On ω-essential mappings onto manifolds

by

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Abstract. Let M be a compact connected manifold. It is shown that if $f\colon X \to M$ is an essential mapping of a compactum X onto M and $\dim X < 2\dim M - 2$, then the product mapping $f \times \mathrm{id}_{I^k}\colon X \times I^k \to M \times I^k$ is essential for each $k = 1, 2, \ldots$ This result gives a partial answer to a problem of J. Krasinkiewicz.

1. Introduction. All spaces considered in this paper are assumed to be metric and all mappings are assumed to be continuous. By dimension we understand the covering dimension dim. Let M be a compact connected manifold (with or without boundary) and let $\mathring{M}$ and ∂M denote its interior and boundary, respectively. Consider a mapping $f\colon X \to M$ of a space X into M. A mapping $g\colon X \to M$ is said to be an *admissible deformation* of f provided there is a homotopy

$$F\colon (X, f^{-1}(\partial M)) \times I \to (M, \partial M)$$

from f to g. A mapping $f\colon X \to M$ is said to be *essential* provided every admissible deformation of f is surjective. Otherwise, f is said to be *inessential* (see [Ho] and comp. [GT], [Kr]). Let I^k be the k-dimensional cube. A mapping $f\colon X \to M$ is said to be *k-essential* ($k = 1, 2, \ldots$) provided the product mapping

$$f \times \mathrm{id}_{I^k}\colon X \times I^k \to M \times I^k$$

is essential, where $f \times \mathrm{id}_{I^k}$ is defined by $(f \times \mathrm{id}_{I^k})(x, y) = (f(x), y)$, for $x \in X$ and $y \in I^k$. Call a mapping *0-essential* if it is essential. A mapping which is k-essential for each $k \geqslant 0$ is called *ω-essential* (comp. [H2]).

In [H1] (see Th. 3.1 and Prop. 1.1) W. Holsztyński using cohomology theory established the following result, here stated only for compacta (see also [M1], Th. 5.2 and [M2], Th. 7):

If $f\colon X \to I^m$ is an essential mapping of a compactum X onto I^m where either $m \leqslant 2$ or $\dim X \leqslant m$, then f is ω-essential.

The aim of this note is to prove the following complementary result (see Section 4).

Main Theorem. *If $f\colon X \to M$ is an essential mapping of a compactum X onto M and $\dim X < 2\dim M - 2$, then f is ω-essential.*

This gives a partial answer to a question posed by J. Krasinkiewicz on a Geometric Topology Seminar in Warsaw (see Section 5). The proof of the Main Theorem is very geometric and uses only certain elementary PL-topology properties (see Section 2) and some expansion techniques (see Section 3). Notice that in [H2] W. Holsztyński gave an example of an essential mapping $f\colon I^4 \to I^3$ which is not 1-essential.

Other results concerning k-essential mappings onto surfaces, cubes and spheres will be presented in forthcoming papers of the author. In particular, using the above-mentioned theorem and certain elementary methods we will prove that an essential mapping $f\colon X \to M$ of a compactum X into M is ω-essential provided either

(a) $M = I^m$ and $m \leqslant 2$, or

(b) $M = S^m$ and either $m = 1$ or $\dim X < 2m-1$.

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2. An auxiliary lemma. Recall that a mapping $f\colon X \to Y$ is *proper* provided $f^{-1}(C)$ is compact for each compact subset $C \subset Y$. We begin with the following lemma (comp. Th. 36 of [Wh, p. 320]).

2.1. LEMMA. *Let $f\colon |K| \to |L|$ be a proper mapping, where K and L are locally finite simplicial complexes. Let L_0 be a finite subcomplex of L. If U is an open subset of $|L|$ such that $|L_0| \subset U$ and $\mathrm{cl}(U)$ is compact, then there exist a subdivision K' of K, a subdivision L' of L which contains L_0 and a mapping $g\colon |K'| \to |L'|$ homotopic to f $\mathrm{rel}\, f^{-1}(|L| \setminus U)$ such that the mapping*

$$g|g^{-1}(L_0)\colon g^{-1}(L_0) \to L_0$$

is simplicial.

Proof. Consider a subdivision L' of L $\mathrm{rel}\, L_0$. Let N_0 be a simplicial neighborhood of L_0 in L' and N_1 a simplicial neighborhood of N_0 in L'. We may assume that

$$(1) \qquad |N_1| \subset U$$

for appropriate L'.

Now consider an open cover $\mathcal{U}$ of $|L|$ consisting of $|L| \setminus |N_1|$ and all open stars $\mathrm{st}(v, L')$, where v runs over all vertices of N_1. From Theorem 35 in [Wh, p. 317] it follows that there exists a subdivision K' of K such that the closed star cover $\overline{\mathrm{st}}\, K'$ refines $f^{-1}(\mathcal{U})$, i.e.

(2) for each vertex w of K', $f(\overline{\mathrm{st}}(w, K')) \subset |L| \setminus |N_1|$ or $f(\overline{\mathrm{st}}(w, K')) \subset \mathrm{st}(v, L')$ for some vertex v of L'.

Put $T_0 = \{\sigma \in K'\colon f(\sigma) \subset |N_0|\}$ and $T_1 = \{\sigma \in K'\colon f(\sigma) \cap |N_1| = \emptyset\}$. Clearly, $|T_0|$ is a compact polyhedron and $|T_0| \cap |T_1| = \emptyset$. Without loss of generality we may assume that

$$(3) \qquad f^{-1}(|L| \setminus U) \subset |T_1|.$$

Since $f(|T_0|) \subset |N_0|$, by (2), it follows that there is a simplicial approximation $\varphi\colon T_0 \to N_1$ of $f|T_0\colon T_0 \to N_1$ and a homotopy $H\colon |T_0| \times I \to |N_1|$ such that if $x \in |T_0|$, $\{H(x, t)\colon t \in I\}$ is the straight line interval from $f(x)$ to $\varphi(x)$ in the carrier of $f(x)$. Observe that

$$(4) \qquad \varphi(\mathrm{bd}\,|T_0|) \subset N_1 \setminus \mathrm{int}\, N_0.$$

Indeed, if $w \in \mathrm{bd}\,|T_0|$ is a vertex of T_0 such that $\varphi(w) \in \mathrm{int}\, N_0$, then $\varphi(w) \in L_0$. Since

$$f(\overline{\mathrm{st}}(w, K')) \subset \mathrm{st}(\varphi(w), L')$$

it follows that $f(\overline{\mathrm{st}}(w, K')) \subset N_0$. Thus $w \in \mathrm{int}\,|T_0|$, a contradiction.

From the construction of H and (4), we infer that $H((\mathrm{bd}\, T_0) \times I) \subset N_1 \setminus \mathrm{int}\, N_0$. So, there is a compact subpolyhedron W of $|L|$ such that

$$(5) \qquad |L_0| \subset \mathrm{int}\, W \subset W \subset |N_0|, \qquad W \cap H((\mathrm{bd}\,|T_0|) \times I) = \emptyset.$$

Now define a homotopy

$$G\colon (T_1 \cup (\mathrm{bd}\,|T_0|)) \times I \to |L| \setminus \mathrm{int}\, W$$

putting $G(x, t) = f(x)$ if $x \in |T_1|$ and $G(x, t) = H(x, t)$ if $x \in \mathrm{bd}\, T_0$, for $t \in I$. Since $|L| \setminus \mathrm{int}\, W \in \mathrm{ANR}$, from the Borsuk Homotopy Extension Theorem, we infer that there is a homotopy

$$G'\colon (|K| \setminus \mathrm{int}\,|T_0|) \times I \to |L| \setminus \mathrm{int}\, W$$

which extends G and $G'(x, 0) = f(x)$ for $x \in |K| \setminus \mathrm{int}\, T_0$. Then it is obvious that

$$F\colon |K| \times I \to |L|$$

given by the formula

$$F(x, t) = \begin{cases} H(x, t) & \text{if } x \in |T_0|, \\ G'(x, t) & \text{if } x \in |K| \setminus \mathrm{int}\,|T_0|, \end{cases}$$

is the desired homotopy. Indeed, F is well defined; $F_0 = f$ by the construction of H and G; F is $\mathrm{rel}\, f^{-1}(|L| \setminus U)$ by (1), (3) and (4). Moreover, $g = F_1$ is simplicial over L_0 because $g^{-1}(L_0) \subset \mathrm{int}\, T_0$ by (5) and $g|T_0 = \varphi$ by the construction of H. ■

Now, we will use Lemma 2.1 to prove

2.2. LEMMA. *Let $f\colon X \to M$ be an essential mapping of a compact polyhedron X onto M. If*

$$(*) \qquad \dim X < 2 \dim M - 2,$$

then $f \times \mathrm{id}_I\colon X \times I \to M \times I$ is essential.

Proof. Set $n = \dim X$ and $m = \dim M$. Let D be an m-dimensional disk in M and pick a point $a \in \mathring{D}$. Suppose, on the contrary, that $f \times \mathrm{id}_I$ is inessential. By [Kr], Lemma I.1.3, there exists a mapping $g_0\colon X \times I \to M \times I$ such that $(a, 1/2) \notin g_0(X \times I)$ and $g_0(x, t) = (f(x), t)$ for each $(x, t) \in (f \times \mathrm{id}_I)^{-1}(\partial(M \times I))$. Set

$$B = M \times I \setminus \mathring{D} \times \mathring{I}.$$

Since $\partial(M\times I)\subset B$ and B is a retract of $M\times I\setminus\{(a, 1/2)\}$, we can assume that $g_0(X\times I)\subset B$. Consider the sets $W_j = g_0^{-1}(\mathring{D}\times(j))$, $j = 0, 1$. Clearly,

(1) $$W_j\cap(X\times(1-j)\cup f^{-1}(\partial M)\times I) = \emptyset.$$

Since $\mathring{D}\times(j)$ is an open subset of B, it follows that W_j is open in $X\times I$ and therefore $W_j = |K_j|$ for some locally finite simplicial complex K_j (in the PL-structure of $X\times I$). Clearly, the mapping $W_j\to\mathring{D}\times(j)$ determined by g_0, is proper. Let L_j be a triangulation of $\mathring{D}\times(j)$ and let σ_j be an m-simplex of L_j. Applying Lemma 2.1 we can construct a mapping $g\colon X\times I\to B$ satisfying the conditions

(2) $g(x, t) = g_0(x, t)$ if $(x, t)\notin W_0\cup W_1$, and $W_j = g^{-1}(\mathring{D}\times(j))$,

(3) $g^{-1}(\sigma_j) = |T_j|$ is a subpolyhedron of $X\times I$ and $g\colon T_j\to\sigma_j$ is simplicial.

Note that

(4) $$g(X\times(0))\subset M\times(0) \quad\text{and}\quad g(x, 0) = (f(x), 0) \quad \text{if } f(x)\notin\mathring{D}.$$

Pick $a_0\in\mathring{\sigma}_0$. Let $p_0\colon M\times(0)\to M$ be a homeomorphism such that $p_0(a_0) = a$ and $p_0(y, 0) = y$ if $y\notin\mathring{D}$. Since $f(x)\neq p_0g(x, 0)$ implies $f(x), (p_0g)(x, 0)\in\mathring{D}$, there exists a homotopy

$$G\colon X\times I\to M \ \operatorname{rel} f^{-1}(\partial M)$$

such that

(5) $$G_0 = f \quad\text{and}\quad G_1(x) = p_0g(x, 0) \quad \text{for each } x\in X.$$

It follows from (3) that $g^{-1}(a_0)$ is a polyhedron and $\dim g^{-1}(a_0)\leqslant(n+1)-m$. By (1) and (2) we have

$$g^{-1}(a_0)\cap(X\times(1)\cup f^{-1}(\partial M)\times I) = \emptyset.$$

Setting $A_0 = q_X(g^{-1}(a_0))$, where q_X is the projection of $X\times I$ on X, we infer that A_0 is a subpolyhedron of X such that

(6) $$\dim A_0\leqslant n+1-m,$$

(7) $$A_0\cap f^{-1}(\partial M) = \emptyset,$$

(8) $$g^{-1}(a_0)\subset A_0\times[0, 1).$$

Observe that $\sigma_1\cap g(A_0\times I) = g(g^{-1}(\sigma_1)\cap A_0\times I)$, hence by (3) and (6) we have

$$\dim(\sigma_1\cap g(A_0\times I))\leqslant n+2-m,$$

which is smaller than m because $n < 2m-2$ by (*). Therefore there exists $a_1\in\sigma_1\setminus g(A_0\times I)$. Put $A_1 = q_X(g^{-1}(a_1))$. One easily sees that

(9) $$A_0\cap A_1 = \emptyset,$$

(10) $$g^{-1}(a_1)\subset A_1\times(0, 1].$$

By (7) and (9) there exists a mapping $\alpha\colon X\to I$ such that

(11) $$\alpha(A_0)\subset(1) \quad\text{and}\quad \alpha(A_1\cup f^{-1}(\partial M))\subset(0).$$

By (8) and (10) we have

(12) $$\{(x, \alpha(x))\colon x\in X\}\cap(g^{-1}(a_0)\cup g^{-1}(a_1)) = \emptyset.$$

Let $p\colon B\to M$ be an extension of p_0 such that

(13) $$p^{-1}(a) = \{a_0, a_1\}.$$

To complete the proof it suffices to define a homotopy $H\colon X\times I\to M \operatorname{rel} f^{-1}(\partial M)$ connecting f with a map that is not surjective. Indeed, this will contradict the assumption about f.

Such a homotopy can be given by the formula

$$H(x, t) = \begin{cases} G(x, 2t) & \text{for } 0\leqslant t\leqslant 1/2,\\ pg(x, (2t-1)\alpha(x)) & \text{for } 1/2\leqslant t\leqslant 1.\end{cases}$$

Indeed, H is well defined by (5); $H_0 = f$ by (5); H is $\operatorname{rel} f^{-1}(\partial M)$ by the construction of G, p_0 and p, and by (4); H_1 is not surjective because $a\notin H_1(X)$ (see (12) and (13)). ■

3. Expansions of essential mappings. Throughout this section by X we denote a compactum and by M a compact connected manifold. Assume that $\mathbf{X} = (X_i, \pi_{ij})$ is an inverse sequence of compacta such that $X = \varprojlim \mathbf{X}$. Let $X^* = \varprojlim \mathbf{X}^*$ where $\mathbf{X}^* = (X_i^*, \tau_{ij})$ is defined as follows (comp. [MS1], p. 48). For each i, put

$$X_i^* = X_1\cup X_2\cup\ldots\cup X_i$$

and for $j > i$ put

$$\tau_{ij} = \tau_{i,i+1}\tau_{i+1,i+2}\ldots\tau_{j-1,j}\colon X_j^*\to X_i^*,$$

where

$$\tau_{i,i+1}(x) = \begin{cases} x & \text{for } x\in X_i^*,\\ \pi_{i,i+1}(x) & \text{for } x\in X_{i+1}.\end{cases}$$

Finally, let $\tau_{i,i} = \mathrm{id}_{X_i^*}$. It is clear that X^* is compact and contains a copy of X as closed subset and mutually disjoint copies of all X_i which approximate X.

Now let $f\colon X\to M$ be a mapping and let $\mathbf{f} = (f_i)$ be a sequence of mappings $f_i\colon X_i\to M$. We define $f^*\colon X^*\to M$ putting

$$f^*(x) = \begin{cases} f(x) & \text{for } x\in X,\\ f_i(x) & \text{for } x\in X_i,\end{cases}$$

The sequence $\mathbf{f}$ is said to be an *expansion* of f (denoted by $\mathbf{f}\colon \mathbf{X}\to M$) provided the following two conditions are satisfied:

(*) $$\pi_i(f^{-1}(\partial M))\subset f_i^{-1}(\partial M) \quad \text{for all } i,$$

(**) $$f^* \text{ is continuous.}$$

An expansion $\boldsymbol{f}$ of f is said to be *essential* provided for each i there is a $j \geqslant i$ such that f_j is essential. This is equivalent to the statement that almost all f_j are essential.

3.1. LEMMA. *A mapping f: $X \to M$ is essential iff its expansion $\boldsymbol{f}$: $\boldsymbol{X} \to M$ is essential.*

Proof. ($\Rightarrow$). From Lemma 6 of [MS2, p. 71], it follows that there exists an $\varepsilon > 0$ such that for any mapping

$$f'\colon (X, f^{-1}(\partial M)) \to (M, \partial M)$$

which is ε-near to f, both f and f' are homotopic as mappings of pairs. Since $\boldsymbol{f} = (f_i)$ is an expansion of f and π_i is an ε_i-push where $\varepsilon_i \to 0$ as $i \to \infty$, there is an i such that for each $j \geqslant i$,

$$\operatorname{dist}(f, f_j\pi_j) < \varepsilon.$$

Since $\pi_j(f^{-1}(\partial M)) \subset f_j^{-1}(\partial M)$, we have

$$f_j\pi_j\colon (X, f^{-1}(\partial M)) \to (M, \partial M).$$

Thus f and $f_j\pi_j$ are homotopic as mappings of pairs. Hence $f_j\pi_j$ is an admissible deformation of f. Since f is essential, we conclude that $f_j\pi_j$, and in consequence f_j, is essential.

($\Leftarrow$). Suppose $f = f^*|X$ is inessential. Then by Theorem I.1.10 of [Kr], there is a neighborhood $U \subset X^*$ of X such that $f^*|U$ is inessential. So, there is an index i such that for each $j \geqslant i$, $X_j \subset U$ and $f_j = f^*|X_j$ is inessential, a contradiction. ■

Let $X = \varprojlim \boldsymbol{X}$ and $Y = \varprojlim \boldsymbol{Y}$, where $\boldsymbol{X} = (X_i, \pi_{ij})$ and $\boldsymbol{Y} = (Y_i, \tau_{ij})$. Assume that $\boldsymbol{f} = (f_i)$: $\boldsymbol{X} \to M$ and $\boldsymbol{g} = (g_i)$: $\boldsymbol{Y} \to N$ are some expansions of the mappings f: $X \to M$ and g: $Y \to N$, respectively. It is easy to check that

$$\boldsymbol{X} \times Y = (X_i \times Y, \pi_{ij} \times \mathrm{id}_Y) \quad \text{and} \quad \boldsymbol{X} \times \boldsymbol{Y} = (X_i \times Y_i, \pi_{ij} \times \tau_{ij})$$

are inverse sequences and

$$X \times Y = \varprojlim (\boldsymbol{X} \times Y) = \varprojlim (\boldsymbol{X} \times \boldsymbol{Y}).$$

Moreover, an easy computation shows that

$$\boldsymbol{f} \times g = (f_i \times g) \quad \text{and} \quad \boldsymbol{f} \times \boldsymbol{g} = (f_i \times g_i)$$

are expansions of $f \times g$. Then by 3.1, we obtain

3.2. COROLLARY. *Assume that $\boldsymbol{f}$ and $\boldsymbol{g}$ are expansions of mappings f: $X \to M$ and g: $Y \to N$, respectively. Then*

(a) *$f \times g$ is essential iff $\boldsymbol{f} \times g$ is essential, and*

(b) *$f \times g$ is essential iff $\boldsymbol{f} \times \boldsymbol{g}$ is essential.* ■

3.3 LEMMA. *Let f: $X \to M$ be a mapping. Then there exists an expansion $\boldsymbol{f}$: $\boldsymbol{X} \to M$ of f, where $\boldsymbol{X}$ is an inverse sequence of compact polyhedra of dimension $\leqslant \dim X$ whose limit is X.*

Proof. By the Theorem of H. Freudenthal (see [E], Th. 1.32.2), X is the limit of an inverse sequence $\boldsymbol{X} = (X_i, \pi_{ij})$ consisting of compact polyhedra of dimension $\leqslant \dim X$. Let $A = f^{-1}(\partial M)$. Since A is a closed subset of X, $A = \varprojlim (A_i, \pi'_{ij})$, where $A_i = \pi_i(A)$ and $\pi'_{ij} = \pi_{ij}|A_j$. Since $\partial M \in \mathrm{ANR}$, there are an open neighborhood V of A in A^* and an extension f_0: $V \to \partial M$ of $f|A$: $A \to \partial M$. Then $A_i \subset V$ for almost all i. Without loss of generality we may assume that $A^* \subset V$. Using the same argument to the mapping $f \cup f_0$: $X \cup A^* \to M$ we get an open neighborhood U of $X \cup A^*$ in X^* and an extension g: $U \to M$ of $f \cup f_0$. As before we may assume that $X^* \subset U$. Now puting $f_j = g|X_j$ we obtain the required expansion of f, because $f_j(\pi_j(A)) = f_j(A_j) = f_0(A_j) \subset \partial M$. ■

4. The main theorem. In this section we prove our theorem mentioned in the introduction.

4.1. THEOREM. *If f: $X \to M$ is an essential mapping of a compactum X onto M and $\dim X < 2\dim M - 2$, then f is ω-essential.*

Proof. We must show that for each $k = 1, 2, \ldots$ the product mapping

$$f \times \mathrm{id}_{I^k}\colon X \times I^k \to M \times I^k$$

is essential. Note that it suffices to prove this for $k = 1$ (the general case follows by an inductive argument together with the Product Theorem 1.5.16 in [E], p. 46). Since f is essential, by Lemmas 3.3 and 3.1, it follows that there exists an inverse sequence $\boldsymbol{X}$ of compact polyhedra of dimension $\leqslant \dim X$ whose limit is X and an essential expansion $\boldsymbol{f}$: $\boldsymbol{X} \to M$ of f. Then Lemma 2.2 implies that $\boldsymbol{f} \times \mathrm{id}_I$ is essential. By Corollary 3.2(a) we conclude that $f \times \mathrm{id}_I$ is essential too. ■

Since $\dim X \times I^k \leqslant k + \dim X$ (see [E], Th. 1.5.16), from Theorem 4.2 we get the following

4.2. COROLLARY. *If f: $X \to M$ is a k-essential mapping of a compactum X onto M and $\dim X < k + 2\dim M - 2$, then f is ω-essential.* ■

5. Comments and problems. We end this paper by stating some problems and giving some characterizations of ω-essentiality.

A. The following problem was posed by J. Krasinkiewicz.

PROBLEM 1. Let f: $X \to M$ be an essential mapping. When is $f \times \mathrm{id}_I$ essential?

The Main Theorem 4.1 gives a partial answer to this problem. However, we do not know an answer to the following problem.

PROBLEM 2. Does Theorem 4.1 remain valid for non-compact spaces?

B. In connection with Problem 1, J. Krasinkiewicz posed the following more general

PROBLEM 3. Let f_i: $X_i \to M_i$ be an essential mapping, $i = 1, 2$. Under what additional conditions is $f_1 \times f_2$ essential?

From Corollary 5.2 below, it follows that the following strong version of problem 3 is interesting.

PROBLEM 4. Let f_i: $X_i \to M_i$ be an ω-essential mapping, $i = 1, 2$. When is $f_1 \times f_2$ (ω-) essential?

5.1. PROPOSITION. *Let* $f_i\colon X_i \to M_i$ *and let* N_i *be a submanifold of a manifold* M_i *such that* $\dim N_i = \dim M_i$, *for* $i = 1, 2$. *If* $f_1 \times f_2\colon X_1 \times X_2 \to M_1 \times M_2$ *is essential, then so is*

$$(f_1 | f_1^{-1}(N_1)) \times \mathrm{id}_{N_2}\colon f_1^{-1}(N_1) \times N_2 \to N_1 \times N_2.$$

Proof. Assume $f_1 \times f_2$ is essential. Since

$$f_1 \times f_2 = (f_1 \times \mathrm{id}_{M_2})(\mathrm{id}_{X_1} \times f_2),$$

it follows that $f_1 \times \mathrm{id}_{M_2}$ is essential. Then its restriction $(f_1 | f_1^{-1}(N_1)) \times \mathrm{id}_{N_2}$ is also essential (see [Kr], Th. I.1.9). ∎

5.2. COROLLARY. *Let* $\dim M_i = m_i$, *for* $i = 1, 2$. *If* $f_1 \times f_2\colon X_1 \times X_2 \to M_1 \times M_2$ *is essential, then* f_1 *is* m_2*-essential.* ∎

5.3. PROPOSITION. *If* $f\colon X \to M$ *is a k-essential mapping* ($k > 1$), *then*

$$f \times \mathrm{id}_{S^{k-1}}\colon X \times S^{k-1} \to M \times S^{k-1}$$

is 1-essential. In particular, $f \times \mathrm{id}_{S^{k-1}}$ *is essential.*

Proof. Consider a collar $N = \partial I^k \times I$ on $\partial I^k = S^{k-1}$ in I^k. Then from [Kr], Theorem I.1.9, it follows that $f \times \mathrm{id}_N$ is essential. Hence $f \times \mathrm{id}_{S^{k-1}}$ is 1-essential. ∎

By Corollary 5.2 and Prop. 5.3 we get the following characterization of ω-essentiality.

5.4. COROLLARY. *A mapping* $f\colon X \to M$ *is* ω*-essential iff* $f \times \mathrm{id}_{S^k}\colon X \times S^k \to M \times S^k$ *is essential, for each* $k = 1, 2, \ldots$ ∎

PROBLEM 5. Let $f\colon X \to M$ be a k-essential mapping and let N be a k-dimensional manifold (for example $N = S^k$). Must $f \times \mathrm{id}_N\colon X \times N \to M \times N$ be essential?

C. Let $M_J = \Pi_{j \in J} M_j$ be a countable product of manifolds. Consider a mapping $(f_j)\colon X \to M_J$. A mapping $(g_j)\colon X \to M_J$ is said to be an *admissible deformation* of (f_j) provided each g_j, $j \in J$, is an admissible deformation of f_j. The mapping (f_j) is said to be *essential* provided every admissible deformation of (f_j) is surjective (see [Kr]). Note that if J is finite, then this definition of essentiality is equivalent to the one given in the introduction.

We have the following characterization of ω-essentiality in the case of mappings of compacta. Here by Q we denote the Hilbert cube.

5.5. THEOREM. *Let* X *be a compactum. A mapping* $f\colon X \to M$ *is* ω*-essential iff* $f \times \mathrm{id}_Q\colon X \times Q \to M \times Q$ *is essential.* ∎

The proof of 5.5 using [Kr], Propositions I.1.2(b) and I.1.1(c), is left to the reader. Notice that the "if" part of Theorem 5.5 remains true without compactness of X; however, the "only if" part is an open problem.

PROBLEM 6. Let $f\colon X \to M$ be an ω-essential mapping of a separable space X onto M. Must $f \times \mathrm{id}_Q$ be essential?

PROBLEM 7. Let $(f_j)\colon X \to M_J$ be an essential mapping of a compactum X onto a product M_J of manifolds, where J is an infinite set. Must (f_j) be 1-essential?

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