Contents of Volume 137, Number 2

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Published by PWN—Polish Scientific Publishers

ISBN 83-01-10242-X ISSN 0016-2736

W R O C Ł A W S K A D R U K A R N I A N A U K O W A

On the representation type of triangular matrix algebras over special algebras

by

Zbigniew Leszczyński (Toruń)

Abstract. In the paper the representation type of triangular matrix algebras over finite-dimensional radical-square-zero algebras is considered. The characterization of algebras with associated triangular algebras of finite (tame) type is given in terms of Gabriel quivers.

1. In this paper all agebras are finite-dimensional over a fixed algebraically closed field k. We call

$$T = T(A) = \begin{bmatrix} A & A \\ 0 & A \end{bmatrix}$$

a *triangular matrix algebra* over an algebra A. Our objective is to show that if $J^2(A) = 0$ (where $J(A)$ is the Jacobson radical of A), then T and $T_0 = T_0(A) = T/J^2(T)$ have the same representation type; this term is restricted in this paper to mean finiteness, tameness, or wildness. We recall that an algebra is *of finite type* [5] if the category A-mod has only finite number of isoclasses of indecomposable modules. A is called *of wild type* if there is an exact embedding $k\langle X, Y\rangle$-mod $\to A$-mod [14]. An algebra is said to be *of tame type* if it is neither of wild nor of finite type. The reader is referred to [6] for the discussion of equivalent formulations of tame type.

One can easily see that the algebra A is basic (connected) if and only if the algebra $T(A)$ is. From [7] we know that a basic radical-square-zero algebra is a bound quiver algebra of the form $A = kQ/(kQ_1^+)^2$, where Q is the Gabriel quiver of the algebra A. So we may assume without loss of generality that our algebras are connected bound quiver algebras of the form $kQ/(kQ_1^+)^2$, for some finite quiver Q. Our main result is

THEOREM 1.1. *Let $A = kQ/(kQ_1^+)^2$, for some finite quiver Q. Then $T(A)$ is of finite (tame) type iff the agebra $T_0(A)$ is of finite (resp. tame) type.*

Let us recall the following useful criterion:

THEOREM 1.2. *If A is an algebra with $J^2(A) = 0$, then*

(a) *A is of finite type iff the separated quiver of A (for the definition see* [1] *and Section 2) is a disjoint union of Dynkin quivers of Fig.* 1 *and their duals.*

(b) *A is of tame type iff the separated quiver of A is a disjoint union of Dynkin quivers, extended Dynkin quivers of Fig. 2 and the duals of the latter, with at least one extended Dynkin quiver in the union (in both cases* $\bullet - \bullet$ *means either* $\bullet \to \bullet$ *or* $\bullet \leftarrow \bullet$ *and each point is either a source or a sink).*

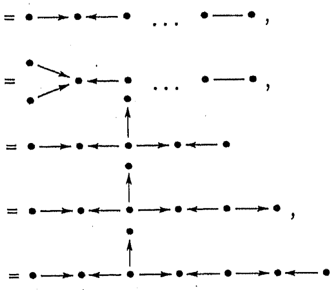


Fig. 1

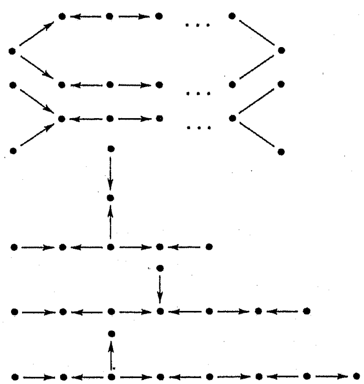


Fig. 2

This together with Theorem 1.1 gives us a diagramatic criterion for determining the representation type of $T(A)$. Moreover, we shall give in Propositions 3.4, 3.5, Remark 3.7 and Lemmas 4.4, 4.5 an effective reduction procedure for description of indecomposable T-modules in case T is not of wild type.

The representation type of triangular matrix algebras has been studied in [1, 2, 11, 12, 15].

2. Preliminaries. By the *Gabriel quiver* of a basic algebra A we mean the quiver $Q(A) = (Q_0, Q_1)$ defined as follows. Let $A/J(A) \simeq k_1 \times \ldots \times k_n$; $k_j = k$; $J(A)/J^2(A)$ $= \bigoplus_{i,j=1}^{n} k_i(J(A)/J^2(A))k_j$. We put $Q_0 = \{1, \ldots, n\}$ and we have f_{ij} arrows from i to j in Q_1, where $f_{ij} = \dim_k k_i(J(A)/J^2(A))k_j$ [7]. With a radical-square-zero algebra A we associate the hereditary algebra

$$H = H(A) = \begin{bmatrix} A/J(A) & J(A) \\ 0 & A/J(A) \end{bmatrix}$$

and define the *separated quiver* of A by $S(A) = Q(H(A))$. The functor

$$M \mapsto \begin{bmatrix} J(A)M \\ M/J(A)M \end{bmatrix}$$

from A-mod to H-mod relates the representation problem for A to the same problem for H [1]. Hence Theorem 1.2 follows from the results in [3, 7, 8].

For a locally finite quiver Q we take $A = kQ/(kQ_1^+)^2$ (for a finite quiver Q this is a f.d. algebra with unit) and we define the *triangular quiver* $T(Q)$ over Q by $T(Q) = Q(T(A))$.

Observe that we obtain $T(Q)$ by splitting every $i \in Q_0$ into two vertices $i, i' \in (T(Q))_0$, every arrow α: $i \to j$ into two arrows α: $i \to j$ and α': $i' \to j'$ in $(T(Q))_1$ and inserting for every $i \in Q_0$ a new arrow γ_i: $i \to i'$ in $(T(Q))_1$. This means that $T(Q)$ is the disjoint union of Q and $Q' = Q$ connected by the arrows γ_i: $i \to i'$.

Fig. 3

Fig. 4

EXAMPLE 2.1. For a loop L (Fig. 3), $T(L)$ is shown in Fig. 4.

LEMMA 2.2. *Let $A = kQ/(kQ_1^+)^2$ for a locally finite quiver Q. Then*

$$Q(T(A^{\mathrm{op}})) = T(Q^{\mathrm{op}}) = (T(Q))^{\mathrm{op}} = Q((T(A))^{\mathrm{op}}).$$

This is straightforward from the definition.

A subquiver C of Q will be called a *ϱ-connected component* if $C_1 \subset Q_1$ is an equivalence class of the minimal equivalence relation such that

(a) if α: $x \to i$, β: $x \to j$ then $\alpha\varrho\beta$,

(b) if α: $i \to y$, β: $j \to y$ then $\alpha\varrho\beta$.

Remark 2.3. The set of ϱ-connected components of a quiver Q is equal to the set of supports of indecomposable representations of $kQ/(kQ_1^+)^2$.

3. T-splitting operation. Let Q denote a locally finite quiver (i.e. every point has only a finite number of neighbors) and let $C = A_n$, $2 \leqslant n \leqslant 4$, denote a ϱ-connected component of Q. The quiver Q is called T-splittable (at the ϱ-connected component C) if there are $x, y \in C_0$ such that x is a source in C but not in Q, and y is a sink in C but not in Q.

In this section we define a T-splitting operation by associating with every T-splittable quiver Q a quiver Q' having exactly one T-splittable ϱ-connected component less than Q. We shall prove that the quivers $S(T(Q))$ and $S(T(Q'))$ $(T(Q)$ and $T(Q'))$ are of the same representation type. In the proofs we use the covering technique (see [4, 5, 10]).

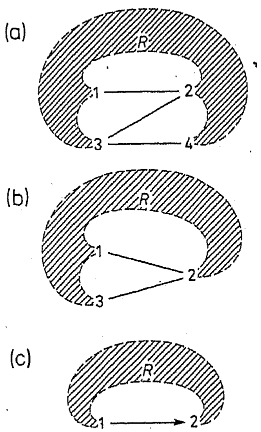


Fig. 5

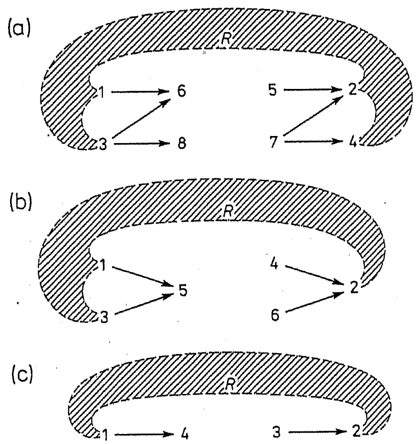


Fig. 6

Let now Q be a T-splittable quiver at the ϱ-connected component $C = A_n$ and let the set Q_0 be indexed by elements of a subposet of the poset $N \setminus \{n+1, \ldots, 2n\}$ (where N denotes the set of natural numbers). Also let Q have the form of Fig. 5 (a), (b), (c) for $n = 4, 3, 2$ respectively, where R is a subquiver of Q such that $Q_1 = R_1 \cup C_1$ and

for (a): $R_0 \cap C_0 \cap \{2, 4\} \neq \emptyset$, $R_0 \cap C_0 \cap \{1, 3\} \neq \emptyset$,
for (b): $R_0 \cap C_0 \cap \{1, 3\} \neq \emptyset$, $R_0 \cap C_0 \supset \{2\}$,
for (c): $R_0 \cap C_0 = \{1, 2\}$.

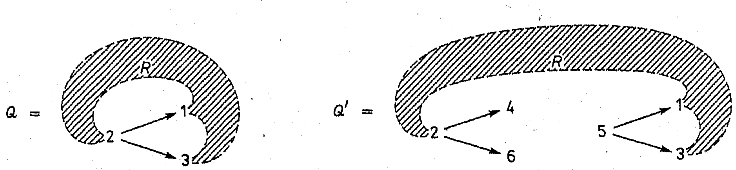


Fig. 7

Then we define the quiver Q' by

$$Q'_0 = Q_0 \cup \{n+1, \ldots, 2n\},$$
$$Q'_1 = R_1 \cup \{(i, j+n), (i+n, j);\ \text{for } (i, j) \in C_1\},$$

i.e. Q' has the respective presentation of Fig. 6.

Moreover, there is a version of the T-splitting operation for the case (b′): $C = A_3^{\mathrm{op}} = 1 \leftarrow 2 \rightarrow 3$ shown in Fig. 7.

Remark 3.1. If the quiver R is not connected (in the usual sense), neither is the quiver Q'.

Remark 3.2. The T-splitting operation is commutative, i.e. for two T-splitting ϱ-connected components, the result of successive T-splitting operations does not depend on the order.

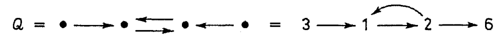


Fig. 8

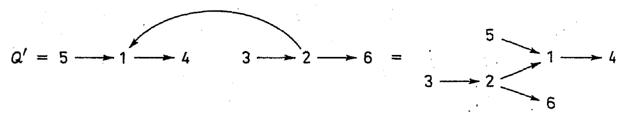


Fig. 9

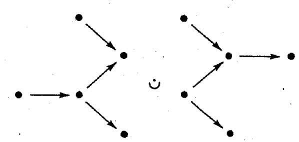


Fig. 10

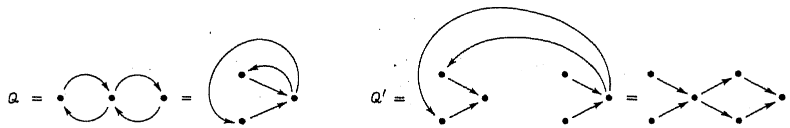


Fig. 11

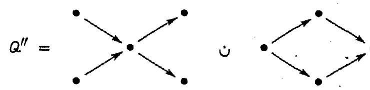


Fig. 12

EXAMPLE 3.3. (a) For Q of Fig. 8, the result of the T-splitting operation for $C = 1 \to 2$ is shown in Fig. 9. Using again the T-splitting operation on the quiver Q' at the ϱ-connected component

$6 \leftarrow 2 \to 1 \leftarrow 5$ we obtain the disconnected quiver of Fig. 10.

(b) Another example is shown in Fig. 11. Using again the T-splitting operation we obtain Fig. 12.

PROPOSITION 3.4. *Suppose the quiver Q is finite or the subquiver R (see above) is disconnected. Let Q' be a quiver obtained by applying the T-splitting operation to Q. Then*

(a) *the quivers $S(T(Q))$, $S(T(Q'))$ are of the same representation type;*

(b) *the quivers $T(Q)$, $T(Q')$ are of the same representation type.*

Proof. (a) follows from the equality $S(T(Q')) = S(T(Q)) \dot{\cup} D \dot{\cup} E$, where D, E are connected components of the quiver $S(T(Q'))$ which are of finite type and are isomorphic to the quiver D_6. Indeed, by the definition of these quivers we have the equalities of Fig. 13.

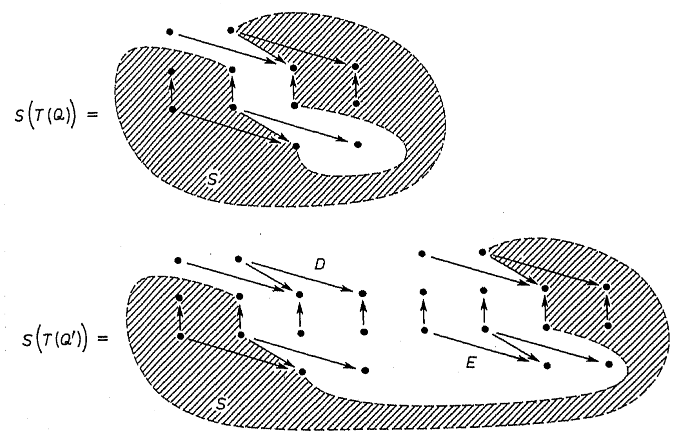


Fig. 13

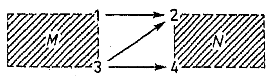


Fig. 14

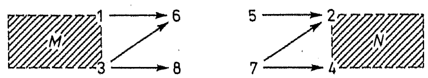


Fig. 15

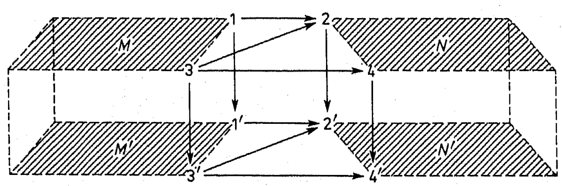


Fig. 16

(b) Let now $C = A_4$ be a T-splitting ϱ-connected component.

(i) First assume that Q is as shown in Fig. 14. Then Q' and $T(Q)$ are illustrated in Figs. 15 and 16 respectively. By straightforward computation or by the one-point extension method [14] one can prove the following statements.

(I) Assume that V is an indecomposable representation of $T(A_4 \cup N)$ such that for some arrow $\alpha \in N_1$ we have either $\tilde{\alpha} \neq 0$ or $\tilde{\alpha}' \neq 0$, where $\tilde{\alpha}(\tilde{\alpha}')$ is the inner linear morphism of V lying on the arrow $\alpha(\alpha')$. Then $V_3 = 0$.

(II) Assume that V is an indecomposable representation of $T(M \cup A_4)$ such that for some arrow $\alpha \in M_1$ we have either $\tilde{\alpha} \neq 0$ or $\tilde{\alpha}' \neq 0$. Then $V_{2'} = 0$.

(III) If V is an indecomposable representation of $T(M \cup A_4 \cup N)$ satisfying $\operatorname{Supp} V \cap T(M) \neq 0$ and $\operatorname{Supp} V \cap T(N) \neq 0$, then $\operatorname{Supp} V \subset T(A_4)$.

(ii) Let now $C = A_4$ be a T-splitting ϱ-connected component of Q shown in Fig. 5 (a). We take the covering quiver $\tilde{Q}$ of Q depicted in Fig. 17. After using twice the T-splitting operation, we know that the support of an indecomposable representation V of $T(\tilde{Q})$ is contained in the triangular quiver over the quiver of Fig. 18. Hence $T(\tilde{Q})$ is locally support finite. Therefore the quivers $T(F)$, $T(\tilde{Q})$, $T(Q)$ are of the same representation type by [4].

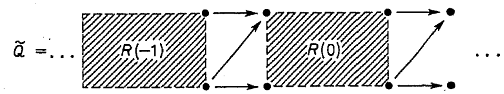


Fig. 17

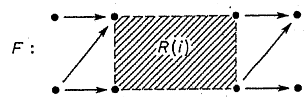


Fig. 18

The version of the proposition for cases (b) and (c) of Fig. 5 follows from the above case (a) because the quivers Q and Q' are subquivers of the corresponding quivers of case (a). Case (b′) is dual to (b).

PROPOSITION 3.5. *Let Q be a finite quiver shown in Fig. 19 with a ϱ-connected component LA_2 (Fig. 20), where the point 2 is a source in the quiver R. We associate with Q the quiver Q' of Fig. 21. Then*

(a) *the quivers $S(T(Q))$, $S(T(Q'))$ are of the same representation type,*

(b) *the quivers $T(Q)$, $T(Q')$ are of the same representation type.*

The correspondence between the quivers Q and Q' will also be called a *T-splitting operation*, so each ϱ-connected component LA_2 is always T-splitting.

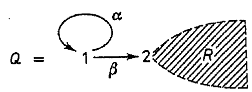


Fig. 19

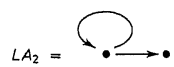


Fig. 20

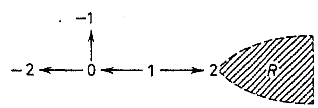


Fig. 21

Proof. (a) Similarly to the proof of Proposition 2.4 one can prove the equality

$$S(T(Q')) = S(T(Q)) \,\dot\cup\, A_5 \,\dot\cup\, D_4.$$

(b) For the quiver Q we denote by $\tilde{Q}$ the covering quiver which unrolls the loop α at the point 1. $\tilde{Q}$ is shown in Fig. 22. For every integer i the ϱ-connected component $\bullet\xleftarrow{\alpha(i)}\bullet\xrightarrow{\beta(i)}\bullet$ is T-splitting in $\tilde{Q}$. After iteration of the T-splitting operation we know that any indecomposable representation in $r(T(\tilde{Q}))$ has support contained in the triangular quiver over the quiver $N(i)$ (Fig. 23) for some integer i. Hence $T(Q')$ is locally support finite. This finishes the proof because $N(i) \simeq Q'$ (see [4]).

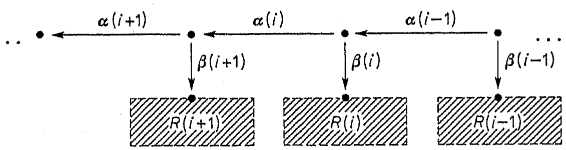


Fig. 22

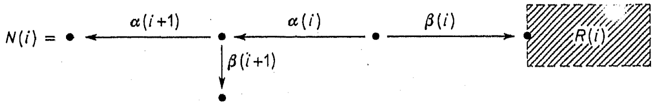


Fig. 23

Remark 3.6. The above two propositions are also true for an arbitrary locally finite quiver Q. One can prove this by straightforward computations (without the use of [4]).

Remark 3.7. By iterated use of T-splitting operations one can describe the supports of indecomposable representations of triangular quivers and simultaneously determine the representation type.

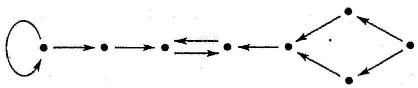


Fig. 24

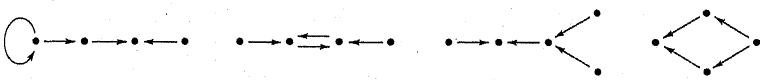


Fig. 25

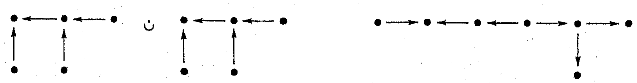


Fig. 26

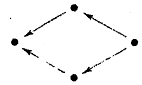


Fig. 27

For example, the quiver Q of Fig. 24 has five ϱ-connected components and for an indecomposable representation $V \in r(T(Q))$ the support of V is contained in the triangular quiver of the quivers of Fig. 25. The triangular quiver over the square (Fig. 27) is of tame type (see [11] and the proof of Theorem 1.1). The other triangular quivers are of finite type. Hence $T(Q)$ is tame.

4. Some necessary conditions for non-wildness. For an algebra A the algebra $T_0(A)$ is a factor algebra of $T(A)$. If $T_0(A)$ is of wild type then so is $T(A)$.

Lemma 4.1. *Let $A = kQ/(kQ_1^+)^2$ be such that $T_0(A)$ is not of wild type. Then the quiver Q has neither double arrows nor double loops.*

Proof. Indeed, if Q has as subquiver one of the quivers of Fig. 28, then $T(Q)$ has as subquiver one of the wild quivers of Fig. 29 and $S(T(Q))$ has a subquiver $\bullet\leftarrow\bullet\rightrightarrows\bullet$.

Fig. 28

Fig. 29

Proposition 4.2. *Let $A = kQ/(kQ_1^+)^2$ be a bound quiver algebra such that $T_0(A)$ is not of wild type.*

(a) *Any ϱ-connected component of Q or Q^{op} has one of the forms*:

$$A_j = 1\to 2\leftarrow 3\to 4\ldots(j-1)—j, \quad 1\leqslant j\leqslant 5$$

where $(j-1)—j$ means $(j-1)\to j$ for $(j-1)$ a source, and $(j-1)\leftarrow j$ for $(j-1)$ a sink, D_4 (see Fig. 1), $\tilde{A}_{1,2}$ (Fig. 30), L (Fig. 3), LA_2 (Fig. 20).

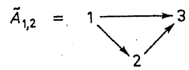


Fig. 30

(b) *Let Q be a connected quiver. If C or C^{op} is a subquiver of Q of one of the forms A_5, D_4, $\tilde{A}_{1,2}$, then $C = Q$.*

Proof. (b) Suppose to the contrary that $C \neq Q$. If C is one of A_5, A_5^{op}, D_4, D_4^{op}, $\tilde{A}_{1,2}$, then $S(T(Q))$ contains as subquiver at least one of one of tame quivers $\tilde{D}_7$, $\tilde{E}_7$, $\tilde{D}_4$, $\tilde{E}_6$, $\tilde{D}_{10}$. It is sufficient to prove that for $C \neq Q$ the quiver $S(T(Q))$ contains an extension of one of the above extended Dynkin quivers which is by Theorem 1.2 of wild type, contrary to our assumption on T_0. We present the proof only in the case $C = D_4$, in the remaining cases the proof is similar. So suppose $C = D_4 \subseteq Q$. If we attach a new arrow then we get a subquiver $D \supset C$ of Q of one of the forms shown in Fig. 31 and $S(T)$ contains $S(T(Q))$ of one of the forms shown in Fig. 32, each of wild type by Theorem 1.2.

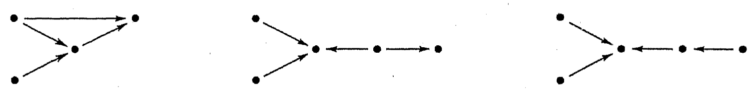


Fig. 31

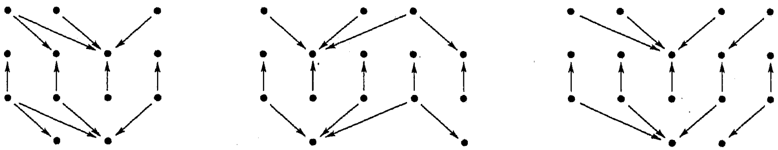
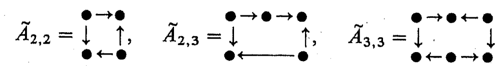


Fig. 32

(a) First we note that Q has no ϱ-connected component C of type $\tilde{A}_m$ with $m \geqslant 4$. For, if Q contains such a subquiver, then from the definition of ϱ it follows that Q contains a subquiver of one of the forms



or $\tilde{A}_{p,q}$ with $p+q \geqslant 7$, where $\tilde{A}_{p,q}$ is a non-splittable oriented cycle such that p (q) arrows have the same orientation. If Q contains a subquiver $\tilde{A}_{m,n}$, then $T(Q)$ contains a subquiver $\tilde{\tilde{A}}_{m,n}$ obtained from $\tilde{A}_{m,n}$ by attaching a new arrow. This contradicts the assumption on T_0 because of Theorem 1.2 and Lemma 2.2. If Q contains $\tilde{A}_{p,q}$ with $p+q \geqslant 7$, then Q contains a subquiver A_6 and T_0 is of wild type because $S(T_0)$ contains $S(T(A_6))$ illustrated in Fig. 33. If Q contains $\tilde{A}_{2,3}$ then $S(T(Q))$ contains the wild quiver of Fig. 34.

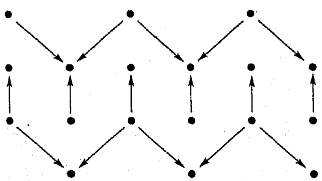


Fig. 33

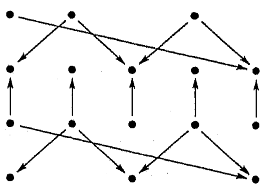


Fig. 34

Let C be a ϱ-connected component of Q. First we suppose that C contains no loop. If C contains a cycle then according to (b) and the statement proved above we have $C = \tilde{A}_{1,2}$. If C has no cycle then either D_4 (D_4^{op}) is contained in C or C is linear. Then (a) follows from (b).

Next suppose C contains a loop L. If C is neither L or LA_2 (nor $(LA_2)^{op}$) then remembering that Q has no double arrows, Q contains one of the quivers of Fig. 35 and therefore $S(T_0(Q))$ contains the respective quiver of Fig. 36. It follows from Theorem 1.2 that T_0 is of wild type contrary to our assumption. This finishes the proof.

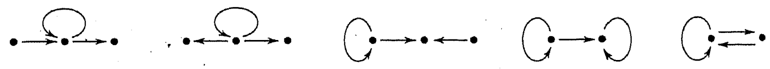


Fig. 35

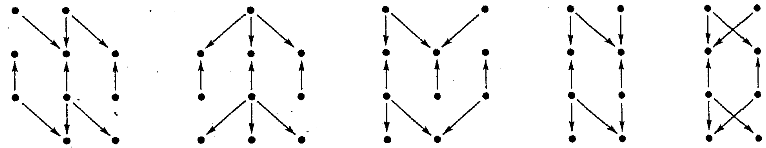
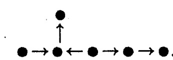


Fig. 36

LEMMA 4.3. *Let Q be one of the quivers A_5, D_4, $\tilde{A}_{1,2}$. Then the quivers $S(T(Q))$, $T(Q)$, $T_0(Q)$ are of tame type.*

Proof. By Theorem 1.2 we know that $T_0(Q)$ has the same representation type as $S(T(Q))$. One can easily see that in all cases $S(T(Q))$ is of tame type. The tameness of $T(A_5)$, $T(D_4)$, $T(\tilde{A}_{1,2})$ has been proved in [11, 15].

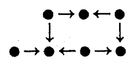
LEMMA 4.4. *Suppose that Q is not a T-splittable quiver and that T_0 is not of wild type. Denote by N the quiver*



(a) *If $C = A_4 = 1 \to 2 \leftarrow 3 \to 4$ is a ϱ-connected component such that neither of the points 2, 4 is a sink in Q, then $Q = N$.*

(b) *If $C = A_4$ as above is a ϱ-connected component such that neither of the points 1, 3 is a source in Q, then $Q = N^{op}$. Moreover, the quiver $T(Q)$ is of finite type.*

Proof. Since 2, 4 are not sinks in Q, we may choose arrows α: $2 \to a$, β: $4 \to b$ with $a, b \in Q_0$. If we have $a = 2, 4$ ($b = 2, 4$), then T_0 is of wild type. If $a \in \{1, 3\}$ ($b \in \{1, 3\}$), then C is splittable. Hence we have $\{a, b\} \cap \{1, 2, 3, 4\} = \emptyset$. Also we have $a \neq b$; otherwise we obtain the quiver $S(T(F))$ of wild type, with F as in Fig. 37. Indeed, $S(T(F))$ is illustrated in Fig. 38 and contains the quiver



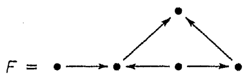


Fig. 37

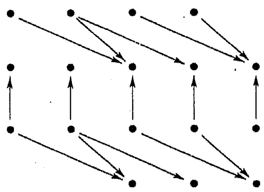


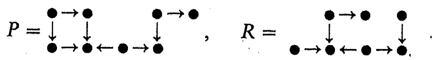
Fig. 38

of wild type. Similarly, one can easily see that, if the quiver Q has another arrow, then either Q is T-splittable or T_0 is of wild type.

The finiteness of the type of $T(N)$ can be proved starting with $r(T(A_4))$ by the one-point extension method. In this way one can show the equality

$$\operatorname{ind}\big(r(T(N))\big) = \operatorname{ind}\big(r(T(A_4))\big) \cup \operatorname{ind}\big(r(P)\big) \cup \operatorname{ind}\big(r(R)\big),$$

where



and hence the representation type of $T(N)$ is the same as that of $T(A_4)$ so it is finite [13, 12].

LEMMA 4.5. *Suppose that Q is not a T-splittable quiver and that T_0 is not of wild type. If Q has a ϱ-connected component $C = A_4 = 1 \to 2 \leftarrow 3 \to 4$ such that the point 2 is not a sink in Q (or 3 is not a source in Q), then Q is isomorphic to a subquiver of the quiver N from Lemma 4.4.*

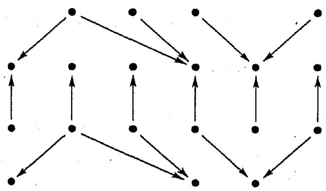


Fig. 39

Proof. Indeed, if Q contains the quiver

$$\begin{array}{ccccccc} & & 5 & \leftarrow & 6 & & \\ & & \uparrow & & & & \\ 1 & \to & 2 & \leftarrow & 3 & \to & 4 \end{array}$$

then $S(T(Q))$ is of wild type, because it contains the quiver of Fig. 39 which is a disjoint union of the quivers D_6, D_4, and the quiver $\tilde{\tilde{E}}_7$ of wild type.

5. The proof of Theorem 1.1. First observe that if Q has a loop then either

(i) $A = k[x]/(x^2)$ and $T(A)$ is of finite type [1], or

(ii) Q has a ϱ-connected component LA_2 or $(LA_2)^{op}$ and is T-splittable, or

(iii) T_0 is of wild type (see Prop. 4.2).

An epimorphism $T \to T_0$ induces an embedding T_0-mod $\subsetneq$ T-mod, hence if T_0 is of wild (not finite) type then so is T.

Another observation is that for a ϱ-connected component C equal to one of A_5, D_4, $\tilde{A}_{1,2}$ the category $r(T(C))$ is of tame type and so is $S(T(C))$ ([11, 15] and Lemma 4.3).

So by the above facts together with Lemma 4.1 and Propositions 4.2, 3.4, 3.5 we may without loss of generality assume that

(1) the quiver Q is connected, not T-splittable, has no double arrows, and has ϱ-connected components A_n only, $n \leqslant 4$,

(2) if $C = A_4 = 1 \to 2 \leftarrow 3 \to 4$ is a ϱ-connected component of Q, then 2 is a sink in Q, 3 is a source in Q (Lemmas 4.4, 4.5) and either the point 1 is a source in Q or 4 is a sink in Q (because Q is not T-splittable),

(3) if $C = A_3 = \bullet \to \bullet \leftarrow \bullet$ is a ϱ-connected component of Q then either the sink in C is a sink in Q or the two sources in C are sources in Q too. (For the dual case $C = A_3^{op} = \bullet \leftarrow \bullet \to \bullet$ either the source in C is a source in Q or there are in C two sinks in Q.)

(4) if $C = A_2 = \bullet \to \bullet$ is a ϱ-connected component, then C_0 contains at least one sink or one source of Q (we know that $T(A_2)$ is of finite type [1]).

Therefore if $C = A_4$ or $C = A_2$, then there is at most one ϱ-connected component D such that $D_0 \cap C_0 \neq \emptyset$.

Similarly, if $C = A_3 = a \to b \leftarrow c$ is a ϱ-connected component in Q (or in Q^{op}) such that

(*) $\qquad$ b is not a sink in Q (resp. a source in Q),

then there is in C at most one ϱ-connected component D such that $D_0 \cap C_0 = \{b\}$.

Let $C^{(1)}, \ldots, C^{(s)}$ denote the set of all ϱ-connected components of the quiver Q. If one of $C^{(i)}$ is equal either to A_4 or to A_2, or to A_3 (A_3^{op}) satisfying the condition (*), then putting $i = 1$ we may order the set of ϱ-connected components in such a way that

$$\operatorname{card}(C_0^{(i)} \cap C_0^{(i+1)}) = 1; \quad i = 1, \ldots, s-1.$$

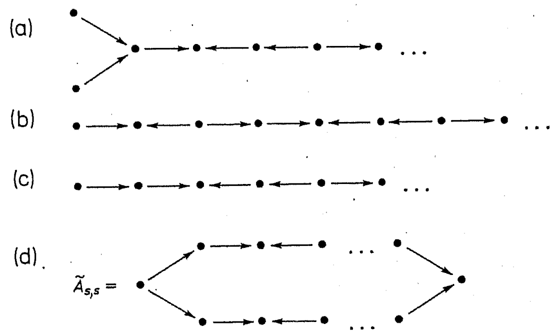


Fig. 40

Hence the starting part of Q (or Q^{op}) is one of those shown in Fig. 40 (a)–(c) and if $C^{(s)}$ is equal either to A_4, or to A_2, or to A_3 (A_3^{op}) satisfying the condition (∗), then the ending part is equal (or dual) to one of (a), (b), (c).

If for each i the quiver $C^{(i)}$ is equal or dual to A_3 and for $C^{(i)}$ the condition (∗) does not hold, then Q is a subquiver of the quiver $\tilde{A}_{s,s}$ of Fig. 40 (d).

We start the proof of the theorem with case (d). Assume that $Q = \tilde{A}_{s,s}$. It is easy to check by the one-point extension method that

$$(**)\qquad \operatorname{ind}\big(r(T(Q))\big) = \operatorname{ind}\big(r(S(T(Q)))\big) \cup \bigcup_{i=1}^{s} \operatorname{ind}\big(r(T(C^{(i)}))\big).$$

Hence $T(Q)$ has the same representation type as $S(T(Q))$ because $T(C^{(i)})$ is of finite type for $i = 1, \ldots, s$ [13, 12]. This type is tame hence $S(T(Q))$ has a connected component equal to D_4 and a component of the form $\tilde{A}_{t,t}$ (see Fig. 41), where $t = 3s+1/2$ (it is easy to see that for $C = \tilde{A}_{s,s}$, s must be even).

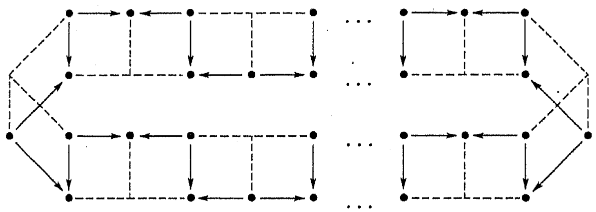


Fig. 41

If now in case (d) $Q \subsetneq \tilde{A}_{s,s}$, then similarly one can show that $T(Q)$ is of finite type.

We leave it to the reader to check that the equality (∗∗) holds for $C^{(1)} \neq A_4 \neq C^{(s)}$. Consequently the theorem follows from (∗∗). Moreover, $T(Q)$ is of finite type except in the case $C^{(1)} = A_3$ (or $C^{(1)} = A_3^{\mathrm{op}}$) and $C^{(s)} = A_3$ ($C^{(s)} = A_3^{\mathrm{op}}$), where both $C^{(1)}$, $C^{(s)}$ satisfy (∗).

Finally, assume that each of $C^{(1)}$, $C^{(s)}$ is equal either to A_4 or to A_3 (A_3^{op}) satisfying (∗). Let us study only some of the relevant cases:

(i) $C^{(1)} = A_4$, $C^{(s)} = A_4$,

(ii) $C^{(1)} = A_4$, $C^{(s)} = A_3$ (satisfying (∗)).

Then by the one-point extension method one can prove the equality

$$\operatorname{ind}\big(r(T(Q))\big) = \operatorname{ind}\big(r(X)\big) \cup \bigcup_{i=1}^{s} \operatorname{ind}\big(r(T(C^{(i)}))\big),$$

where, for (i), X is the quiver with two commutativity relations shown in Fig. 42, and, for (ii), X is the quiver with one commutativity relation of Fig. 43. In these two cases the quiver X (of type $\tilde{D}_n$) is of tame type [14] ($n = 3s+5$ for (i) and $n = 3s+2$ for (ii)) and $S(T(Q))$ has only one connected component not of finite type which is equal to $\tilde{D}_{3s+3}$ for (i) and to $\tilde{D}_{3s+1}$ for (ii). Hence, $S(T(Q))$ is also of tame type. Similarly one can prove the other cases.

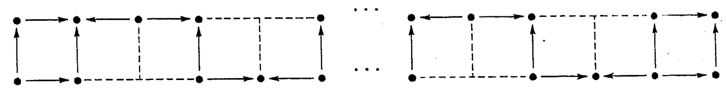


Fig. 42

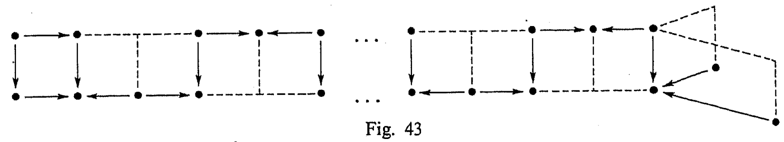


Fig. 43

We have the following corollary of the above proof:

COROLLARY 5.1. *Suppose the quiver Q is not T-splittable and $T(Q)$ is of tame type. Then either Q is equal to $\tilde{A}_{s,s}$ for some natural number s or each of $C^{(1)}$, $C^{(s)}$ (in the notation above) is equal either to A_4 or to A_3 (or to A_3^{op}) satisfying (∗).*

The author wishes to thank Prof. A. Skowroński for fruitful discussions and remarks concerning the constructions of proofs with the application of the covering technique and to Prof. D. Simson for friendly care during the preparation of the manuscript. He is also grateful to the referee for suggestions and comments.

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Received 7 March 1988;
in revised form 3 November 1988 and 25 September 1989

Homogeneous cohomology manifolds which are inverse limits

by

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Abstract. We describe a class of homogeneous cohomology manifolds.

1. Introduction. The aim of this paper is to consider a certain class of compact, finite-dimensional, homogeneous spaces which are inverse limits of topological manifolds. We say that a space X is *homogeneous* if for every $x, y \in X$ there exists a homeomorphism $h\colon X \to X$ with $h(x) = y$. The spaces that we consider depend on an orientable n-manifold L^n (with possibly empty boundary) and on a countable or finite family $\mathscr{M}^n$ of closed orientable manifolds of the same dimension n. We will denote them by $X(L^n, \mathscr{M}^n)$. A first such space was constructed in [J] for $L^3 = S^3$ and the one-element family $\mathscr{M}^3 = \{H\}$ where H was a homotopy 3-sphere $\neq S^3$, as a potential counterexample to the Bing–Borsuk conjecture([1]). Earlier similarly constructed spaces were considered in a different context in [P] and [W]. Then Ancel and Siebenmann ([A–S]) noticed that $X(S^3, H')$ where H' is some homology 3-sphere can be identified with a compactification of the Davis contractible 4-manifold which covers a closed 4-manifold [D]. They also introduced axioms describing spaces $X(L^n, \{M\})$ for the families $\mathscr{M}^n = \{M\}$ consisting of one element. Axiomatic description seems particularly useful when applied to spaces $X(L^n, \mathscr{M}^n)$ with an infinite family $\mathscr{M}$. Our axioms for general spaces $X(L^n, \mathscr{M}^n)$ presented in Section 2 differ only slightly from those of Ancel and Siebenmann. They play an important role in the proof of m-homogeneity of $X(L^n, \mathscr{M}^n)$ given in Sections 7 and 8. In Section 4 we use a lemma proved by Toruńczyk to show that the spaces $X(L^n, \mathscr{M}^n)$ depend only on L^n and $\mathscr{M}^n$, and in Section 5 we give a construction of $X(L^n, \mathscr{M}^n)$. If a family $\mathscr{M}^n$ consists of homology n-spheres then $X(L^n, \mathscr{M}^n)$ is a cohomology manifold. In this case $X(L^n, \mathscr{M}^n)$ can often be identified with the fixed-point set of a topological action on a manifold or a cohomology manifold. The theory of such actions was developed in [B]. Such homogeneous cohomology manifolds also appear as compactifications of contractible 4-manifolds, or orbit spaces of actions of 0-dimensional infinite compact groups. We give some examples in Sections 9 and 10.

([1]) I have been informed by several people that J. Martin also considered a similar construction.