y) \in \phi\}$. If Ω is a recursive operator and the recursively enumerable set W determines Ω, then every recursive approximation of finite sets $\{W^t: t \in \omega\}$ to W gives an *approximation* $\{\Omega^t: t \in \omega\}$ *to* Ω, where, for every t, W^t determines Ω^t. Following [MS], partial continuous operators defined through recursive operators will be called *partial recursive B-operators*. It is shown in [MS] that the domains of the partial recursive B-operators are exactly the $\prod_2^0$ sets of functions of the arithmetical hierarchy. A partial recursive B-operator having domain ω^ω is called a *total recursive B-operator*: we recall ([RO]) that a *general recursive operator* is a recursive operator F such that $F(\omega^\omega) \subseteq \omega^\omega$; it follows that Ψ is a total recursive B-operator if and only if Ψ is the restriction to the Baire space of some general recursive operator. Since the set of partial continuous operators and the set of partial recursive B-operators are both closed under composition, we are led to the following definition (see [MS]):

DEFINITION 1.1. Let **Bo** be the monoid given by the partial continuous operators with the operation of composition, and let **B** be the submonoid of **Bo** consisting of the partial recursive B-operators.

It turns out that **B** and **Bo**, regarded as categories of partial morphisms, are in fact recursion categories and have been investigated in [MS] in the context of recursion categories. If we identify the *domains* (in the technical sense of [DPH, Definition 3.1]) of these categories with their set-theoretic domains, then, as we have already remarked, the domains of **Bo** are exactly the G_δ sets of the Baire space and the domains of **B** are exactly the $\prod_2^0$ sets of the Baire space. Once we are given a recursion category, it seems worthwhile to investigate the notion of reducibility between domains, which can be worked out from [DPH, Definition 3.6]. In the classical recursion category of partial recursive functions, this notion of reducibility is exemplified by many-one reducibility between recursively enumerable sets. The interpretation in **Bo** of this reducibility corresponds to the restriction to the G_δ sets of the Baire space of the so-called *Wadge reducibility* $\leqslant_w$ on subsets of the Baire space (see [WA], [VW], [LO]), defined by $\mathscr{A} \leqslant_w \mathscr{B}$ if there exists a continuous mapping F from the Baire space into itself such that $\mathscr{A} = F^{-1}(\mathscr{B})$. The ordering of the *Wadge degrees* (i.e. equivalence classes of the equivalence relation $\mathscr{A} \equiv_w \mathscr{B}$ if $\mathscr{A} \leqslant_w \mathscr{B}$ and $\mathscr{B} \leqslant_w \mathscr{A}$) is studied for instance in [VW], [LO]: from Theorem 3.1 of [VW], we get a complete picture of the partial order of the equivalence classes of the domains of **Bo**.

Via the aforementioned indentification of the domains of **B** with the $\prod_2^0$ sets of the Baire space, and since the total morphisms of **B** are the total recursive B-operators, in **B** the reducibility of [DPH, Definition 3.6] becomes the relation $\leqslant$ given by the following .

DEFINITION 1.2. For all domains $\mathscr{A}$, $\mathscr{B}$ of **B** let $\mathscr{A} \leqslant \mathscr{B}$ if there exists a total recursive B-operator F such that $\mathscr{A} = F^{-1}(\mathscr{B})$.

According to [DPH, Definition 3.8], $\mathscr{B}$ is a *complete domain* of **B** if $\mathscr{B}$ is a domain of **B**, and, for every domain $\mathscr{A}$ of **B**, $\mathscr{A} \leqslant \mathscr{B}$. Let $\equiv$ be the equivalence relation generated by $\leqslant$: dividing $\leqslant$ by $\equiv$ we get a partial ordering (for simplicity still denoted by $\leqslant$) on the set of equivalence classes; the partial order thus obtained will be denoted by $\mathfrak{P}_{\mathbf{B}}$. If $\mathscr{A}$ is a domain of **B** (i.e. $\mathscr{A} \in \prod_2^0$), let $[\mathscr{A}]$ denote the equivalence class of $\mathscr{A}$ under $\equiv$. Elements of $\mathfrak{P}_{\mathbf{B}}$ will be called *degrees* and denoted by the symbols $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}, \ldots$

The purpose of this note is to make some remarks on the relation $\leqslant$ and the partial order $\mathfrak{P}_{\mathbf{B}}$: for instance we characterize the complete domains of **B** (a characterization of a different type, showing that in **B** creative domains (see [DPH, Definition 8.1]) and complete domains coincide, is given in [MS]). We also study in some detail a substructure of $\mathfrak{P}_{\mathbf{B}}$, called $\mathfrak{B}$. Several facts about $\mathfrak{B}$ are established: we show that $\mathfrak{B}$ is a distributive upper semilattice; we characterize the finite ideals of $\mathfrak{B}$, so that we derive as a corollary that the first order theory of $\mathfrak{B}$ is undecidable and we show that $\mathfrak{B}$ is not a lattice. Of course, the reducibility studied in this paper can be extended to the class of all subsets of the Baire space or, if one just wishes, to hierarchy-theoretic levels different from $\prod_2^0$. There are at least two reasons why, for the moment, we confine ourselves to $\prod_2^0$ sets. One reason is that some of the most interesting results about $\mathfrak{P}_{\mathbf{B}}$ (for instance, Theorem 2.14 and Corollary 2.16) immediately extend, by the same proofs, to more general situations, for example when we consider all subsets of the Baire space. A more important reason is that the $\prod_2^0$ sets of functions constitute the domains of a recursion category and, as such, allow for a use of the machinery developed in [DPH] (for instance, the recusion theorem — see [DPH], § 4.5 — has been used in an essential way in [MS] to show that in **B** (and in **Bo**) every creative domain is complete). Full use of this machinery may constitute the basis for subsequent work, for instance structure inside degrees, 1-degrees etc.

FACT 1.3. (1) *All complete domains of* **B** *are equivalent under* $\equiv$;

(2) *All recursive subsets* $\mathscr{C} \subseteq \omega^\omega$ *such that* $\mathscr{C} \neq \varnothing$ *and* $\mathscr{C} \neq \omega^\omega$ *are equivalent under* $\equiv$.

Proof. Immediate. ■

Fact 1.3 suggests that we introduce the following notation: let $\mathbf{1} = \{\mathscr{C} \subseteq \omega^\omega\colon \mathscr{C}$ is a complete domain of $\mathbf{B}\}$, and let $\mathbf{0} = \{\mathscr{C} \subseteq \omega^\omega\colon \mathscr{C}$ recursive and $\mathscr{C} \neq \varnothing$, $\mathscr{C} \neq \omega^\omega\}$.

For any partial order $\mathfrak{P} = \langle P, \leqslant\rangle$ and $a \in P$, let $\mathfrak{P}(\leqslant a) = \{b \in P\colon b \leqslant a\}$; in this fashion are also defined $\mathfrak{P}(< a)$, $\mathfrak{P}(\geqslant a)$, $\mathfrak{P}(> a)$. Also, for every $\mathscr{A} \subseteq \omega^\omega$, let $\mathscr{A}^c = \omega^\omega - \mathscr{A}$.

FACT 1.4. $\mathfrak{P}_{\mathbf{B}}(> [\varnothing]) - \mathfrak{P}_{\mathbf{B}}(\geqslant \mathbf{0}) \neq \varnothing$ *and* $\mathfrak{P}_{\mathbf{B}}(> [\omega^\omega]) - \mathfrak{P}_{\mathbf{B}}(\geqslant \mathbf{0}) \neq \varnothing$.

Proof. The following are almost immediate: for every $\mathscr{A} \in \prod_2^0$,

$[\mathscr{A}] \in \mathfrak{P}_{\mathbf{B}}(> [\varnothing])$ if and only if $\mathscr{A} \neq \varnothing$ and $\mathscr{A}^c$ contains at least a recursive function;

$[\mathscr{A}] \in \mathfrak{P}_{\mathbf{B}}(> [\omega^\omega])$ if and only if $\mathscr{A}^c \neq \varnothing$ and $\mathscr{A}$ contains at least a recursive function;

$[\mathscr{A}] \in \mathfrak{P}_{\mathbf{B}}(\geqslant \mathbf{0})$ if and only if both $\mathscr{A}$ and $\mathscr{A}^c$ contain a recursive function.

For instance, to show that $[\mathscr{A}] \in \mathfrak{P}_{\mathbf{B}}(> [\varnothing])$ if and only if $\mathscr{A} \neq \varnothing$ and $\mathscr{A}^c$ contains a recursive function, let $\mathscr{A} \in \prod_2^0$ be given such that $\varnothing < \mathscr{A}$; then clearly $\mathscr{A} \neq \varnothing$ and there exists a total recursive B-operator F such that $F(\omega^\omega) \subseteq \mathscr{A}^c$; but, clearly, $F(f)$ is recursive for every recursive f. On the other hand, if $f \in \mathscr{A}^c$ and f is recursive, then the constant mapping $F(g) = f$, for every $g \in \omega^\omega$, is a total recursive B-operator such that $\varnothing = F^{-1}(\mathscr{A})$.

Thus, to show that $\mathfrak{P}_{\mathbf{B}}(> [\varnothing]) - \mathfrak{P}_{\mathbf{B}}(\geqslant \mathbf{0}) \neq \varnothing$ it suffices to find a non-empty $\mathscr{A} \in \prod_2^0$ such that $\mathscr{A}$ does not contain any recursive function; Corollary III.4.5 of [HI] shows that in fact there exists $\mathscr{A} \in \prod_1^0$ of the form $\mathscr{A} = \{f\}$ such that f is not recursive.

Example 2.2 (1) below provides an instance of a set $\mathscr{A} \in \prod_2^0$ such that $[\mathscr{A}] \in \mathfrak{P}_{\mathbf{B}}(> [\omega^\omega]) - \mathfrak{P}_{\mathbf{B}}(\geqslant \mathbf{0}) \neq \varnothing$. ■

From the point of view of classical recursion theory and hierarchy theory, each of

$\mathfrak{P}_{\mathbf{B}}(>[\emptyset])$, $\mathfrak{P}_{\mathbf{B}}(>[\omega^\omega])$ and $\mathfrak{P}_{\mathbf{B}}(\geqslant \mathbf{0})$ is interesting, since each of them is related to basis questions: note for instance that by [HI], Lemma III.4.12, every equivalence class of a dense $\prod_2^0$ set, different from ω^ω, lies in $\mathfrak{P}_{\mathbf{B}}(>[\omega^\omega])$.

It seems appropriate to give special attention to $\mathfrak{P}_{\mathbf{B}}(\geqslant \mathbf{0})$: indeed, $\mathbf{0}$ becomes the least element of the structure and this corresponds, in a sense, to our intuition of the recursive subsets of ω^ω, as being the "least difficult" domains of **B**.

DEFINITION 1.5. Let $\mathfrak{B} = \mathfrak{P}_{\mathbf{B}}(\geqslant \mathbf{0})$.

Notice (see proof of Fact 1.4) that a degree A belongs to $\mathfrak{B}$ if and only if A contains an element $\mathscr{A}$ such that both $\mathscr{A}$ and $\mathscr{A}^c$ contain some recursive function. The elements of $\mathfrak{B}$ will be called *degrees of* $\mathfrak{B}$.

2. The structure of $\mathfrak{B}$. We begin this section with some remarks on the complete domains of **B**.

FACT 2.1. *If $\mathscr{A}$ is a complete domain of* **B** *then $\mathscr{A}$ is strictly $\prod_2^0$ (i.e. $\mathscr{A} \in \prod_2^0 - \sum_2^0$).*

Proof. Immediate, since if F is a total recursive B-operator and $\mathscr{A} \in \sum_2^0$ then $F^{-1}(\mathscr{A}) \in \sum_2^0$. ∎

Contrary to what happens for **Bo** (where every domain $\mathscr{A}$ which is strictly G_δ, i.e. $\mathscr{A} \in G_\delta - F_\sigma$, is complete: [WA]), it is not true that every strictly $\prod_2^0$ domain is complete, as is shown by the following example.

Let ϕ_x denote the partial recursive function with index x, in some standard enumeration of the partial recursive functions (see e.g. [RO]).

EXAMPLE 2.2. (1) ([HI], p. 111) Let $\mathscr{A}_{n,a} = \{f\colon (\forall m < n+a+1)[\phi_a(m)\downarrow = f(m)]\}$ and let $\mathscr{A} = \bigcap_n \bigcup_a \mathscr{A}_{n,a}$. Clearly $\mathscr{A} \in \prod_2^0$, i.e. $\mathscr{A}$ is a domain of **B**. It is easy to see that $\mathscr{A}$ and $\mathscr{A}^c$ are dense, thus, by [KU, § 12.V.(i)], $\mathscr{A} \in G_\delta - F_\sigma$. Therefore $\mathscr{A}$, being strictly G_δ, is a fortiori strictly $\prod_2^0$. On the other hand, $\mathscr{A}$ cannot be a complete domain of **B** since $\mathscr{A}^c$ does not contain any recursive function and, thus, for no $\mathscr{B}$ such that $\mathbf{0} \leqslant [\mathscr{B}]$ can we have $\mathscr{B} \leqslant \mathscr{A}$.

(2) If $A \subseteq \omega$ and $A \in \prod_2^0 - \sum_2^0$ in the arithmetical hierarchy of sets of integers, then $\mathscr{A}_A \in \prod_2^0 - \sum_2^0$ (see [RO, Theorem 15.XXIV]) in the arithmetical hierarchy of sets of functions, where $\mathscr{A}_A = \{f\colon f(0) \in A\}$. Nevertheless $\mathscr{A}_A$ cannot be complete in **B**, since $\mathscr{A}_A$ is clopen in the Baire topology: thus for no $\mathscr{A}$ which is not clopen (for instance, the set $\mathscr{A}$ in part (1) of this example), can we have $\mathscr{A} \leqslant \mathscr{A}_A$.

Now, for every subset $\mathscr{A}$ of the Baire space, let $\bar{\mathscr{A}}$ denote the closure of $\mathscr{A}$. The above mentioned result quoted in [KU, § 12.V.(i)] stating that for every $\mathscr{A} \subseteq \omega^\omega$, $\mathscr{A} \notin G_\delta \cap F_\sigma$ if and only if there exists a closed set $\mathscr{F} \neq \emptyset$ such that

$$(*) \qquad \mathscr{F} = \overline{\mathscr{F} \cap \mathscr{A}} = \overline{\mathscr{F} \cap \mathscr{A}^c}$$

gives a topological characterization of the complete domains of **Bo** (furnishing a topological characterization of the property of being G_δ but not F_σ).

Example 2.2 (1) shows that there exist domains $\mathscr{A} \in \prod_2^0$ satisfying $(*)$ but not complete in **B**.

We are able, however, to show that validity of an effective version of $(*)$ is equivalent to completeness in **B**.

We need two preliminary lemmas:

LEMMA 2.3. *$\mathscr{D} = \{f \in \omega^\omega\colon (\forall x)(\exists y > x)[f(y) \neq 0]\}$ is a complete domain of* **B**.

Proof. See for instance [MS]. ∎

Let 2^ω be the subspace of the Baire space consisting of all 0-1 valued functions, and let Fis be the set of finite initial segments, where a *finite initial segment* is a partial function from ω into ω whose domain is a finite initial segment of ω; if $\bar{f} \in \mathrm{Fis}$, then $\mathrm{lh}(\bar{f})$ is defined to be the least number not in the domain of $\bar{f}$. We suppose that we have fixed also a one-one recursive encoding of Fis onto ω. Given any function g and $x \in \omega$, let $g_{\upharpoonright x}$ denote the restriction of g to $\{y \in \omega\colon y < x\}$. We have:

LEMMA 2.4. *Suppose that Ψ is a partial recursive B-operator such that $2^\omega \subseteq \mathrm{domain}(\Psi)$. Then $\{f\colon (\exists g \in 2^\omega)[\Psi(g) = f]\}$ is a $\prod_1^0$ set in the arithmetical hierarchy.*

Proof. Let Ψ be a partial recursive B-operator such that $2^\omega \subseteq \mathrm{domain}(\Psi)$ and let Ω be a recursive operator such that Ψ is defined through Ω. Let $\mathscr{R} \subseteq \omega \times \omega^\omega$ be the relation defined by

$$\mathscr{R}(x, f) \Leftrightarrow (\exists \bar{g} \in \mathrm{Fis})[\bar{g} \text{ is 0-1 valued } \& \,\mathrm{lh}(\bar{g}) = x \,\&\, \Omega(\bar{g}) \subseteq f].$$

Clearly $\mathscr{R} \in \prod_1^0$.

Let now $\mathscr{F} = \{f\colon (\exists g \in 2^\omega)[\Psi(g) = f]\}$; we want to show that, for every f,

$$f \in \mathscr{F} \Leftrightarrow (\forall x)\mathscr{R}(x, f).$$

Clearly, if $f \in \mathscr{F}$ then $(\forall x)\mathscr{R}(x, f)$. Suppose that f is a function such that $(\forall x)\mathscr{R}(x, f)$ and, for any given $x \in \omega$, let $T_x = \{g \in 2^\omega\colon \Omega(g_{\upharpoonright x}) \subseteq f\}$. Since $(\forall x)\mathscr{R}(x, f)$, each T_x is non-empty, and, since 2^ω is compact in the Baire topology and $\Omega(h)$ is total for every $h \in 2^\omega$, it follows that $\bigcap_x T_x \neq \emptyset$. Let $g \in \bigcap_x T_x$; then $\Psi(g) = \Omega(g) = f$, i.e. $f \in \mathscr{F}$ as desired. ∎

THEOREM 2.5. *A domain $\mathscr{A}$ of* **B** *is complete if and only if there exist a non-empty $\prod_1^0$ set $\mathscr{F}$ and partial recursive functions ϱ and χ such that, letting $\mathscr{C} = \mathscr{F} \cap \mathscr{A}$ and $\mathscr{C}' = \mathscr{F} \cap \mathscr{A}^c$, the following holds:*

$(\forall \bar{f} \in \mathrm{Fis})[S_{\bar{f}} \cap \mathscr{F} \neq \emptyset \Rightarrow \varrho(\bar{f}), \chi(\bar{f})$ are defined & $\phi_{\varrho(\bar{f})}$ is total & $\phi_{\varrho(\bar{f})} \in S_{\bar{f}} \cap \mathscr{C}'$ & $\phi_{\chi(\bar{f})}$ is total & $\phi_{\chi(\bar{f})} \in S_{\bar{f}} \cap \mathscr{C}]$

(via the encoding of Fis, *ϱ and χ are viewed as partial functions from* Fis *into ω). Notice that the existence of such ϱ and χ implies of course that $\mathscr{F} = \overline{\mathscr{F} \cap \mathscr{A}} = \overline{\mathscr{F} \cap \mathscr{A}^c}$.*

Proof. ($\Leftarrow$) Let $\mathscr{F}$, $\mathscr{C}$, $\mathscr{C}'$ be as in the statement of the theorem; then $\mathscr{C} \in \prod_2^0$, $\mathscr{C}' \in \sum_2^0$. Let $\mathscr{C} = \{f\colon (\forall x)(\exists y)\mathscr{R}(x, y, f)\}$, where $\mathscr{R} \subseteq \omega^2 \times \omega^\omega$ is recursive; let $\mathscr{C}_{x,y} = \{f\colon \mathscr{R}(x, y, f)\}$, so that $\mathscr{C} = \bigcap_x \bigcup_y \mathscr{C}_{x,y}$.

Let $\mathrm{Seq} = \bigcup(\omega^n\colon n \in \omega)$, i.e. the set of all finite sequences of natural numbers. If $s \in \omega^n$, then n is the *length* of s, denoted by $\mathrm{lh}(s)$; if $s = (x_0, \ldots, x_n)$ and $i \leqslant n$, then $(s)_i = x_i$; the empty sequence is denoted by $(\;)$; if $s, t \in \mathrm{Seq}$ then $s * t$ is the concatenation of s and t; for every $i \in \omega$, the symbol (i) denotes the image of the number i under the obvious embedding $\omega \to \mathrm{Seq}$. Assume also that $s \mapsto \mathrm{cd}(s)$ is a 1-1 recursive encoding of Seq onto ω. Seq is ordered by the lexicographical ordering. We define a family $\{h_s, k_s\colon s \in \mathrm{Seq}\}$ of functions from ω into Fis as follows.

Definition of $k_{(\ )}$: Let

$$k_{(\ )}(i) = (\phi_{\varrho(\emptyset)}) \upharpoonright i+1.$$

Since, clearly, $S^0_\emptyset \cap \mathscr{F} \neq \emptyset$, thus $\phi_{\varrho(\emptyset)}$ is a total function in $\mathscr{C}'$ and thus $\bigcup_i k_{(\ )}(i) \in \mathscr{C}'$.

Definition of h_s *starting from* k_s: assume that $\bigcup_i k_s(i) \in \mathscr{C}'$, thus $S_{k_s(i)} \cap \mathscr{C}' \neq \emptyset$ for every $i \in \omega$ (hence $\chi(k_s(i))$ is defined) and let $i \in \omega$ be given. Search for a finite initial segment $\tilde{f}$ such that

$$\tilde{f} \supseteq k_s(i) \;\&\; \tilde{f} \subseteq \phi_{\chi(k_s(i))} \;\&\; S_{\tilde{f}} \subseteq \bigcap \{\bigcup_y \mathscr{C}_{x,y} \colon x \leqslant \mathrm{cd}(s)\}.$$

Notice that, starting from $k_s(i)$ one can always find such an $\tilde{f} \supseteq k_s(i)$: we are using, of course, the basic property of $\mathscr{C}$, $\mathscr{C}'$ and the fact that $\mathscr{R}(x, y, f)$ is recursive. Let $h_s(i)$ be the first such $\tilde{f}$ to appear in our search.

Definition of $k_{s*(i)}$ *starting from* h_s: Let $i \in \omega$ be given and assume that $S_{h_s(i)} \cap \mathscr{C} \neq \emptyset$ (hence $\varrho(h_s(i))$ is defined). Again by the property of $\mathscr{C}$ and $\mathscr{C}'$ above, $\phi_{\varrho(h_s(i))}$ is total and $\phi_{\varrho(h_s(i))} \in S_{h_s(i)} \cap \mathscr{C}'$. For every $j \in \omega$ define

$$k_{s*(i)}(j) = (\phi_{\varrho(h_s(i))}) \upharpoonright \mathrm{lh}(h_s(i))+j+1.$$

Thus $\bigcup_j k_{s*(i)}(j) \in \mathscr{C}'$ and, a fortiori, $\bigcup_j k_{s*(i)}(j) \in \mathscr{A}^c$.

The family $\{h_s, k_s \colon s \in \mathrm{Seq}\}$ satisfies the following conditions:

(1) $$k_s(i) \subset k_s(i+1);$$

(2) $$k_s(i) \subseteq h_s(i);$$

(3) $$h_s(i) \subset k_{s*(i)}(0);$$

(4) $$\bigcup_i k_s(i) \in \mathscr{A}^c;$$

(5) $$S_{h_s(i)} \subseteq \bigcap \{\bigcup_y \mathscr{A}_{x,y} \colon x \leqslant \mathrm{cd}(s)\};$$

(6) h_s, k_s are recursive (via the encoding of Fis);

(7) there exist recursive functions u, v such that, for every $x \in \omega$, $k_s = \phi_x \Rightarrow h_s = \phi_{u(x)}$, and $h_s = \phi_x \Rightarrow k_{s*(i)} = \phi_{v(x,i)}$.

We are now ready to show that $\mathscr{A}$ is complete. Let $\mathscr{D}$ be as in Lemma 2.3; thus it suffices to show that there exists a total recursive B-operator F such that $\mathscr{D} = F^{-1}(\mathscr{A})$. We shall define F in the following way: for every $f \in \omega^\omega$, in order to compute $F(f)$ we define by induction a sequence of finite initial segments $\{\tilde{f}_n \colon n \in \omega\}$ such that, eventually, $F(f) = \bigcup_n \tilde{f}_n$. The definition of the sequence is as follows:

Step 0. If $f(0) \neq 0$ then $\tilde{f}_0 = h_{(\ )}(0)$; if $f(0) = 0$ then $\tilde{f}_0 = k_{(\ )}(0)$.

Step $n+1$. If $f(n+1) \neq 0$ then we distinguish two cases:

Case 1. If $f(n) \neq 0$ and, say, $\tilde{f}_n = h_s(i)$, then $\tilde{f}_{n+1} = h_{s*(i)}(0)$: notice that $h_s(i) \subset k_{s*(i)}(0) \subseteq h_{s*(i)}(0)$;

Case 2. If $f(n) = 0$ and, say, $\tilde{f}_n = k_s(i)$, then $\tilde{f}_{n+1} = h_s(i)$.

If $f(n+1) = 0$ then we still distinguish two cases:

Case 1. If $f(n) \neq 0$ and $\tilde{f}_n = h_s(i)$ then $\tilde{f}_{n+1} = k_{s*(i)}(0)$;

Case 2. If $f(n) = 0$ and $\tilde{f}_n = k_s(i)$ then $\tilde{f}_{n+1} = k_s(i+1)$.

It is clear that F is a total recursive B-operator.

To show that $\mathscr{D} = F^{-1}(\mathscr{A})$, we distinguish two cases:

1) If $f \in \mathscr{D}$ then we observe that the set $S = \{s \in \mathrm{Seq} \colon (\exists i)(\exists n)[\tilde{f}_n = h_s(i)]\}$ is infinite and in fact contains sequences of arbitrarily large length: to see this, notice that, for every n, if $f(n) \neq 0$ then there exist $s \in \mathrm{Seq}$ and a number i such that $\tilde{f}_n = h_s(i)$; on the other hand, if $f(n) \neq 0$ and $\tilde{f}_n = h_s(i)$ (thus $s \in S$) and r is the least number such that $r > n$ and $f(r) \neq 0$, then, for some $j \in \omega$, $\tilde{f}_r = {}_{s*(i)}(j)$, hence $s*(i) \in S$. Therefore, by (5), for infinitely many s, $F(f) \in \bigcap \{\bigcup_y \mathscr{A}_{x,y} \colon x \leqslant \mathrm{cd}(s)\}$, i.e. $F(f) \in \mathscr{A}$.

2) If $f \notin \mathscr{D}$ then, for some $s \in \mathrm{Seq}$, we have $F(f) = \bigcup_i k_s(i)$ and thus $F(f) \in \mathscr{A}^c$.

Proof of $\Rightarrow$. Throughout the remainder of this proof, $\underline{0}$ and $\underline{1}$ denote the functions $\lambda x \cdot 0$ and $\lambda x \cdot 1$, respectively; also, given $\tilde{f} \in \mathrm{Fis}$ and a function g, let $\tilde{f}*g$ be the function

$$\tilde{f}*g(x) = \begin{cases} \tilde{f}(x) & \text{if } x < \mathrm{lh}(\tilde{f}), \\ h(x - \mathrm{lh}(\tilde{f})) & \text{otherwise.} \end{cases}$$

Now, let $\mathscr{D}$ denote again the domain of Lemma 2.3 and let $\mathscr{A}$ be a complete domain of **B**; then there exists a total morphism F of **B** such that $\mathscr{D} = F^{-1}(\mathscr{A})$. Let also $\mathscr{F} = \{f \colon (\exists g \in 2^\omega)[F(g) = f]\}$; by Lemma 2.4, $\mathscr{F} \in \prod^0_1$. We want to show that, associated with $\mathscr{F}$, we can define suitable partial recursive functions ϱ and χ, so that the theorem is true.

To this end, let $\mathscr{C} = \mathscr{F} \cap \mathscr{A}$ and $\mathscr{C}' = \mathscr{F} \cap \mathscr{A}^c$. Since F is a recursive B-operator, let us fix a recursive operator Ω through which F is defined. Let $\{\Omega^t \colon t \in \omega\}$ be an approximation to Ω. Let now $\tilde{f} \in \mathrm{Fis}$ be given; in order to define $\varrho(\tilde{f})$ and $\chi(\tilde{f})$ search for $t \in \omega$ and a 0-1 valued $\tilde{g} \in \mathrm{Fis}$ such that $\tilde{f} \subseteq \Omega^t(\tilde{g})$; if no such t, $\tilde{g}$ exist then $\varrho(\tilde{f})$ and $\chi(\tilde{f})$ are both undefined; otherwise, let t, $\tilde{g}$ be the first to appear in our search and define $\varrho(\tilde{f})$ to be an index of $\Omega(\tilde{g}*\underline{0})$ and $\chi(\tilde{f})$ to be an index of $\Omega(\tilde{g}*\underline{1})$: clearly $\tilde{f} \subseteq \Omega(\tilde{g}*\underline{0})$, $\tilde{f} \subseteq \Omega(\tilde{g}*\underline{1})$; moreover, $\Omega(\tilde{g}*\underline{0}) = F(\tilde{g}*\underline{0}) \in \mathscr{C}'$ since $\tilde{g}*\underline{0} \in \mathscr{D}^c \cap 2^\omega$, and $\Omega(\tilde{g}*\underline{1}) = F(\tilde{g}*\underline{1}) \in \mathscr{C}$, since $\tilde{g}*\underline{1} \in \mathscr{D} \cap 2^\omega$. ■

Corollary 2.6. *For every complete domain* $\mathscr{A}$ *of* **B**, *there exists a complete domain* $\mathscr{B}$ *of* **B** *such that* $\mathscr{A} \subseteq \mathscr{B}$ *and* $\mathscr{B}^c$ *is countable.*

Proof. Given a complete domain $\mathscr{A}$, let F be a recursive B-operator such that $\mathscr{D} = F^{-1}(\mathscr{A})$, where $\mathscr{D}$ is again the domain of Lemma 2.3. Let $\mathscr{B} = (\mathscr{C}')^c$ where $\mathscr{C}'$ is as in the proof of the second part (i.e. proof of $\Rightarrow$) of the proof of Theorem 2.5. Now, $\mathscr{A} \subseteq \mathscr{B}$; moreover, $\mathscr{B} \in \prod^0_2$, since $\mathscr{C}' \in \sum^0_2$; also, $\mathscr{B}^c$ is countable (since $\mathscr{C}' \subseteq F(\mathscr{D}^c)$, and $F(\mathscr{D}^c)$ is countable, $\mathscr{D}^c$ being so). To show that $\mathscr{B}$ is complete, notice that $\mathscr{F} \cap \mathscr{B} = \mathscr{C}$ and $\mathscr{F} \cap \mathscr{B}^c = \mathscr{C}'$; thus $\mathscr{F}$, ϱ, χ satisfy the property mentioned in the statement of Theorem 2.5, and so $\mathscr{B}$ is complete. ■

The proof of Theorem 2.5 actually yields the following corollary which furnishes a useful sufficient condition for testing whether a $\prod^0_2$ set is a complete domain of **B**.

COROLLARY 2.7. *Let* $\mathscr{A}\in\prod_2^0$. *If there exist* $\mathscr{C}\subseteq\mathscr{A}$ *and* $\mathscr{C}'\subseteq\mathscr{A}^c$ *such that*

(1) $$\mathscr{C},\mathscr{C}'\neq\emptyset;\qquad \mathscr{C}\in\prod_2^0;$$

(2) *there exist partial recursive functions* ϱ, χ *such that, for every* $\bar{f}\in\mathrm{Fis}$, $[S_{\bar{f}}\cap\mathscr{C}\neq\emptyset\Rightarrow\varrho(\bar{f})$ *is defined* & $\phi_{\varrho(\bar{f})}$ *is total* & $\phi_{\varrho(\bar{f})}\in S_{\bar{f}}\cap\mathscr{C}']$ *and* $[S_{\bar{f}}\cap\mathscr{C}'\neq\emptyset\Rightarrow\chi(\bar{f})$ *is defined* & $\phi_{\chi(\bar{f})}$ *is total* & $\phi_{\chi(\bar{f})}\in S_{\bar{f}}\cap\mathscr{C}]$;

then $\mathscr{A}$ *is a complete domain of* **B**.

Proof. See the proof of the $\Leftarrow$ part of Theorem 2.5. ∎

EXAMPLE 2.8 (of how Theorem 2.5 or Corollary 2.7 can be applied).

(1) If X is a recursive subset of ω, $X\neq\emptyset$ and $X\neq\omega$, then $\mathscr{A}=\{f\colon(\forall x)(\exists y>x)[f(y)\in X]\}$ is a complete domain of **B**: apply Corollary 2.7 by taking $\mathscr{C}=\mathscr{A}$ and $\mathscr{C}'=\{f\colon(\exists x)(\forall y>x)[f(y)=a]\}$, where $a\notin X$ is fixed.

Notice that if $\omega-X$ has more than one element, then $\mathscr{A}$ and $\mathscr{A}^c$ are uncountable.

(2) If X is an infinite recursively enumerable set and $X\neq\omega$, then $\mathscr{A}=\{f\colon \mathrm{range}(f)=X\}$ is a complete domain of **B**: indeed, $\mathscr{A}$ is $\prod_2^0$, as is easily seen; moreover, Corollary 2.7 can be applied to $\mathscr{A}$, taking $\mathscr{C}=\mathscr{A}$ and $\mathscr{C}'=\{f\colon(\exists x)[(\forall y\geqslant x)[f(y)=a]\,\&\,(\forall y\leqslant x)[f(y)\in X]]\}$, where $a\in X$ is fixed.

The following notation is employed in Lemma 2.9 below: for every $f\in\omega^\omega$ and $x\in\omega$, let $x*f$ denote the function such that $x*f(0)=x$, and $x*f(y)=f(y-1)$, for every $y>0$. For every $x\in\omega$ and $\mathscr{A}\subseteq\omega^\omega$, let also $x*\mathscr{A}=\{x*f\colon f\in\mathscr{A}\}$.

LEMMA 2.9. $\mathfrak{B}$ *is a bottomed and topped upper semilattice.*

Proof. Clearly $\mathfrak{B}$ is bottomed. As to the rest of the proof, notice that one can easily show that, in fact, $\mathfrak{P}_{\mathbf{B}}$ is a topped upper semilattice. Indeed, by the very definition of complete domain, it immediately follows that **1** is the greatest element of $\mathfrak{P}_{\mathbf{B}}$.

Moreover, if $\boldsymbol{A}=[\mathscr{A}]$ and $\boldsymbol{B}=[\mathscr{B}]$ are degrees then the least upper bound $\boldsymbol{A}\vee\boldsymbol{B}$ of $\boldsymbol{A}$ and $\boldsymbol{B}$ exists and

$$\boldsymbol{A}\vee\boldsymbol{B}=\big[\bigcup\{x*\mathscr{A}\colon x\text{ even}\}\cup\bigcup\{x*\mathscr{B}\colon x\text{ odd}\}\big].$$

Moreover, if $\boldsymbol{A},\boldsymbol{B}\in\mathfrak{B}$ then $\boldsymbol{A}\vee\boldsymbol{B}\in\mathfrak{B}$. The proof is complete. ∎

An upper semilattice $\mathfrak{P}=\langle P,\leqslant,\vee\rangle$ is *distributive* if

$$(\forall a,b,c\in P)\big[a\leqslant b\vee c\Rightarrow(\exists b_0)(\exists c_0)[b_0\leqslant b\,\&\,c_0\leqslant c\,\&\,b_0\vee c_0=a]\big].$$

THEOREM 2.10. $\mathfrak{B}$ *is distributive.*

Proof. Let $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}$ be degrees of $\mathfrak{B}$ such that $\boldsymbol{A}\leqslant\boldsymbol{B}\vee\boldsymbol{C}$ and let $\mathscr{A}\in\boldsymbol{A}$, $\mathscr{B}\in\boldsymbol{B}$, $\mathscr{C}\in\boldsymbol{C}$; thus, for some total recursive B-operator F,

$$(\forall f)\big[f\in\mathscr{A}\Leftrightarrow F(f)\in\bigcup\{x*\mathscr{B}\colon x\text{ even}\}\cup\bigcup\{x*\mathscr{C}\colon x\text{ odd}\}\big].$$

Let $\mathscr{B}_0=F^{-1}(\bigcup\{x*\mathscr{B}\colon x\text{ even}\}$ and $\mathscr{C}_0=F^{-1}(\bigcup\{x*\mathscr{C}\colon x\text{ odd}\})$. Clearly $\mathscr{B}_0$, $\mathscr{C}_0\in\prod_2^0$. Moreover, $\mathscr{B}_0\leqslant\mathscr{B}$ and $\mathscr{C}_0\leqslant\mathscr{C}$; let us show for instance that $\mathscr{B}_0\leqslant\mathscr{B}$. To this end, notice that both $\mathscr{B}$ and $\mathscr{B}^c$ contain a recursive function, since $[\mathscr{B}]$ is a degree in $\mathfrak{B}$; in particular, let $h\in\mathscr{B}^c$ be recursive, and let

$$G(f)=\begin{cases}\lambda x\cdot F(f)(x+1) & \text{if } F(f)(0)\text{ is even,}\\ h & \text{if } F(f)(0)\text{ is odd.}\end{cases}$$

Then G is a total recursive B-operator such that $\mathscr{B}_0=G^{-1}(\mathscr{B})$.

We claim that, letting $\boldsymbol{B}_0=[\mathscr{B}_0]$ and $\boldsymbol{C}_0=[\mathscr{C}_0]$, we have $\boldsymbol{A}=\boldsymbol{B}_0\vee\boldsymbol{C}_0$. Indeed, $\mathscr{A}\leqslant\bigcup\{x*\mathscr{B}_0\colon x\text{ even}\}\cup\bigcup\{x*\mathscr{C}_0\colon x\text{ odd}\}$ via H where $H(f)=(F(f)(0))*f$. Conversely, one has to show that $\mathscr{B}_0\leqslant\mathscr{A}$ and $\mathscr{C}_0\leqslant\mathscr{A}$; for instance, to show that $\mathscr{B}_0\leqslant\mathscr{A}$, let $g\in\mathscr{A}^c$, g recursive, and define $K\colon\omega^\omega\to\omega^\omega$ by

$$K(f)=\begin{cases}f & \text{if } F(f)(0)\text{ is even,}\\ h & \text{if } F(f)(0)\text{ is odd.}\end{cases}$$

Clearly K is a total recursive **B**-operator that does the required job. Since clearly $\boldsymbol{B}_0$, $\boldsymbol{C}_0\in\mathfrak{B}$, this concludes the proof. ∎

THEOREM 2.11. *For any degrees* $\boldsymbol{A}$ *and* $\boldsymbol{B}$, $\boldsymbol{A}\vee\boldsymbol{B}=\mathbf{1}$ *if and only if* $\boldsymbol{A}=\mathbf{1}$ *or* $\boldsymbol{B}=\mathbf{1}$, *i.e.* **1** *is join irreducible.*

Proof. Let $\boldsymbol{A}\vee\boldsymbol{B}=\mathbf{1}$ and let $\mathscr{A}\in\boldsymbol{A}$ and $\mathscr{B}\in\boldsymbol{B}$. Then, by Lemma 2.9, the domain $\mathscr{E}=\bigcup\{x*\mathscr{A}\colon x\text{ even}\}\cup\bigcup\{x*\mathscr{B}\colon x\text{ odd}\}$ is complete and therefore, by Theorem 2.5, there exist a non-empty set $\mathscr{F}\in\prod_1^0$ and partial recursive functions ϱ,χ such that, letting $\mathscr{C}=\mathscr{F}\cap\mathscr{E}$ and $\mathscr{C}'=\mathscr{F}\cap\mathscr{E}^c$, we have:

$(\forall\bar{f}\in\mathrm{Fis})[S_{\bar{f}}\cap\mathscr{F}\neq\emptyset\Rightarrow\varrho(\bar{f}),\ \chi(\bar{f})$ are defined & $\phi_{\varrho(\bar{f})}$ is total & $\phi_{\varrho(\bar{f})}\in S_{\bar{f}}\cap\mathscr{C}'$ & $\phi_{\chi(\bar{f})}$ is total & $\phi_{\chi(\bar{f})}\in S_{\bar{f}}\cap\mathscr{C}]$;

First suppose that, for some even $x\in\omega$, $\mathscr{F}\cap(x*\mathscr{A})\neq\emptyset$; we will show that, in this case, $\mathscr{A}$ is complete. Thus, let x be an even number such that $\mathscr{F}\cap(x*\mathscr{A})\neq\emptyset$: let $\mathscr{F}^+=\{f\colon x*f\in\mathscr{F}\}$; clearly $\mathscr{F}^+\in\prod_1^0$. Let $\gamma\colon\omega\to\omega$ be a recursive function such that, for every $z\in\omega$, $\gamma(z)$ is an index of $\lambda x\cdot\phi_z(x+1)$. Associated with $\mathscr{F}^+$ let us define two partial recursive functions ϱ^+, χ^+ by

$$\varrho^+(\bar{f})=\begin{cases}\gamma\big(\varrho((x)*\bar{f})\big) & \text{if } \varrho((x)*\bar{f})\downarrow,\\ \text{undefined} & \text{otherwise,}\end{cases}\qquad \chi^+(\bar{f})=\begin{cases}\gamma\big(\chi((x)*\bar{f})\big) & \text{if } \chi((x)*\bar{f})\downarrow,\\ \text{undefined} & \text{otherwise.}\end{cases}$$

Now, $\mathscr{F}^+$, ϱ^+, χ^+ satisfy the hypotheses of Theorem 2.5, showing that $\mathscr{A}$ is complete.

We have shown that if, for some even $x\in\omega$, $\mathscr{F}\cap(x*\mathscr{A})\neq\emptyset$, then $\mathscr{A}$ is complete; in a similar way we conclude that if $\mathscr{F}\cap x*\mathscr{B}\neq\emptyset$, for some odd $x\in\omega$, then $\mathscr{B}$ is complete. Since at least one of these two cases has to occur, the conclusion follows. ∎

For every $A\subseteq\omega$, let $\mathscr{A}_A=\{f\colon f(0)\in A\}$. In the following, the symbol $\leqslant_m$ denotes m-reducibility between subsets of ω.

LEMMA 2.12. *For every* $A,B\subseteq\omega$, $A\leqslant_m B$ *if and only if there exists a total recursive* B-*operator* F *such that* $\mathscr{A}_A=F^{-1}(\mathscr{A}_B)$.

Proof. ($\Rightarrow$) Let $A\leqslant_m B$ via a recursive function f and define $F\colon\omega^\omega\to\omega^\omega$ by $F(g)=f(g(0))*g$. Thus $\mathscr{A}_A=F^{-1}(\mathscr{A}_B)$.

($\Leftarrow$) Suppose that $\mathscr{A}_A = F^{-1}(\mathscr{A}_B)$ where F is a total recursive B-operator; let Ω be a recursive operator through which F is defined and let $\{\Omega^t: t\in\omega\}$ be an approximation to Ω. As in the proof of Theorem 2.5, let $\underline{0} = \lambda x\cdot 0$. Define a recursive function f as follows: for every $x\in\omega$, search for $\tilde{g}\in \mathrm{Fis}$ and $t\in\omega$ such that $\tilde{g}\subseteq x*\underline{0}$ and $\Omega^t(\tilde{g})(0)$ is defined; for the first $\tilde{g}$ and t to appear in our search let $f(x) = \Omega^t(\tilde{g})(0)$. Then it follows that $x\in A \Leftrightarrow x*\underline{0}\in\mathscr{A}_A \Leftrightarrow F(x*\underline{0})\in\mathscr{A}_B \Leftrightarrow f(x)\in B$. ■

Since $\mathscr{A}_A\in\prod_2^0$ for every $\prod_2^0$ set of numbers A, the partial order of the m-degrees of $\prod_2^0$ sets of numbers is embeddable in $\mathfrak{B}$. In particular, the m-degrees of recursively enumerable sets are embeddable in $\mathfrak{B}$.

LEMMA 2.13. *Let A be a degree of $\mathfrak{B}$ that contains an element of the form $\mathscr{A}_A$, for some recursively enumerable set A. Then, for every degree $\mathbf{B}$, if $\mathbf{B}\leqslant \mathbf{A}$ then $\mathbf{B}$ also contains an element $\mathscr{A}_B$, for some recursively enumerable set B.*

Proof. Let A be a given recursively enumerable set and let $\mathscr{B} = F^{-1}(\mathscr{A}_A)$ where F is a total recursive B-operator. Let Ω be a recursive operator through which F is defined and let

$$W = \{\tilde{f}\colon \Omega(\tilde{f})(0) \text{ is defined}\}.$$

Clearly W is an infinite recursively enumerable set of finite initial segments; let $\xi\colon W\to\omega$ be a 1-1 onto partial recursive function. Finally, let

$$B = \{x\in\omega\colon \Omega(\xi^{-1}(x))(0)\in A\}$$

(notice that, for every x, $\Omega(\xi^{-1}(x))(0)$ is defined, since $\xi^{-1}(x)$ is a finite initial segment that belongs to W). Clearly the set B is recursively enumerable and we claim that $\mathscr{A}_B\equiv\mathscr{B}$.

Given $\tilde{f}\in\mathrm{Fis}$ and a function g, let $\tilde{f}*g$ be the function defined as in the second part of the proof of Theorem 2.5. To show that $\mathscr{A}_B\leqslant\mathscr{B}$, consider the total recursive B-operator $G\colon \omega^\omega\to\omega^\omega$, $G(f) = (\xi^{-1}(f(0)))*f$; clearly, for every f, if $f\in\mathscr{A}_B$ then $G(f)\in\mathscr{B}$; conversely, if $f\notin\mathscr{A}_B$ then $f(0)\notin B$, thus $\Omega(\xi^{-1}(f(0)))(0)\notin A$; since $\xi^{-1}(f(0))\subseteq G(f)$, we have $\Omega(\xi^{-1}(f(0)))\subseteq\Omega(G(f)) = F(G(f))$, hence $F(G(f))\notin\mathscr{A}_A$ and, finally, $G(f)\notin\mathscr{B}$.

To show that $\mathscr{B}\leqslant\mathscr{A}_B$, define $H\colon\omega^\omega\to\omega^\omega$ by $H(f) = \xi(\tilde{f})*f$ where $\tilde{f}$ is the first element to appear in a fixed recursively enumerable enumeration of W such that $\tilde{f}\subseteq f$. ■

THEOREM 2.14. (1) *$\mathfrak{B}$ is not a lattice;*

(2) *in $\mathfrak{B}$ a countable family of degrees need not have a least upper bound.*

Proof. Let $\{\boldsymbol{a}_n\colon n\in\omega\}$ be an infinite ascending sequence of recursively enumerable m-degrees with an exact pair $\boldsymbol{a}$, $\boldsymbol{b}$ such that (see [ER], [LA])

$$(\forall \text{ recursively enumerable } m\text{-degree } \boldsymbol{c})\ [\boldsymbol{c}\leqslant_m\boldsymbol{a}\ \&\ \boldsymbol{c}\leqslant_m\boldsymbol{b}\Rightarrow(\exists n)[\boldsymbol{c}\leqslant_m\boldsymbol{a}_n]].$$

Let $A_n\in\boldsymbol{a}_n$, $A\in\boldsymbol{a}$, $B\in\boldsymbol{b}$; for every $n\in\omega$, let $\mathbf{A}_n = [\mathscr{A}_{A_n}]$ and let $\mathbf{A} = [\mathscr{A}_A]$, $\mathbf{B} = [\mathscr{A}_B]$. Then, by Lemmas 2.12 and 2.13, the sequence $\{\mathbf{A}_n\colon n\in\omega\}$ of degrees of $\mathfrak{B}$ does not have a least upper bound nor does the pair $\mathbf{A}$, $\mathbf{B}$ have a greatest lower bound. ■

THEOREM 2.15. *The finite ideals of $\mathfrak{B}$ are exactly the finite distributive lattices.*

Proof. Since $\mathfrak{B}$ is a distributive upper semilattice (see Theorem 2.10), every finite ideal $\mathfrak{B}$, being a lattice, is a distributive lattice. The other half of the theorem follows from Lemma 2.12, Lemma 2.13 and the fact that every finite distributive lattice can be embedded as an ideal in the upper semilattice of the recursively enumerable m-degrees (see [ER], [LA]). ■

COROLLARY 2.16. *The first order theory of $\mathfrak{B}$ (in the language having signature $\langle\leqslant\rangle$) is undecidable.*

Proof. It is known that if $\mathfrak{P}$ is an upper semilattice in which every finite distributive lattice is embeddable as an ideal, then the first order theory of $\mathfrak{P}$, in the language having signature $\langle\leqslant\rangle$, is undecidable (see for instance [LE], p. 137). Thus the claim follows from Theorem 2.15. ■

Inspection of the proofs of Theorem 2.14 and Corollary 2.16 shows that these results hold in fact for the substructures of $\mathfrak{P}$ containing the degrees of the subsets of the Baire space of the form $\mathscr{A}_A$ for some recursively enumerable set A. In particular, let $\mathfrak{D}$ be the restriction of $\mathfrak{P}$ to the degrees of decidable domains (in the sense of [DPH, Definition 3.9]; in $\mathbf{B}$ the decidable domains are the Δ_2^0 sets of the Baire space): one point of difference between $\mathfrak{P}$ and the classical theory of the recursively enumerable m-degrees, in which all decidable sets fall in a unique equivalence class (provided that they are not empty, with non-empty complement) is the following

COROLLARY 2.17. (1) *The finite ideals of $\mathfrak{D}$ are exactly the finite distributive lattices;*

(2) *the first order theory of $\mathfrak{D}$ (in the language with signature $\langle\leqslant\rangle$) is undecidable.*

Proof. Immediate since $\mathscr{A}_A\in\sum_1^0$, for every recursively enumerable set A. Thus every $\mathscr{A}_A$ is decidable. ■

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The Hahn–Banach theorem implies the existence of a non-Lebesgue measurable set

by

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Abstract. In this paper we show that the Axioms of Zermelo–Fraenkel set theory together with the Hahn–Banach theorem imply the existence of a non-Lebesgue measurable set. Our construction does not make any use of the Axiom of Choice.

§ 0. Introduction. Few methods are known to construct non-Lebesgue measurable sets of reals: most standard ones start from a well-ordering of $\boldsymbol{R}$, or from the existence of a nontrivial ultrafilter over ω, and thus need the axiom of choice AC or at least the Boolean Prime Ideal theorem (BPI, see [5]). In this paper we present a new way for proving the existence of nonmeasurable sets using a convenient operation of a discrete group on the Euclidean sphere. The only choice assumption used in this construction is the Hahn–Banach theorem, a weaker hypothesis than BPI (see [9]). Our construction proves that the Hahn–Banach theorem implies the existence of a nonmeasurable set of reals. This answers questions in [9], [10]. (Since we do not even use the countable axiom of choice, we cannot assume the countable additivity of Lebesgue measure, e.g. the real numbers could be a countable union of countable sets.)

In fact we prove (under the Hahn–Banach theorem) that there is no finitely additive, rotation invariant extension of Lebesgue measure to $\mathscr{P}(\boldsymbol{R}^3)$. Recall that the Hahn–Banach theorem implies the existence of a finitely additive, isometry invariant extension of Lebesgue measure to $\mathscr{P}(\boldsymbol{R}^2)$ (see [14]).

We use standard set-theoretical notation and terminology. For example, if X is any set, $\mathscr{P}(X)$ is the power set of X. If $A \subseteq X$ and $f\colon X \to Y$ is a map, then $f[A]$ is the image of A under f. ω is the set of all natural numbers.

We assume ZF throughout this paper; no choice assumption (even countable) is made.

§ 1. Definitions. First, let us give one of the many equivalent statements of the Hahn–Banach theorem. We use the version [11]:

THE HAHN–BANACH THEOREM. *Let E be a vector space over the reals, let S be a subspace of E, and f be a linear functional on S. Let p be a map $E \to \boldsymbol{R}$ such that whenever $x, y \in E$ and $\lambda \geqslant 0$, we have $p(\lambda x) = \lambda p(x)$ and $p(x+y) \leqslant p(x)+p(y)$ and for all*