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Homogeneous cohomology manifolds which are inverse limits

by

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Abstract. We describe a class of homogeneous cohomology manifolds.

1. Introduction. The aim of this paper is to consider a certain class of compact, finite-dimensional, homogeneous spaces which are inverse limits of topological manifolds. We say that a space X is *homogeneous* if for every $x, y \in X$ there exists a homeomorphism $h: X \to X$ with $h(x) = y$. The spaces that we consider depend on an orientable n-manifold L^n (with possibly empty boundary) and on a countable or finite family $\mathcal{M}^n$ of closed orientable manifolds of the same dimension n. We will denote them by $X(L^n, \mathcal{M}^n)$. A first such space was constructed in [J] for $L^3 = S^3$ and the one-element family $\mathcal{M}^3 = \{H\}$ where H was a homotopy 3-sphere $\neq S^3$, as a potential counterexample to the Bing–Borsuk conjecture(¹). Earlier similarly constructed spaces were considered in a different context in [P] and [W]. Then Ancel and Siebenmann ([A–S]) noticed that $X(S^3, H')$ where H' is some homology 3-sphere can be identified with a compactification of the Davis contractible 4-manifold which covers a closed 4-manifold [D]. They also introduced axioms describing spaces $X(L^n, \{M\})$ for the families $\mathcal{M}^n = \{M\}$ consisting of one element. Axiomatic description seems particularly useful when applied to spaces $X(L^n, \mathcal{M}^n)$ with an infinite family $\mathcal{M}$. Our axioms for general spaces $X(L^n, \mathcal{M}^n)$ presented in Section 2 differ only slightly from those of Ancel and Siebenmann. They play an important role in the proof of m-homogeneity of $X(L^n, \mathcal{M}^n)$ given in Sections 7 and 8. In Section 4 we use a lemma proved by Toruńczyk to show that the spaces $X(L^n, \mathcal{M}^n)$ depend only on L^n and $\mathcal{M}^n$, and in Section 5 we give a construction of $X(L^n, \mathcal{M}^n)$. If a family $\mathcal{M}^n$ consists of homology n-spheres then $X(L^n, \mathcal{M}^n)$ is a cohomology manifold. In this case $X(L^n, \mathcal{M}^n)$ can often be identified with the fixed-point set of a topological action on a manifold or a cohomology manifold. The theory of such actions was developed in [B]. Such homogeneous cohomology manifolds also appear as compactifications of contractible 4-manifolds, or orbit spaces of actions of 0-dimensional infinite compact groups. We give some examples in Sections 9 and 10.

(¹) I have been informed by several people that J. Martin also considered a similar construction.

The spaces $X(L^n, \mathcal{M}^n)$ have many properties of manifolds besides being homogeneous and being cohomology manifolds if $\mathcal{M}^n$ consists of homology spheres. In [J–R], for example, it is shown that they have certain general position properties for $n = 3$. Finally, there are many such spaces: in Section 11 it is shown that $X(L^3, \mathcal{M}) \neq X(L^3, \mathcal{M}')$ if $\mathcal{M}$ and $\mathcal{M}'$ are infinite families consisting of irreducible 3-manifolds and $\mathcal{M} \neq \mathcal{M}'$. In particular, there exists an uncountable family of non-homeomorphic, homogeneous 3-dimensional cohomology manifolds $X(S^3, \mathcal{M}^3)$.

2. Axiomatic description of $X(L^n, \mathcal{M}^n)$. Let L^n be an orientable n-manifold with (possibly empty) boundary ∂L^n and let $\mathcal{M}^n = \{M_1, M_2, \ldots\}$ be a finite or countable family of closed, distinct n-manifolds. We define a class $X(L^n, \mathcal{M}^n)$ of compact spaces as follows: $X \in X(L^n, \mathcal{M}^n)$ if and only if $X = \varprojlim \{L_i, \alpha_{i,i+1}\}$ and the spaces L_i and maps $\alpha_{i,i+1}\colon L_{i+1} \to L_i$, $i \in N$, satisfy the following axioms:

(1) $L_1 = L^n$ and every L_i is a connected sum of L^n and finitely many n-manifolds, each homeomorphic to some member of $\mathcal{M}^n$, with $\partial L_i = \partial L^n$.

(2) There exists a finite collection Ω_i of pairwise disjoint bicollared n-cells in $L_i \backslash \partial L_i$.

(3) $\alpha_{i,i+1}\colon L_{i+1} \to L_i$ is a homeomorphism over the set $L_i \backslash \bigcup \{\operatorname{int} Y\colon Y \in \Omega_i\}$.

(4) For every $Y \in \Omega_i$, $\alpha_{i,i+1}^{-1}(Y)$ is homeomorphic to $M \backslash \mathring{D}^n$, where $M \in \mathcal{M}^n$ and D^n is a bicollared disk in M.

(5) For every $j > i$ if $Y \in \Omega_i$ and $Y' \in \Omega_j$, then $\partial Y \cap \alpha_{ij}(Y') = \emptyset$ (here $\alpha_{ij} = \alpha_{i,i+1} \circ \ldots \circ \alpha_{j-1,j}\colon L_j \to L_i$. We also put $\alpha_{i,i} = \mathrm{id}_{L_i}$).

(6) The collection of sets $\{\alpha_{ij}(Y)\colon j \geqslant i,\ Y \in \Omega_j\}$ is a *null family*, that is, for every $\varepsilon > 0$, only a finite number of elements of the family have diameter $\geqslant \varepsilon$.

(7) The sum $\bigcup \Omega_i$ of the collection of cells $\Omega_i = \{\alpha_{ij}(Y)\colon j \geqslant i,\ Y \in \Omega_j$ and $\alpha_{ij}(Y)$ is not contained in $\alpha_{ik}(Y')$ for any $i \leqslant k < j$ and $Y' \in \Omega_k\}$ is dense in L_i.

(8) For $M \in \mathcal{M}^n$ let $\Omega'_i(M) = \{\alpha_{ij}(Y) \in \Omega_i\colon Y \in \Omega_i$ and $\alpha_{j,j+1}^{-1}(Y) \approx M \backslash \mathring{D}^n\}$. Then the sum $\bigcup \Omega'_i(M_j)$ of $\Omega'_i(M_j)$ is dense in $L_i \backslash \bigcup_{k \in N \backslash \{j\}} (\bigcup \Omega'_i(M_k))$, for every $i, j \in N$.

We will denote by $\alpha_i\colon X \to L_i$ the natural inverse limit projection.

The axiomatic description of $X(L^n, \mathcal{M}^n)$ presented here was first given by Ancel and Siebenmann for a one-element family $\mathcal{M}^n = \{M\}$. The families Ω'_i and $\Omega'_j(M_i)$ defined in axioms (7) and (8) are determined by Ω_i, so the spaces in $X(L^n, \mathcal{M}^n)$ depend on $\mathcal{M}^n$, the spaces L_i, the maps $\alpha_{i,i+1}$ and the families Ω_i. By a *defining system* for X we will mean a family $\{\mathcal{M}^n, L_i, \alpha_{i,i+1}, \Omega_i\}$ such that $X = \varprojlim \{L_i, \alpha_{i,i+1}\}$. In Section 4 we will show that there is only one space $X \in X(L^n, \mathcal{M}^n)$, so we will often write $X = X(L^n, \mathcal{M}^n)$. This will show that X depends only on L_n and $\mathcal{M}^n$, no matter which defining system for X we choose.

Let us note the following:

PROPOSITION (2.1). *If every $M \in \mathcal{M}^n$ is a homology n-sphere, then $X \in X(L^n, \mathcal{M}^n)$ is a cohomology n-manifold.*

Proof. In Section 8 we will prove that $X \in X(L^n, \mathcal{M}^n)$ is homogeneous so it is enough to compute the local Betti numbers $p^i(x, X)$ (see [B], pp. 7–9) around $x = \bigcap_{k=1}^{\infty} \alpha_{i_k}^{-1}(Y_{i_k})$ where $\alpha_i\colon X \to L_i$ is the inverse limit projection, Y_{i_k} is an element of some Ω_{i_k} and $\{i_k\}$ is a sequence such that $\alpha_{i_k, i_{k+1}}(Y_{i_{k+1}}) \subset Y_{i_k}$. By continuity

$$\check{H}^i(X, X \backslash \alpha_{i_k}^{-1}(\mathring{Y}_{i_k})) \approx \varinjlim H^i(L_j, L_j \backslash \alpha_{i_k,j}^{-1}(\mathring{Y}_{i_k}))$$

and these groups are 0 for $i \neq n, 0$ and $\boldsymbol{Z}$ for $i = 0$ or n. It is easy to see that the natural homomorphism $j_{kl}^i\colon \check{H}^i(X, X \backslash \alpha_{i_k}^{-1}(\mathring{Y}_{i_k})) \to \check{H}^i(X, X \backslash \alpha_{i_l}^{-1}(\mathring{Y}_{i_l}))$ is an isomorphism, so $p^i(x, X) = 0$ for $i \neq n, 0$ and $p^n(x, X) = 1$. X is locally orientable, so by [B], p. 9, it is a cohomology manifold.

PROPOSITION (2.2). *If $X \in X(L^n, \mathcal{M}^n)$, then* $\dim X = n$.

Proof is the same as in [J], p. 134.

3. Auxiliary lemma. Let $\mathcal{Z}$ be a family of n-cells contained in the interior of a given n-manifold M. By $S(\mathcal{Z})$ we will denote the sum of interiors of all n-cells $Z \in \mathcal{Z}$. We assume that we have fixed an orientation on M and the induced orientation on ∂M, and also on Z and ∂Z for every $Z \in \mathcal{Z}$.

We will say that a countable family $\mathcal{Z}$ of n-cells in the interior of a given n-manifold M is a *good stratified family* if the following conditions are satisfied:

(1) $\mathcal{Z} = \mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \ldots$, where each $\mathcal{Z}_i$ is a countable subfamily of $\mathcal{Z}$, and there is a countable or finite number of the families $\mathcal{Z}_i$.

(2) Each $S(\mathcal{Z}_i)$ is dense in $M \backslash S(\mathcal{Z}_1 \cup \ldots \cup \mathcal{Z}_{i-1} \cup \mathcal{Z}_{i+1} \cup \ldots)$.

(3) ∂Z is bicollared in M for every $Z \in \mathcal{Z}$.

(4) For every $Z_1, Z_2 \in \mathcal{Z}$, $Z_1 \neq Z_2$, we have $Z_1 \cap Z_2 = \emptyset$.

(5) $\mathcal{Z}$ is a null family in M, i.e. for every $\varepsilon > 0$ the set $\{Z \in \mathcal{Z}\colon \operatorname{diam} Z > \varepsilon\}$ is finite.

The following lemma (and its proof) is a simple extension of Toruńczyk's Lemma from [J].

LEMMA (3.1). *Let M and N be orientable n-manifolds and let $h\colon M \to N$ be an orientation-preserving homeomorphism. Let $\mathcal{Y} = \mathcal{Y}^1 \cup \mathcal{Y}^2 \cup \ldots$ and $\mathcal{Z} = \mathcal{Z}^1 \cup \mathcal{Z}^2 \cup \ldots$ be two good stratified families (each containing the same number, finite or infinite, of subfamilies) of n-cells in the interior of M and N respectively. For every $(Y, Z) \in \mathcal{Y}^i \times \mathcal{Z}^i$ let $\varphi_Y^Z\colon \partial Y \to \partial Z$ be an orientation-preserving homeomorphism. Then there exist bijective functions $p_i\colon \mathcal{Y}^i \to \mathcal{Z}^i$ and a homeomorphism $h'\colon M \backslash S(\mathcal{Y}) \to N \backslash S(\mathcal{Z})$ such that $h'|\partial M = h|\partial M$ and $h'|\partial Y = \varphi_Y^{p_i(Y)}$ for every $Y \in \mathcal{Y}^i$ and $i \in N$.*

We set $p = \bigcup_{i=1}^{\infty} p_i\colon \mathcal{Y} \to \mathcal{Z}$.

Sketch of the proof. We assume (without loss of generality) that $M = N$, $h = \mathrm{id}_M$ and $\operatorname{diam} M \leqslant 1$. Each φ_Y^Z can be extended to a homeomorphism $\psi_Y^Z\colon Y \to Z$ for $Y \in \mathcal{Y}^i$ and $Z \in \mathcal{Z}^i$, $i \in N$. Let $\Psi_i = \{\psi_Y^Z\colon Y \in \mathcal{Y}^i,\ Z \in \mathcal{Z}^i\}$, and let $H(M)$ be the set of all homeomorphisms of M which are identity on ∂M. Set $\mathcal{Z}_n^i = \{Z \in \mathcal{Z}^i\colon \operatorname{diam} Z \geqslant 2^{-n}\}$, $\mathcal{Y}_n^i = \{Y \in \mathcal{Y}^i\colon \operatorname{diam} Y \geqslant 2^{-n}\}$,

$$\mathcal{Z}_n = \bigcup_{i=1}^{\infty} \mathcal{Z}_n^i, \quad \mathcal{Y}_n = \bigcup_{i=1}^{\infty} \mathcal{Y}_n^i.$$

For any $f \in H(M)$ and any family $\mathcal{T}$ of subsets of M, let $f(\mathcal{T}) = \{f(T)\colon T \in \mathcal{T}\}$. By (5), $\mathcal{Z}_n$ and $\mathcal{Y}_n$ are finite families. We construct inductively homeomorphisms $f_n, g_n \in H(M)$,

$n = 1, 2, \ldots$, such that the following conditions are satisfied:

(a_n) If $Y \in \mathscr{Y}^i_n$ then there is $Z \in \mathscr{Z}^i$ such that $f_n(Y) = g_n(Z)$ and $g_n^{-1} f_n | Y \in \Psi_i$ for every i.

$(a_n)'$ If $Z \in \mathscr{Z}^i_n$ then there is $Y \in \mathscr{Y}^i$ such that $f_n(Y) = g_n(Z)$ and $g_n^{-1} f_n | Y \in \Psi_i$ for every i.

(b_n) $\operatorname{diam} f_n(Y) < 2^{-n}$ for every $Y \in \mathscr{Y} \setminus (\mathscr{Y}_n \cup f_n^{-1} g_n(\mathscr{Z}_n))$.

$(b_n)'$ $\operatorname{diam} g_n(Z) < 2^{-n}$ for every $Z \in \mathscr{Z} \setminus (\mathscr{Z}_n \cup g_n^{-1} f_n(\mathscr{Y}_n))$.

(c_n) $f_n | Y = f_{n-1} | Y$ for every $Y \in \mathscr{Y}_{n-1} \cup f_{n-1}^{-1} g_{n-1}(\mathscr{Z}_{n-1})$.

$(c_n)'$ $g_n | Z = g_{n-1} | Z$ for every $Z \in \mathscr{Z}_{n-1} \cup g_{n-1}^{-1} f_{n-1}(\mathscr{Y}_{n-1})$.

(d_n) $\operatorname{dist}(f_n, f_{n-1}) \leqslant 2^{-n+2}$, $\operatorname{dist}(f_n^{-1}, f_{n-1}^{-1}) \leqslant 2^{-n+2}$.

$(d_n)'$ $\operatorname{dist}(g_n, g_{n-1}) \leqslant 2^{-n+3}$, $\operatorname{dist}(g_n^{-1}, g_{n-1}^{-1}) \leqslant 2^{-n+3}$.

The construction is exactly the same as in [J], pp. 129–130, so we omit it. The only difference is that we have to choose the elements Z_Y and Y_Z [J] (p. 130) in the appropriate families $\mathscr{Z}^i$ or $\mathscr{Y}^i$. We can also use the annulus theorem for any dimension by [K] and [Q]. Having f_n and g_n we put $f = \lim f_n$, $g = \lim g_n$; they are both in $H(M)$, and $h'' = g^{-1} f$ is a homeomorphism such that $h''(M \setminus S(\mathscr{Y})) = M \setminus S(\mathscr{Z})$, $h''(S(\mathscr{Y}^i)) = S(\mathscr{Z}^i)$ for $i = 1, 2, \ldots$, and $h'' | Y \in \Psi_i$ for every $Y \in \mathscr{Y}^i$. So we can take $h' = h'' | M \setminus S(\mathscr{Y})$ and $p(Y) = h''(Y)$ for every $Y \in \mathscr{Y}$.

4. The uniqueness of $X \in X(L^n, \mathscr{M}^n)$. Let $X \in X(L^n, \mathscr{M}^n)$ where $\mathscr{M}^n = \{M_1, M_2, \ldots\}$ and let $\{\mathscr{M}^n, L_i, \alpha_{i,i+1}, \Omega_i\}$ be a defining system for X. We set $\mathscr{Y}^{\emptyset} = \mathscr{Y}^{\emptyset}_1 \cup \mathscr{Y}^{\emptyset}_2 \cup \ldots = \Omega'_1$ where $Y^{\emptyset}_j = \Omega'_1(M_j)$. Let us index the elements of the family $\mathscr{Y}^{\emptyset}$: $\mathscr{Y}^{\emptyset} = \{Y_{i_1}\}_{i_1 \in N}$, and for every index $i_1 \in N$, let us define integers $j_1 = j_1(i_1)$ and $k_1 = k_1(i_1)$ as follows: j_1 is an integer such that $Y_{i_1} \in \mathscr{Y}^{\emptyset}_{j_1}$ and k_1 is the uniquely determined integer such that $\alpha_{1,k_1}^{-1}(Y_{i_1}) \in \Omega_{k_1}$, or equivalently, $\alpha_{1,k_1}^{-1}(Y_{i_1}) \approx D^n$ and $\alpha_{1,k_1+1}^{-1}(Y_{i_1}) \approx M_{j_1} \setminus \mathring{D}^n$.

Now we make the following:

Inductive assumption. Let $\mathscr{I} = \{i_1, \ldots, i_m\}$ be a finite sequence of positive integers and that with every $l \leqslant m$ we have associated two positive integers

$$j_l = j_l(i_1, \ldots, i_l) \quad \text{and} \quad k_l = k_l(i_1, \ldots, i_l)$$

so that $1 \leqslant k_1 < \ldots < k_m$, and that there is a sequence of n-cells

$$(4.1) \qquad Y_{i_1} \subset L_1, \quad Y_{i_1,i_2} \subset \alpha_{1,k_1+1}^{-1}(Y_{i_1}), \ldots, \quad Y_{i_1,\ldots,i_m} \subset \alpha_{k_{m-2},k_{m-1}+1}^{-1}(Y_{i_1,\ldots,i_{m-1}});$$

and that

$$(4.2) \qquad \alpha_{1,k_1}^{-1}(Y_{i_1}) \in \Omega_{k_1}, \quad \alpha_{k_1+1,k_2}^{-1}(Y_{i_1,i_2}) \in \Omega_{k_2}, \ldots, \quad \alpha_{k_{m-1}+1,k_m}^{-1}(Y_{i_1,\ldots,i_m}) \in \Omega_{k_m};$$

$$(4.3) \qquad Y_{i_1,\ldots,i_l} \in \Omega_{k_l}(M_{j_l}) \quad \text{for } l \leqslant m.$$

With this inductive assumption, we now define a family $\mathscr{Y}^{\mathscr{I}} = \mathscr{Y}^{\mathscr{I}}_1 \cup \mathscr{Y}^{\mathscr{I}}_2 \cup \ldots$ by

$$\mathscr{Y}^{\mathscr{I}} = \{Y \in \Omega'_{k_m+1}\colon\ Y \subset \alpha_{k_{m-1}+1,k_m+1}^{-1}(Y_{i_1,\ldots,i_m})\}$$

and then $\mathscr{Y}^{\mathscr{I}}_j = \mathscr{Y}^{\mathscr{I}} \cap \Omega'_{k_m+1}(M_j)$. We can write $\mathscr{Y}^{\mathscr{I}} = \{Y_{i_1,i_2,\ldots,i_{m+1}}\}_{i_{m+1} \in N}$, where $i_1, \ldots, i_m$ are fixed by the inductive assumption, and $i_{m+1} \in N$ is a varying index which distinguishes an element of $\mathscr{Y}^{\mathscr{I}}$. We will also use the notation $Y_{\mathscr{I},i_{m+1}}$ for $Y_{i_1,\ldots,i_{m+1}}$ and

$Y_{\mathscr{I}}$ for $Y_{i_1,\ldots,i_m}$. Then we define numbers

$$j_{m+1} = j_{m+1}(i_1, \ldots, i_{m+1}) \quad \text{and} \quad k_{m+1} = k_{m+1}(i_1, \ldots, i_{m+1})$$

as the unique integers such that $\alpha_{k_m+1,k_{m+1}}^{-1}(Y_{\mathscr{I},i_{m+1}}) \in \Omega_{k_{m+1}}$ and $Y_{\mathscr{I},i_{m+1}} \in \mathscr{Y}^{\mathscr{I}}_{j_{m+1}}$. Of course then $k_{m+1} > k_m$, and conditions (4.1–3) are satisfied with $m+1$ in place of m.

Starting the induction with $\mathscr{Y}^{\emptyset}$ we get the families $\mathscr{Y}^{\mathscr{I}} = \{Y_{\mathscr{I},i_{m+1}}\}_{i_{m+1} \in N}$ of n-cells for any finite ordered set $\mathscr{I} = \{i_1, \ldots, i_m\}$ of positive integers so that conditions (4.1–3) are satisfied, and for every $Y_{i_1,\ldots,i_m}$ we have a good stratified family $\mathscr{Y}^{\mathscr{I}} = \mathscr{Y}^{\mathscr{I}}_1 \cup \mathscr{Y}^{\mathscr{I}}_2 \cup \ldots$ of n-cells in $\alpha_{k_{m-1}+1,k_{m+1}}^{-1}(Y_{i_1,\ldots,i_m}) \approx M_{j_m} \setminus \mathring{D}^n$.

We can also define the families

$$\mathscr{Y}^m = \bigcup_{i_1,\ldots,i_m \in N} \mathscr{Y}^{\{i_1,\ldots,i_m\}} \quad \text{and} \quad \mathscr{Y}^m_j = \bigcup_{i_1,\ldots,i_n \in N} \mathscr{Y}^{\{i_1,\ldots,i_m\}}_j;$$

of course then $\mathscr{Y}^m = \mathscr{Y}^m_1 \cup \mathscr{Y}^m_2 \cup \ldots$

Notice that the members of $\mathscr{Y}^m$ are cells which are contained in many different manifolds L_i.

Then let us consider the family $\mathscr{Y}^* = \{\alpha_{k_{m-1}+1}^{-1}(Y_{i_1,\ldots,i_m})\colon m \in N, i_1, \ldots i_m \in N\} = \{Y^*_{i_1,\ldots,i_m}\}$ consisting of closed subsets of $X = \varprojlim \{L_i, \alpha_{i,i+1}\}$. We have

$$(4.4) \qquad Y^*_{i_1} \supset Y^*_{i_1,i_2} \supset Y^*_{i_1,i_2,i_3} \supset \ldots$$

for every infinite sequence of integers $\{i_1, i_2, \ldots\}$.

Moreover, by axiom (6), for every such sequence

$$\tilde{Y}_{\{i_1,i_2,\ldots\}} = \bigcap_{l=1}^{\infty} Y^*_{i_1,\ldots,i_l}$$

is a point.

The set $\tilde{Y}$ consisting of all such intersection points is dense in X, with one point corresponding to one sequence $\{i_1, i_2, \ldots\}$. Using axiom (3) and the projections $\alpha_i\colon X \to L_i$ to make the necessary identifications in L_i's and X we have the equality

$$(4.5) \qquad X \setminus \tilde{Y} = (L_1 \setminus S(\mathscr{Y}^{\emptyset})) \cup \bigcup_{i_1 \in N} (\alpha_{1,k_1+1}^{-1}(Y_{i_1}) \setminus S(\mathscr{Y}^{\{i_1\}})) \cup \bigcup_{i_1,i_2 \in N} (\alpha_{k_1+1,k_2+1}^{-1}(Y_{i_1,i_2}) \setminus S(\mathscr{Y}^{\{i_1,i_2\}})) \cup \ldots$$

Here in each summand $k_m = k_m(i_1, \ldots, i_m)$, where $i_1, \ldots, i_m$ are the integers appearing in this particular summand. Also using the fact that $\alpha_i | \alpha_i^{-1}(\partial L_1)\colon \alpha_i^{-1}(\partial L_1) \to \partial L_1$ is a homeomorphism, we can assume that $\partial L = \partial L_1 \subset X$. Now we prove the following:

THEOREM (4.6). *Let $X \in X(L^n, \mathscr{M}^n)$ and $\bar{X} \in X(\bar{L}^n, \mathscr{M}^n)$ and let $h\colon L^n \to \bar{L}^n$ be an orientation-preserving homeomorphism. Then there exists a homeomorphism $h^*\colon X \to \bar{X}$ such that $h^* | \partial L^n = h | \partial L^n$.*

Proof. Let $\{\mathscr{M}^n, L_i, \alpha_{i,i+1}, \Omega_i\}$ and $\{\mathscr{M}^n, \bar{L}_i, \bar{\alpha}_{i,i+1}, \bar{\Omega}_i\}$ be defining sequences for X and $\bar{X}$. For X we will use all the notation established until now, that is we have families Ω_i, $\Omega'_i(M_j)$, $\mathscr{Y}^{\mathscr{I}}$, $\mathscr{Y}^{\mathscr{I}}_j$, $\mathscr{Y}^m$, $\mathscr{Y}^m_j$, $\mathscr{Y}^*$ and the spaces $Y_{i_1,\ldots,i_m}$, $Y^*_{i_1,\ldots,i_m}$, $\tilde{Y}$. Accordingly for $\bar{X}$ we have families $\bar{\Omega}'_i$, $\bar{\Omega}'_i(M_j)$, $\mathscr{Z}^{\mathscr{I}}$, $\mathscr{Z}_j^{\mathscr{I}}$, $\mathscr{Z}^m$, $\mathscr{Z}_j^m$, $\mathscr{Z}^*$ and spaces $Z_{i_1,\ldots,i_m}$, $Z^*_{i_1,\ldots,i_m}$, $\tilde{Z}$ etc.

We want to prove that X and $\bar{X}$ are homeomorphic. First we use Lemma (3.1) with $M = L^n = L_1$, $N = \bar{L}^n = \bar{L}_1$, and the stratified families $\mathscr{Y} = \mathscr{Y}^{\emptyset} = \mathscr{Y}_1^{\emptyset} \cup \mathscr{Y}_2^{\emptyset} \cup \ldots$ and $\mathscr{Z} = \mathscr{Z}^{\emptyset} = \mathscr{Z}_1^{\emptyset} \cup \mathscr{Z}_2^{\emptyset} \cup \ldots$ of n-cells in L^n and $\bar{L}^n$ respectively, to find a homeomorphism

$$h_1^*\colon L_1 \setminus S(\mathscr{Y}^{\emptyset}) \to \bar{L}_1 \setminus S(\mathscr{Z}^{\emptyset})$$

with $h_1^*|\partial L_1 = h|\partial L^n$ and bijective functions $p_j\colon \mathscr{Y}_j^{\emptyset} \to \mathscr{Z}_j^{\emptyset}$ such that for every $Y \in \mathscr{Y}_j^{\emptyset}$, $h_1^*(\partial Y) = \partial(p_j(Y))$.

Assume inductively that we have defined a homeomorphism h_m^* which takes the union of the first m summands of (4.5):

$$(4.7)\qquad (L_1 \setminus S(\mathscr{Y}^{\emptyset})) \cup \ldots \cup \bigcup_{i_1,\ldots,i_m \in N} (\alpha^{-1}_{k_{m-1}+1,k_m+1}(Y_{i_1,\ldots,i_m}) \setminus S(\mathscr{Y}^{\{i_1,\ldots,i_m\}}))$$

onto the corresponding union

$$(4.8)\qquad (\bar{L}_1 \setminus S(\mathscr{Z}^{\emptyset})) \cup \ldots \cup \bigcup_{i_1,\ldots,i_m \in N} (\bar{\alpha}^{-1}_{k_{m-1}+1,k_m+1}(Z_{i_1,\ldots,i_m}) \setminus S(\mathscr{Z}^{\{i_1,\ldots,i_n\}})$$

and bijective functions $p_j^m\colon \mathscr{Y}_j^m \to \mathscr{Z}_j^m$ so that $h_m^*|\partial L^n = h|\partial L^n$ and for every $Y \in \mathscr{Y}_j^m$ we have $h_m^*(\partial Y) = \partial(p_j^m(Y))$; we also denote by p^m the map defined by $p^m(Y) = p_j^m(Y)$ for any $Y \in \mathscr{Y}_j^m$ and any $j \in N$. Take any $Y = Y_{i_1,\ldots,i_{m+1}} \in \mathscr{Y}^m$, and let $Z = Z_{i'_1,\ldots,i'_{m+1}} = p^m(Y)$. Of course $Z \in \mathscr{Z}^m$. We have the numbers $j_{m+1} = j_{m+1}(i_j, \ldots, i_{m+1})$ and $k_{m+1} = k_{m+1}(i_1, \ldots, i_{m+1})$ defined uniquely for $Y_{i_1,\ldots,i_{m+1}}$ and the numbers $j'_{m+1} = j'_{m+1}(i'_1, \ldots, i'_{m+1})$ and $k'_{m+1} = k'_{m+1}(i'_1, \ldots, i'_{m+1})$ defined for $Z_{i_1,\ldots,i_{m+1}}$. By the definition of p^m as $p^m = \bigcup p_j^m$, we have $j_{m+1} = j'_{m+1}$. Let $\mathscr{I} = \{i_1, \ldots, i_{m+1}\}$ and $\mathscr{I}' = \{i'_1, \ldots, i'_{m+1}\}$.

Now we use Lemma (3.1) again with $M = \alpha^{-1}_{k_m+1,k_{m+1}+1}(Y)$ and $N = \bar{\alpha}^{-1}_{k'_m+1,k'_{m+1}+1}(Z)$. Because $j_{m+1} = j'_{m+1}$, we have a homeomorphism $h\colon M \to N$, and we have good stratified families $\mathscr{Y}^{\mathscr{I}} = \mathscr{Y}_1^{\mathscr{I}} \cup \mathscr{Y}_2^{\mathscr{I}} \cup \ldots$ and $\mathscr{Z}^{\mathscr{I}} = \mathscr{Z}_1 \cup \mathscr{Z}_2 \cup \ldots$ We can also assume that $h|\partial M = h_m^*|\alpha^{-1}_{k_m+1,k_{m+1}+1}(\partial Y)$. Now using Lemma (3.1) we get a homeomorphism

$$h_{\mathscr{I}}^*\colon \alpha^{-1}_{k_m+1,k_{m+1}+1}(Y) \setminus S(\mathscr{Y}^{\mathscr{I}}) \to \bar{\alpha}^{-1}_{k'_m+1,k'_{m+1}+1}(Z) \setminus S(\mathscr{Z}^{\mathscr{I}'})$$

which extends h_m^*, and functions $p_j^{\mathscr{I}}\colon \mathscr{Y}_j^{\mathscr{I}} \to \mathscr{Z}_j^{\mathscr{I}'}$ and $p^{\mathscr{I}} = \bigcup_j p_j^{\mathscr{I}}\colon \mathscr{Y}^{\mathscr{I}} \to \mathscr{Z}^{\mathscr{I}}$ so that $h^*(\partial Y) = \partial(p^{\mathscr{I}}(Y))$ for $Y \in \mathscr{Y}^{\mathscr{I}}$.

Finally, we can define h_{m+1}^* to be h_m^* on the union (4.7), and $h_{\mathscr{I}}^*$ on every $Y_{\mathscr{I}} \in \mathscr{Y}^m$; p_j^{m+1} can be defined by $p_j^{m+1} = \bigcup_{\mathscr{I}} p_j^{\mathscr{I}}$ where $\mathscr{I}$ runs over all $(m+1)$-element sets $\mathscr{I} = \{i_1, \ldots, i_{m+1}\}$ of integers. This completes the inductive definition of h_m^*.

It is easy to see that now we can define a homeomorphism

$$\bar{h}^* = \bigcup_{m=1}^{\infty} h_m^*\colon X \setminus \tilde{Y} \to \bar{X} \setminus \tilde{Z}.$$

Then we can uniquely extend $\bar{h}^*$ to $h^*\colon X \to \bar{X}$ by $h^*(\tilde{Y}_{\{i_1,i_2,\ldots\}}) = \tilde{Z}_{\{i'_1,i'_2,\ldots\}}$ where $\{i'_1, i'_2, \ldots\}$ is the uniquely defined sequence of integers such that for every m, $p^m(Y_{i_1,\ldots,i_{m+1}}) = Z_{i'_1,\ldots,i'_{m+1}}$.

COROLLARY (4.7). *Let $X \in X(L^n \# N, \mathscr{M})$ and let $X' \in X(L^n \# N', \mathscr{M})$, where N and N' are connected sums of a finite number of manifolds homeomorphic to elements of $\mathscr{M}^n$. Then X and X' are homeomorphic.*

Proof. We can construct systems $\{\mathscr{M}^n, L_i, \alpha_{i,i+1}, \Omega_i\}$ and $\{\mathscr{M}^n, \bar{L}_i, \bar{\alpha}_{i,i+1}, \bar{\Omega}_i\}$ with $L_1 = \bar{L}_1 = L^n$ such that $L_{i'} \approx L^n \# N$ and $\bar{L}_{i''} \approx L^n \# N'$ for some integers i', i'', so that both systems define a space from $X(L^n, \mathscr{M}^n)$ but after restricting the first of them to L_i with $i > i'$ and the second to $\bar{L}_i$ with $i > i''$ they define spaces from $X(L^n \# N, \mathscr{M}^n)$ and $X(L^n \# N', \mathscr{M}^n)$ respectively.

5. Construction of $X(L^n, \mathscr{M}^n)$. We now know that for every L^n and $\mathscr{M}^n$ there exists at most one space $X \in X(L^n, \mathscr{M}^n)$. We will denote it simply $X(L^n, \mathscr{M}^n)$. In this section we will construct a space $X(L^n, \mathscr{M}^n)$ for any L^n and $\mathscr{M}^n$, thus proving that for every L^n and $\mathscr{M}^n$ there exists precisely one space $X(L^n, \mathscr{M}^n)$. We assume the following:

Data (5.1). If $\mathscr{M}^n$ is infinite, then $\{j_1, j_2 \ldots\}$ is an infinite sequence of integers in which every integer appears infinitely many times. If $\mathscr{M}^n$ is finite, then $\{j_1, j_2, \ldots\}$ is an infinite sequence of integers $\leqslant s$, where s is the cardinality of $\mathscr{M}^n$, in which every such integer appears infinitely many times.

For any L^n and $\mathscr{M}^n = \{M_i\}_{i \in N}$ we will construct a system $\{\mathscr{M}^n, L_i, \alpha_{i,i+1}, \Omega_i\}$ defining a space $X = X(L^n, \mathscr{M}^n)$ which moreover satisfies the following condition:

> A($\{j_1, j_2, \ldots\}$): For every $k \in N$ and every $Y \in \Omega_k$ we have $\alpha^{-1}_{k,k+1}(Y) \approx M_{j_k} \setminus \mathring{D}^n$. Moreover, for every $\varepsilon > 0$ there exists k_0 such that for every $k > k_0$ we have $\operatorname{diam}(\alpha_k^{-1}(Y)) < \varepsilon$ for every $Y \in \Omega_k$ and for every $x \in X(L^n, \mathscr{M}^n)$ there exists $Y \in \Omega_k$ with $\operatorname{dist}(x, \alpha_k^{-1}(Y)) < \varepsilon$.

The construction is inductive: let $\{\varepsilon_1, \varepsilon_2, \ldots\}$ be a sequence of positive numbers converging to 0. We take $L_1 = L^n$ and for Ω_1 we take any finite collection of disjoint, bicollared n-cells in L_1 having diameters $< \varepsilon_1$ and such that for every $x \in L_1$ there exists $Y \in \Omega_1$ with $\operatorname{dist}(x, Y) < \varepsilon_1$. Then assume inductively that we have defined spaces $L_1, \ldots, L_k$, maps $\alpha_{i,i+1}$ for $i < k$ and families Ω_i for $i \leqslant k$ so that axioms (1), (2) are satisfied for $i < k$, axioms (3), (4) are satisfied for $i < k$ and axiom (5) is satisfied for $i, j \leqslant k$. Moreover, we require that $\operatorname{diam}(Y) \leqslant \varepsilon_i$ for $i \leqslant k$ and $Y \in \Omega_i$, for any $x \in L_i$ there exists $Y \in \Omega_i$ with $\operatorname{dist}(x, Y) < \varepsilon_i$, and $\alpha^{-1}_{i,i+1}(Y) \approx M_{j_i} \setminus \mathring{D}^n$ for every $i < k$ and $Y \in \Omega_i$.

Then we construct L_{k+1} as follows: we remove the interior of every $Y \in \Omega_k$ from L_k, and instead of it we glue in a copy of the manifold $M_{j_k} \setminus \mathring{D}^n$, identifying its boundary with ∂Y. We choose a metric on L_{k+1} to coincide with the metric of L_k on $L_k \cap L_{k+1}$, and on the attached manifolds we choose it so that each attached copy of $M_{j_k} \setminus \mathring{D}^n$ has diameter $< \varepsilon_k$. Then we define $\alpha_{k,k+1}$ to be identity on $L_k \cap L_{k+1}$; on every attached copy of $M_{j_k} \setminus \mathring{D}^n$ we let $\alpha_{k,k+1}$ be any map onto the corresponding $Y \in \Omega_k$ which extends identity on ∂Y and for which $\alpha_{k,k+1}|\alpha^{-1}_{k,k+1}(\partial Y)\colon \alpha^{-1}_{k,k+1}(\partial Y) \to \partial Y$ is a homeomorphism. The set $Z_{k+1} = \bigcup_{i \leqslant k} \alpha^{-1}_{i,k+1}(\bigcup \{\partial Y\colon\ Y \in \Omega_i\})$ is closed and nowhere dense in L_{k+1}. Therefore we can define Ω_{k+1} to be any finite family of disjoint, bicollared n-cells in L_{k+1} such that for any $Y \in \Omega_{k+1}$ we have $Y \cap Z_{k+1} = \emptyset$, $\operatorname{diam}(Y) < \varepsilon_{k+1}$, and for any $x \in L_{k+1}$ there exists $Y \in \Omega_{k+1}$ with $\operatorname{dist}(x, Y) < \varepsilon_{k+1}$. It is easy to see that then the inductive hypothesis is met with k replaced by $k+1$. If we perform all the steps of the induction, we get a system $\{\mathcal{M}^n, L_i, \alpha_{i,i+1}, \Omega_i\}$ which satisfies axioms (1)–(5). Axioms (6) and (7) are satisfied because of the way in which the families Ω_i depend on $\{\varepsilon_i\}_{i\in N}$. Finally, the system satisfies $A(\{i_1, i_2, \ldots\})$, so axiom (8) is satisfied because of the properties of the sequence $\{i_1, i_2, \ldots\}$.

6. Embeddings of $X(L^n, \mathcal{M}^n)$ into manifolds. Let $X = X(L^n, \mathcal{M}^n)$, let $\{\mathcal{M}^n, L_i, \alpha_{i,i+1}, \Omega_i\}$ be a defining system for X, and let P be a manifold (possibly with boundary) of dimension $k > n$. We will use all the notation introduced in Section 4. In particular, we have $\mathcal{M}^n = \{M_j\}_{j\in N}$, $\Omega_1 = \mathcal{Y}^{\emptyset}$, $\Omega_1'(M_j) = \mathcal{Y}_j^{\emptyset}$ and we have families $\mathcal{Y}^{\mathcal{I}} = \mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \ldots$ for $\mathcal{I} = \{i_1, \ldots, i_m\}$. We assume the following:

Data (6.1). There exists a bicollared topological embedding $e_L\colon L^n \to P$ with $e_L(\partial L^n) = e_L(L^n) \cap \partial P$, and for every $M \in \mathcal{M}^n$ there exists a bicollared embedding $e_M\colon M\setminus \mathring{D}^n \to D^k$ (where $D^n \subset M$) such that $e_M(\partial D^n) = e_M(M\setminus\mathring{D}^n) \cap \partial D^k$ and $e_M(\partial D^n)$ is a standard $(n-1)$-sphere $S^{n-1} \subset S^{k-1} = \partial D^k$.

Assuming (6.1) we will construct an embedding of X into P. First, having $e_L\colon L \to P$ and any fixed sequence $\delta_1 > \delta_2 > \ldots$ of positive numbers converging to 0, we construct inductively a sequence of embeddings

(6.2) $$e_L^m\colon L_m \to P,$$

where $L_0 = L$, and

$$L_m = (L\setminus S(\mathcal{Y}^{\emptyset})) \cup \bigcup_{i_1\in N} (\alpha_{1,k_1+1}^{-1}(Y_{i_1})\setminus S(\mathcal{Y}^{\{i_1\}})) \cup \ldots$$
$$\ldots \cup \bigcup_{i_1,\ldots,i_m\in N} (\alpha_{k_{m-1}+1,k_m+1}^{-1}(Y_{i_1,\ldots,i_m})\setminus S(\mathcal{Y}^{\{i_1,\ldots,i_m\}})) \cup \bigcup_{i_1,\ldots,i_m\in N} S(\mathcal{Y}^{\{i_1,\ldots,i_m\}}),$$

where $k_m = k_m(i_1, \ldots, i_m)$ in every summand.

We claim that these embeddings have the following properties:

a(m): For every m the maps e_L^m and e_L^{m-1} coincide on the space

$$L_{m-1} \cap L_m = (L^n\setminus S(\mathcal{Y}^{\emptyset})) \cup \bigcup_{i_1\in N} (\alpha_{1,k_1+1}^{-1}(Y_{i_1})\setminus S(\mathcal{Y}^{\{i_1\}})) \cup \ldots$$
$$\ldots \cup \bigcup_{i_1,\ldots,i_{m-1}\in N} (\alpha_{k_{m-2}+1,k_{m-1}+1}^{-1}(Y_{i_1,\ldots,i_{m-1}})\setminus S(\mathcal{Y}^{\{i_1,\ldots,i_{m-1}\}}))$$

b(m): For every m and every set $\mathcal{I} = \{i_1, \ldots, i_m\}$ consisting of m integers there is a bicollared embedding

$$e^{\mathcal{I}}\colon D^k \to P$$

such that $e^{\{i_1,\ldots,i_{m-1}\}}(D^k) \supset e^{\{i_1,\ldots,i_m\}}(D^k)$ and $e^{\mathcal{I}}(D^k) \cap e^{\mathcal{I}'}(D_k) = \emptyset$ for any two m-element sets $\mathcal{I} \neq \mathcal{I}'$; moreover:

c(m): $e^{\mathcal{I}}(D^k) \cap e_L^m(L_m) = e_L^m(\alpha_{k_{m-1}+1,k_m+1}^{-1}(Y_{i_1,\ldots,i_m}))$ is a topological, bicollared submanifold of $e^{\mathcal{I}}(D^k)$ for $\mathcal{I} = \{i_1, \ldots, i_m\}$ and $(e^{\mathcal{I}})^{-1}(e^{\mathcal{I}}(\partial D^k) \cap e_L^m(L_m)) = S^{n-1} \subset S^{k-1}$ is the standard $(n-1)$-sphere in $\partial D^k = S^{k-1}$.

d(m): $\operatorname{diam}(e^{\mathcal{I}}(D^k)) < \delta_m$ for $\mathcal{I} = \{i_1, \ldots, i_m\}$.

e(m): $\operatorname{diam}(e_L^m(Y_{i_1,\ldots,i_{m+1}})) < \delta_{m+1}$ for every $Y_{i_1,\ldots,i_{m+1}} \in \mathcal{Y}^{\{i_1,\ldots,i_m\}}$.

Let us take the map $e_L\colon L^n \to P$. Using Bing's shrinking criterion described for example in [F], p. 417, Lemma (7.1), we can find a homeomorphism $\xi\colon L^n \to L^n$ such that the embedding $e_L^0 = e_L\xi\colon L^n \to P$ has the following property: $\{e_L^0(Y_{i_1})\colon Y_{i_1} \in \mathcal{Y}^{\emptyset}\}$ is a null family with all elements of diameter $< \delta_1$. We start induction with this map.

Assume now that we have constructed e_L^m satisfying a(m) – e(m). Let $\mathcal{I} = \{i_1, \ldots, i_m\}$ and consider the manifold $e_L^m(\alpha_{k_{m-1}+1,k_m+1}^{-1}(Y_{i_1,\ldots,i_m}))$ which is a bicollared topological submanifold of $e^{\mathcal{I}}(D^k)$ (by c(m)). Using e(m) we can easily produce embeddings $e^{\{i_1,\ldots,i_{m+1}\}}\colon D^k \to P$ for every $i_{m+1} \in N$ so that b$(m+1)$ and d$(m+1)$ are satisfied. Then let $\eta\colon \alpha_{k_{m+1},k_{m+1}+1}^{-1}(Y_{i_1,\ldots,i_{m+1}}) \to M_{j_{m+1}}$ be a homeomorphism, where $j_{m+1} = j_{m+1}(i_1, \ldots, i_{m+1})$. We define an embedding

$$f_{\mathcal{I},i_{m+1}} = e^{\{i_1,\ldots,i_{m+1}\}} \circ \eta\colon \alpha_{k_m+1,k_{m+1}+1}^{-1}(Y_{i_1,\ldots,i_{m+1}}) \to e^{\{i_1,\ldots,i_{m+1}\}}(D^k).$$

The map $f_{\mathcal{I},i_{m+1}}$ extends $e_L^m | L_{m+1} \cap L_m$ if η chosen properly on the boundary, but we cannot put $e_L^{m+1} = f_{\mathcal{I},i_{m+1}}$, because e$(m)$ would not be satisfied. But again we can compose $f_{\mathcal{I},i_{m+1}}$ with an automorphism ξ of $\alpha_{k_m+1,k_{m+1}+1}^{-1}(Y_{i_1,\ldots,i_{m+1}})$ obtained by Bing's shrinking criterion so that if we put $e_L^{m+1} | \alpha_{k_m+1,k_{m+1}+1}^{-1}(Y_{i_1,\ldots,i_{m+1}}) = f_{\mathcal{I},i_{m+1}} \circ \xi$ then both e$(m+1)$ and c$(m+1)$ are satisfied.

So we have defined e_L^{m+1} as equal to e_L^m on $L_m \cap L_{m+1}$ and to $f_{\mathcal{I},i_{m+1}} \circ \xi$ on $\alpha_{k_m+1,k_{m+1}+1}^{-1}(Y_{i_1,\ldots,i_{m+1}})$. Now, notice that $L_m \cap L_{m+1}$ is the union of the first $m+1$ summands of (4.5). This implies that we can define an embedding

$$e_{X\setminus\tilde{Y}}\colon X\setminus\tilde{Y} \to P$$

by $e_{X\setminus\tilde{Y}} | L_m \cap L_{m+1} = e_L^m | L_m \cap L_{m+1}$. This is well defined because $X\setminus\tilde{Y} = \bigcup_m (L_m \cap L_{m+1})$ by (4.5). Now, we extend $e_{X\setminus\tilde{Y}}$ to a map $e_X\colon X \to P$ putting

$$e_X(Y_{\{i_1,i_2,\ldots\}}) = \bigcap_{m=1}^{\infty} e^{\{i_1,\ldots,i_m\}}(D^k).$$

It can easily be seen from a(m)–e(m) that e_X is an embedding.

7. Defining systems satisfying a special condition. Let $X = X(L^n, \mathcal{M}^n)$ for some L^n and $\mathcal{M}^n$ and let $\{\mathcal{M}^n, L_i, \alpha_{i,i+1}, \Omega_i\}$ be a defining system for X. Of course we have much freedom in choosing a defining system. Let $p_1, \ldots, p_m \in X$ be any finite sequence of distinct points in X. We are interested whether $\{\mathcal{M}^n, L_i, \alpha_{i,i+1}, \Omega_i\}$ satisfies the following condition:

B$(p_1, \ldots, p_m)$: There exists an increasing sequence $k_1, k_2, \ldots$ of integers, a manifold $M \in \mathcal{M}^n$, and a family of n-cells $\{Y_1^j, Y_2^j, \ldots\}$ for every $j \leqslant m$ such that $Y_i^j \in \Omega_{k_i}$, $\alpha_{k_i,k_i+1}^{-1}(Y_i^j) \approx M\setminus\mathring{D}^n$, $\alpha_{k_i,k_i+1}(Y_{i+1}^j) \subset Y_i^j$ for any $i \in N$, and $p_j = \bigcap_{i=1}^{\infty} \alpha_{k_i}^{-1}(Y_i^j)$.

LEMMA (7.1). *For every L^n and $\mathcal{M}^n$ and any points $p_1, \ldots, p_m \in X = X(L^n, \mathcal{M}^n)$, there exists a system $\{\mathcal{M}^n, L_i, \alpha_{i,i+1}, \Omega_i\}$ defining X which satisfies condition* B$(p_1, \ldots, p_m)$.

Proof. By the uniqueness theorem (4.6) and by the construction described in Section 5, we can find a system $\{\mathcal{M}^n, L_i, \alpha_{i,i+1}, \bar{\Omega}_i\}$ defining X and satisfying condition $A(\{j_1, j_2, \ldots\})$ for some sequence $\{j_1, j_2, \ldots\}$ which satisfies (5.1). Note that in $A(\{j_1, j_2, \ldots\})$ we assume our family $\mathcal{M}^n$ to be indexed: $\mathcal{M}^n = \{M_i\}_{i\in N}$. For every $j \leqslant m$ we can choose a family $\{U_i^j\}_{i\in N}$ of open subsets of X such that $U_1^j \supset U_2^j \supset \ldots$,

$U_1^j \cap U_1^{j'} = \emptyset$ for $j \neq j'$ and $p_j = \bigcap_{i=1}^{\infty} U_i^j$. Moreover, we can assume that $V_{k,i}^j = \alpha_k(U_i^j)$ is an open $\bar{\Omega}_k'$-saturated subset of L_k for any $k \geqslant i$, i.e. for every $Y \in \bar{\Omega}_k'$ either $Y \cap V_{k,i}^j = \emptyset$ or $Y \subset V_{k,i}^j$.

We will modify inductively the families $\bar{\Omega}_i$ in the system $\{\mathcal{M}^n, L_i, \alpha_{i,i+1}, \bar{\Omega}_i\}$ so as to get a new system $\{\mathcal{M}^n, L_i, \alpha_{i,i+1}, \Omega_i\}$ defining X and satisfying $\mathrm{B}(p_1, \ldots, p_m)$.

By $\mathrm{A}(\{j_1, j_2, \ldots\})$ we can find an integer $k_1 > 1$ such that for $j \leqslant m$ there are n-cells $Y_j \in \bar{\Omega}_{k_1}$ such that $\alpha_{k_1}^{-1}(Y_j) \subset U_1^j$ and $M_{jk_1} = M$, and consequently $\alpha_{k_1,k_1+1}^{-1}(Y_j) \approx M \backslash \mathring{D}^n$.

Moreover, if $\alpha_{k_1}(p_j) \in Y$ for some $Y \in \bar{\Omega}_{k_1}$ then we can assume that $Y_j = Y$.

Now for $j \leqslant m$ we will define n-cells $Y_1^j \subset V_{k_1,1}^j = \alpha_{k_1}(U_1^j)$ with the following properties: $\mathrm{Int}(Y_1^j) \supset Y_j \cup \{\alpha_{k_1}(p_j)\}$, ∂Y_1^j does not intersect any element of $\bar{\Omega}_{k_1}'$ and Y_1^j does not intersect any element of the family

$$\{\alpha_{k,k_1}^{-1}(T) \colon\ k \leqslant k_1 \text{ and } T \in \bar{\Omega}_k\} \backslash \{Y_j\}.$$

We construct the sets Y_1^j as follows: we take a decomposition $\pi \colon L_{k_1} \to L_{k_1}/\bar{\Omega}_{k_1}' = \tilde{L}_{k_1}$. By a theorem of Bing (Theorem (7.2) in [F]) $\tilde{L}_{k_1}$ is homeomorphic to L_{k_1} and the non-degenerate elements of this decomposition form a dense countable subset in $\tilde{L}_{k_1}$. Then for every $j \leqslant m$ we find a bicollared n-cell $\tilde{Y} \subset \tilde{L}_{k_1}$ such that $\mathrm{Int}\,\tilde{Y}$ contains $\pi(\alpha_{k_1}(p_1)) \cup \pi(Y_j)$. Because of our choice of Y_j we can assume that

$$\tilde{Y} \cap \pi(\{\alpha_{k,k_1}^{-1}(T) \colon\ k \leqslant k_1 \text{ and } T \in \bar{\Omega}_k\}) = \pi(Y_j).$$

By [B–P], p. 140, Theorem (7.2), we can assume that $\partial \tilde{Y} \cap \pi(\bar{\Omega}_{k_1}') = \emptyset$. Finally, we can choose $\tilde{Y}$ so small that $\tilde{Y} \subset \pi(V_{k,1}^j)$. Then we define $Y_1^j = \pi^{-1}(\tilde{Y})$. It is easy to see that Y_1^j so constructed has all required properties. To complete the first step of induction we put $\Omega_i = \bar{\Omega}_i$ for $i \neq k_1$ and $\Omega_{k_1} = \bar{\Omega}_{k_1} \cup \{Y_1^1, \ldots, Y_1^m\} \backslash \{Y_1, \ldots, Y_m\}$.

Now assume that we have integers $k_1 < \ldots < k_s$ and a system $\{\mathcal{M}^n, L_i, \alpha_{i,i+1}, \bar{\Omega}_i\}$ defining X such that for every $j \leqslant m$ there exists a family of n-cells $\{Y_1^j, \ldots, Y_s^j\}$ with $Y_i^j \in \bar{\Omega}_{k_i}$, $\alpha_{k_i,k_i+1}^{-1}(Y_i^j) \approx M \backslash \mathring{D}^n$ for $i \leqslant s$. $\alpha_{k_i,k_{i+1}}(Y_{i+1}^j) \in V_{k_i,i+1}^j \cap Y_i^j$ for $i < s$ and $Y_s^j \ni \alpha_{k_s}(p_j)$ (the definition of $V_{k,i}^j$ remains unchanged). By $\mathrm{A}(\{j_1, j_2, \ldots\})$ we can find an integer $k_{s+1} > k_s$ such that for $j \leqslant m$ there are n-cells $Y_j \in \bar{\Omega}_{k_{s+1}}$ such that $\alpha_{k_{s+1}}^{-1}(Y_j) \subset U_{s+1}^j \cap \mathrm{Int}(\alpha_{k_s}^{-1}(Y_s^j))$ and that $M_{jk_{s+1}} = M$.

Moreover, we can assume, that if $\alpha_{k_{s+1}}(p_j) \in Y$ for some $Y \in \bar{\Omega}_{k_{s+1}}$ then $Y_j = Y$.

Now we define n-cells $Y_{s+1}^j \subset V_{k_{s+1},s+1}^j \cap \alpha_{k_s,k_{s+1}}^{-1}(Y_s^j)$ in the same way as for Y_1^j. We complete the induction by putting $\Omega_i = \bar{\Omega}_i$ for $i \neq k_{s+1}$ and $\Omega_{k_{s+1}} = \bar{\Omega}_{k_{s+1}} \cup \{Y_{s+1}^1, \ldots, Y_{s+1}^m\} \backslash \{Y_1, \ldots, Y_m\}$.

Every step of this induction changes only one family Ω_i. This implies that performing all the steps we get a system $\{\mathcal{M}^n, L_i, \alpha_{i,i+1}, \Omega_i\}$ which still satisfies axioms (1)–(8) and $\mathrm{A}(\{j_1, j_2, \ldots\})$ and additionally it satisfies $\mathrm{B}(p_1, \ldots, p_m)$.

8. The m-homogeneity of $X(L^n, \mathcal{M}^n)$. A space X is said to be *m-homogeneous* if for any two m-element collections of distinct points $\{p_1, \ldots, p_m\}$ and $\{\bar{p}_1, \ldots, \bar{p}_m\}$ in X there exists a homeomorphism $h \colon X \to X$ such that $h(p_i) = \bar{p}_i$ for $i \leqslant m$. We prove the following:

THEOREM (8.1). *For every closed manifold L^n, every family $\mathcal{M}^n$ and every positive integer m the space $X = X(L^n, \mathcal{M}^n)$ is m-homogeneous.*

Proof. Let $\{p_1, \ldots, p_m\}$ and $\{\bar{p}_1, \ldots, \bar{p}_m\}$ be two collections of points. Without loss of generality we can assume that the collections are disjoint. By (7.1) there exists a system $\{\mathcal{M}^n, L_i, \alpha_{i,i+1}, \Omega_i\}$ defining X and satisfying $\mathrm{B}(p_1, \ldots, p_m, \bar{p}_1, \ldots, \bar{p}_m)$. Then of course the conditions $\mathrm{B}(p_1, \ldots, p_m)$ and $B(\bar{p}_1, \ldots, \bar{p}_m)$ are also satisfied, and moreover an increasing sequence $k_1 < k_2 < \ldots$ determined by these two conditions is the same.

This implies that for every $j \leqslant$ m there are families $\{Y_1^j, Y_2^j, \ldots\}$ and $\{\bar{Y}_1^1, \bar{Y}_2^j, \ldots\}$, with $Y_i^j, \bar{Y}_i^j \in \Omega_{k_i}$, $\alpha_{k_i,k_i+1}^{-1}(Y_i^j) \approx \alpha_{k_i,k_i+1}^{-1}(\bar{Y}_i^j) \approx M \backslash \mathring{D}^n$ for some fixed $M \in \mathcal{M}^n$, $\alpha_{k_i,k_{i+1}}(Y_{i+1}^j) \subset Y_i^j$, $\alpha_{k_i,k_{i+1}}(\bar{Y}_{i+1}^j) \subset \bar{Y}_i^j$ and $p_j = \bigcap_{i=1}^{\infty} \alpha_{k_i}^{-1}(Y_i^j)$, $\bar{p}_j = \bigcap_{i=1}^{\infty} \alpha_{k_i}^{-1}(\bar{Y}_i^j)$.

We construct inductively a homeomorphism $h \colon X \to X$ with $h(p_j) = \bar{p}_j$, $j \leqslant m$. First we take the spaces

$$X_1 = X \backslash \alpha_{k_1}^{-1}\Big(\bigcup_{j=1}^{m} Y_1^j\Big) \quad \text{and} \quad \bar{X}_1 = X \backslash \alpha_{k_1}^{-1}\Big(\bigcup_{j=1}^{m} \bar{Y}_1^j\Big).$$

Obviously

$$X_1 \in X\Big(L_{k_1} \backslash \bigcup_{j=1}^{m} \mathrm{Int}\, Y_1^j,\ \mathcal{M}\Big) \quad \text{and} \quad \bar{X}_1 \in X\Big(L_{k_1} \backslash \bigcup_{j=1}^{m} \mathrm{Int}\, \bar{Y}_1^j,\ \mathcal{M}\Big)$$

so by Theorem (4.6) there is a homeomorphism $h_1 \colon X_1 \to \bar{X}_1$ such that

$$h_1(\alpha_{k_1}^{-1}(\partial Y_1^j)) = \alpha_{k_1}^{-1}(\partial \bar{Y}_1^j) \quad \text{for } j \leqslant m.$$

Then we set

$$X_s = X \backslash \bigcup_{j=1}^{m} \alpha_{k_s}^{-1}(\mathrm{Int}\, Y_s^j) \quad \text{and} \quad \bar{X}_s = X \backslash \bigcup_{j=1}^{m} \alpha_{k_s}^{-1}(\mathrm{Int}\, \bar{Y}_s^j)$$

and assume inductively that we have a homeomorphism $h_s \colon X_s \to \bar{X}_s$ such that

$$h_s(\alpha_{k_s}^{-1}(\partial Y_s^j)) = \alpha_{k_s}^{-1}(\partial \bar{Y}_s^j) \quad \text{for } j \leqslant s.$$

For every $j \leqslant m$ we find a homeomorphism

$$h_{s+1}^j \colon\ \alpha_{k_s}^{-1}(Y_s^j) \backslash \alpha_{k_{s+1}}^{-1}(\mathrm{Int}\, Y_{s+1}^j) \to \alpha_{k_s}^{-1}(\bar{Y}_s^j) \backslash \alpha_{k_{s+1}}^{-1}(\mathrm{Int}\, \bar{Y}_{s+1}^j)$$

which extends h_s, i.e. agrees with h_s on $\alpha_{k_s}^{-1}(\partial Y_s^j)$. Indeed, the spaces $\alpha_{k_s}^{-1}(Y_s^j) \backslash \alpha_{k_{s+1}}^{-1}(\mathrm{Int}\, Y_{s+1}^j)$ and $\alpha_{k_s}^{-1}(\bar{Y}_s^j) \backslash \alpha_{k_{s+1}}^{-1}(\mathrm{Int}\, \bar{Y}_{s+1}^j)$ belong to the classes $X((M \backslash \mathring{D}_1^n \backslash \mathring{D}_2^n) \# N, \mathcal{M})$ and $X((M \backslash \mathring{D}_1^n \backslash \mathring{D}_2^n) \# \bar{N}, \mathcal{M})$ respectively where D_1^n and D_2^n are two disjoint n-disks in M, and N and $\bar{N}$ are both connected sums of a finite number of manifolds from $\mathcal{M}$ (even though not necessarily $N = \bar{N}$). So by Corollary (4.7) the required homeomorphism h_{s+1}^j exists. Then we define $h_{s+1} \colon X_{s+1} \to \bar{X}_{s+1}$ by $h_{s+1}|X_s = h_s$ and $h_{s+1}|\alpha_{k_s}^{-1}(Y_s^j) \backslash \alpha_{k_{s+1}}^{-1}(\mathrm{Int}\, Y_{s+1}^j) = h_{s+1}^j$ for $j \leqslant m$.

Finally, we define $h \colon X \to X$ by $h|X_s = h_s$ and $h(p_j) = \bar{p}_j$ for $j \leqslant m$.

9. The spaces $X(L^n, \mathcal{M}^n)$ as fixed-point sets of topological Lie group actions on manifolds. In this section we show how the spaces $X(L^n, \mathcal{M}^n)$ can naturally appear as fixed-point sets of topological actions of compact Lie groups on manifolds.

EXAMPLE (9.1). Let $\mathcal{M}^3$ be any finite or countable family of homology 3-spheres, and let $L^3 = S^3$. We take $e_L\colon S^3 \subsetneq S^4$ to be the standard inclusion. By [F], Theorem (1.4), p. 367, for every $M \in \mathcal{M}^3$ there exsits a topological, bicollared (but not always smooth) embedding $e_M\colon \mathring{M}\backslash D^3 \to D^4$ such that there exists an involution $i_M\colon D^4 \to D^4$ with $\mathrm{Fix}(i_M) = e_M(M\backslash \mathring{D}^3)$. Now, applying the construction from Section 6 to the space $X = X(S^3, \mathcal{M}^3)$, we get an embedding

$$e_X\colon X \to S^4.$$

We have the standard orientation-reversing involution $i\colon S^4 \to S^4$ with $\mathrm{Fix}(i) = S^3$, and for every $\mathcal{I} = \{i_1, \ldots, i_m\}$ we have an involution

$$i_{\mathcal{I}}\colon e^{\mathcal{I}}(D^4) \to e^{\mathcal{I}}(D^4)$$

with $\mathrm{Fix}(i_{\mathcal{I}}) = e^{\mathcal{I}}(D^4) \cap e_L^m(L_m) = e_L^m(\alpha_{k_{m-1}+1,k_m+1}^{-1}(Y_{i_1,\ldots,i_m}))$ (see c(m), Section 6). Moreover, we can assume that for any $\mathcal{I}$ we have

$$i_{\mathcal{I}}|e^{\mathcal{I}}(\partial D^4) = i_{\{i_1,\ldots,i_{m-1}\}}|e^{\mathcal{I}}(D^4).$$

Of course $i_{\mathcal{I}}$ is an involution induced by $i_{M_{j_m}}$, where $j_m = j_m(i_1, \ldots, i_m)$.

Now we can define an involution $i_X\colon S^4 \to S^4$ by

$$i_X = i \quad \text{on} \quad S^4 \backslash \bigcup_{i_1 \in N} e^{\{i_1\}}(D^4),$$

$$i_X = i_{\mathcal{I}} \quad \text{on} \quad e^{\mathcal{I}}(D^4) \backslash \bigcup_{i_{m+1} \in N} e^{\{i_1,\ldots,i_m,i_{m+1}\}}(D^4),$$

where $\mathcal{I} = \{i_1, \ldots, i_m\}$. Of course $e_X(X) = \mathrm{Fix}(i_X)$. The quotient space S^4/i_X consists of a contractible, non-compact 4-manifold and a "compactifying" cohomology manifold X. As was discovered by Ancel and Siebenmann, for certain homology spheres M, $X(S^3, \{M\})$ compactifies a simply connected 4-manifold of Davis which covers a closed 4-dimensional manifold.

EXAMPLE (9.2). We take again $L^3 = S^3$, and let $\mathcal{M}^3$ be a family of homology 3-spheres. Let $M \in \mathcal{M}^3$ and let $e_M\colon M\backslash \mathring{D}^3 \to D^5$ be a smooth embedding with $e_M(\partial D^3) \subset \partial D^5$. By [M–Y] if every $M \in \mathcal{M}^3$ bounds a contractible manifold, then there is an action φ_m of S^1 on D^5, standard on ∂D^5, and such that $e_M(M\backslash \mathring{D}^3) = \mathrm{Fix}(\varphi_M)$. Using the same procedure as in (9.1) with involutions replaced by S^1-actions, we get an S^1-action φ_X on S^5 with $\mathrm{Fix}(\varphi_X) = e_X(X)$ where $e_X\colon X \to S^5$ is the embedding of the space $X = X(S^3, \mathcal{M}^3)$ described in Section 6.

The two examples show the way of obtaining a great variety of topological Lie group actions on manifolds with fixed-point sets $X(L^n, \mathcal{M}^n)$ and orbit spaces quotients which are manifolds compactified by $X(L^n, \mathcal{M}^n)$.

10. The space $X(S^3, \{H^3\})$ as the orbit space of an action of an infinite 0-dimensional compact group. In [Ko] Kolmogorov gives an example of a 2-adic group acting effectively on a 1-dimensional locally connected continuum so that the orbit space is the 2-dimensional continuum of Pontryagin (see also [W]).

Here we give another similar example of an effective action of an infinite compact 0-dimensional group on a locally connected continuum P such that the orbit space is the cohomology manifold $X(S^3, \{H^3\})$. Here H^3 is a Poincaré homology sphere (see [K–S]). First we construct P: let φ be an action of the binary icosahedral group I^* on S^3 so that S^3/I^* is a Poincaré 3-sphere H^3, and the projection onto the orbit space of this action $\pi_\varphi\colon S^3 \to S^3/I^* = H^3$ is a 120-fold covering. Let $D^3 \subset H^3$ be a bicollared 3-cell. Then we take the space $R' = S^3\backslash \pi_\varphi^{-1}(\mathring{D}^3)$ which is a 3-sphere with 120 holes. We identify all the components of $\partial R'$ which consists of 120 copies of a 2-sphere, to one 2-sphere. We get a projection $\pi\colon R' \to R$ onto a space R. R admits an action φ' of I^* with quotient map $\pi_{\varphi'}\colon R \to H^3\backslash \mathring{D}^3$ such that the following diagram commutes:

(10.1)
$$\begin{array}{ccc} R' & \xrightarrow{\ \pi\ } & R \\ & {\scriptstyle \pi_\varphi|R'} \searrow \quad \swarrow {\scriptstyle \pi_{\varphi'}} & \\ & H^3\backslash \mathring{D}^3 & \end{array}$$

Let $\partial R = \pi(\partial R')$. Now we construct P in the following way: we remove from S^3 the union of interiors of a dense, countable null family of bicollared 3-cells. We get a space Z_1 with "boundary" consisting of a countable family of 2-spheres. Then we attach to Z_1 a countable family $\{R_{i_1}\}_{i_1 \in N}$ of copies of R so that each ∂R_{i_1} is identified with one of the 2-spheres in the "boundary" of Z_1. We also require that the diameters of the sets R_{i_1} converge to 0, so that $\{R_{i_1}\}_{i_1 \in N}$ is a null family. The action φ' of I^* on R gives rise to an action φ_{i_1} on each R_{i_1} with $\mathrm{Fix}(\varphi_{i_1}) = \partial R_{i_1}$. We get a space $P_1 = Z_1 \cup \bigcup_{i_1 \in N} R_{i_1}$ and an action φ_1' of I^* on P_1 with $\varphi_1'|R_{i_1} = \varphi_{i_1}$ and $\mathrm{Fix}(\varphi_1') = Z_1$. Then from each manifold $R_{i_1}\backslash \partial R_{i_1}$ we remove the union of interiors of a dense, countable null family of 3-cells. This time we also require that this family be φ_{i_1}-invariant. We get a new space Z_2 and a countable number of 2-spheres in the boundary, and to each of the 2-spheres contained in R_{i_1}, $i_1 \in N$, we attach a copy R_{i_1,i_2} of R. Of course we must ensure that $\{R_{i_1,i_2}\}$ is a null family for every $i_1 \in N$. Again the action φ' of I^* on R gives rise to an action φ_{i_1,i_2} of I^* on every R_{i_1,i_2} with $\mathrm{Fix}(\varphi_{i_1,i_2}) = \partial R_{i_1,i_2}$. These actions, together with the actions φ_{i_1}, $i_1 \in N$, give an effective action φ_2' of $I^* \times I^*$ on the space $P_2 = Z_2 \cup \bigcup_{i_1,i_2 \in N} R_{i_1,i_2}$.

We continue the same procedure infinitely many times, obtaining the spaces P_n with an effective action φ_n' of $I^* \times \ldots \times I^*$. Then we put $P = \varprojlim \{P_n, \alpha_{n,n+1}\}$, where $\alpha_{n,n+1}$ is equal to id on Z_{n+1}. It is easy to see that we have a natural action φ_∞ of A_∞ on P, where

$$A_\infty = \varprojlim \{(I^*)^n, p_{n,n+1}\}, \quad (I^*)^n = I^* \times \ldots \times I^*$$

and $p_{n,n+1}\colon (I^*)^{n+1} \to (I^*)^n$ is given by $p_{n,n+1}(q_1, \ldots, q_n, q_{n+1}) = (q_1, \ldots, q_n)$.

It follows from the commutativity of (10.1) that the orbit space of φ_∞ is homeomorphic to $X(S^3, \{H^3\})$, where H^3 is the homology sphere of Poincaré.

11. An uncountable family of homogeneous cohomology 3-manifolds. Let now $\mathcal{M}^3$ be a countable family of 3-dimensional irreducible homology spheres such that no two elements of $\mathcal{M}^3$ have the same fundamental group. There is an abundance of such families. Brieskorn homology spheres [M2] can serve as a good example. The aim of this section is to prove the following theorem, yielding the existence of uncountably many different spaces $X(S^3, \mathcal{M}^3)$ which are cohomology manifolds.

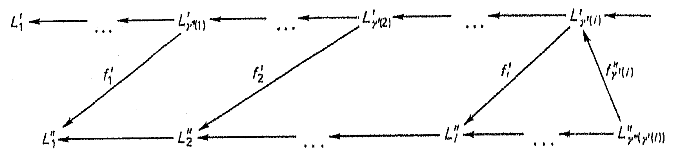
THEOREM (11.1). *If $\mathcal{M}'$ and $\mathcal{M}''$ are two subfamilies of $\mathcal{M}^3$ and $\mathcal{M}' \neq \mathcal{M}''$ then the spaces $X' = X(S^3, \mathcal{M}')$ and $X'' = X(S^3, \mathcal{M}'')$ are not homeomorphic.*

Proof. We have either $\mathcal{M}' \setminus \mathcal{M}'' \neq \emptyset$ or $\mathcal{M}'' \setminus \mathcal{M}' \neq \emptyset$, so suppose that $M \in \mathcal{M}'' \setminus \mathcal{M}'$, and let $\{\mathcal{M}', L_i', \alpha_{i,i+1}', \Omega_i'\}$ and $\{\mathcal{M}'', L_i'', \alpha_{i,i+1}'', \Omega_i''\}$ be defining systems for X' and X'' respectively.

Suppose that X' and X'' are homeomorphic, and consequently have the same Borsuk shape ([B] or [M–S]). That means that there exist increasing maps γ': $N \to N$, γ'': $N \to N$ of the set N of positive integers, and families of maps

$$\underline{f}' = \{f_n' \colon L_{\gamma'(n)}' \to L_n''\}_{n \in N} \quad \text{and} \quad \underline{f}'' = \{f_n \colon L_{\gamma''(n)}'' \to L_n'\}_{n \in N}$$

such that $f_k' \alpha_{\gamma'(k),\gamma'(k+1)}' \simeq \alpha_{k,k+1}'' f_{k+1}'$, $f_k'' \alpha_{\gamma''(k),\gamma''(k+1)}'' \simeq \alpha_{k,k+1}' f_{k+1}''$ for any $k \in N$, and for every $i \in N$ there exists $k \in N$ such that $f_i' f_{\gamma'(i)}'' \alpha_{\gamma''(\gamma'(i)),k}'' \simeq \alpha_{i,k}''$ and $f_i'' f_{\gamma''(i)}' \alpha_{\gamma'(\gamma''(i)),k}' \simeq \alpha_{i,k}'$.



Take any i and $k > i$ such that the last two homotopies hold. Then $L_k'' = L_i'' \# P$ for some homology manifold P, and consequently $\pi_1(L_k'') = \pi_1(L_i'') * \pi_1(P)$ (the choice of base points is irrelevent in our discussion). The map

$$\alpha = (\alpha_{i,k})_\# \colon \pi_1(L_k'') \to \pi_1(L_i'')$$

is a contraction of $\pi_1(L_i'') * \pi_1(P)$ onto $\pi_1(L_i'')$ given by $\alpha(a_1 b_1\ a_2 b_2 \ldots) = a_1 a_2 \ldots$, and of course it is surjective. This implies that $(f_i')_\# (f_{\gamma'(i)}'')_\# (\alpha_{\gamma''(\gamma'(i)),k}'')_\# = (\alpha_{i,k}'')_\#$ is surjective, and consequently $(f_i')_\#$ is surjective. So we have proved that for any i, $(f_i')_\#$ and analogously $(f_i'')_\#$ are surjective. Set $H_M = \pi_1(M)$. Every L_i'' is a connected sum of elements of $\mathcal{M}''$, and there must exist an index i such that L_i'' contains a summand homeomorphic to M. Again, let $k \in N$ be such that $f_i' f_{\gamma'(i)}'' \alpha_{\gamma''(\gamma'(i)),k}'' \simeq \alpha_{i,k}''$; then $L_i'' = M \# \bar{P}$ and let $L_k'' = M \# P \# \bar{P}$, and so $\pi_1(L_i'') = H_M * \bar{H}$, $\pi_1(L_k'') = H_M * \bar{H} * \bar{\bar{H}}$, where $\bar{H} = \pi_1(P)$, $\bar{\bar{H}} = \pi_1(\bar{P})$. Set $\pi_1(L_{\gamma'(i)}') = G$. Then we have maps

$$H_M * \bar{H} \xleftarrow{\beta} G \xleftarrow{\xi} H_M * \bar{H} * \bar{\bar{H}}$$

where $\beta = (f_i')_\#$, $\xi = (f_{\gamma'(i)}'')_\# (\alpha_{\gamma''(\gamma'(i)),k}'')_\#$. As we have mentioned both maps are surjective, and $\alpha = \beta\xi$ is a contraction of $(H_N * \bar{H}) * \bar{\bar{H}}$ onto $H_M * \bar{H}$. Set $G_M = \xi(H_M)$ and $\bar{G} = \xi(\bar{H} * \bar{\bar{H}})$. We have $\alpha | H_M = \mathrm{id}_{H_M}$ and $\beta | G_M \colon G_M \to H_M$ is an isomorphism. Also $G_M \cap \bar{G} = \{1\}$, because $\alpha(\bar{H} * \bar{\bar{H}}) = \beta(G) = \bar{H}$. The fact that ξ is surjective now implies that $G = G_M * \bar{G} \approx H_M * \bar{G}$, so $\pi_1(L_{\gamma'(i)}') \approx G * H_M$. The Kneser conjucture ([H]), yields

that $L_{\gamma'(i)}' = M \# M'$ for some closed 3-manifold M'. But there is no manifold homeomorphic to M in $\mathcal{M}'$, so using Milnor's theorem about the uniqueness of a decomposition of a 3-manifold into a connected sum we get a contradiction.

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