We shall show that although the interior of $\varrho(F)$ is non-empty, there is no measure $\mu \in \mathcal{M}_E(F)$ with $\int \varphi_F \, d\mu = v$ when v belongs to the boundary of $\varrho(F)$, unless v is an extreme point of $\varrho(F)$ (i.e. one of the vertices of the square $[0, 1]^2$). In particular, if v belongs to the boundary of $\varrho(F)$ but is not a vertex of $[0, 1]$ and both coordinates of v are rational then there is no point $z \in \mathbf{R}^2$ with $\pi(z)$ periodic and $\varrho(F, z) = v$.

Suppose that $\mu \in \mathcal{M}_E(F)$ and

$$\int \varphi_F \, d\mu = (\alpha, 0). \tag{10}$$

Let $\varphi_F = (\varphi_1, \varphi_2)$. By the definition of F, we have $\varphi_2 \geqslant 0$, which together with (10) gives $\varphi_2 = 0$ μ-almost everywhere. This means that

$$\pi^{-1}(\operatorname{supp} \mu) \subset \mathbf{Z} \times \mathbf{R}.$$

However, $\operatorname{supp} \mu$ is f_F-invariant, so $\pi^{-1}(\operatorname{supp} \mu)$ is F-invariant. The only points of $\mathbf{Z} \times \mathbf{R}$ for which also their image belongs to $\mathbf{Z} \times \mathbf{R}$ are the points of the form p or $p + (0, \frac{1}{2})$ where $p \in \mathbf{Z}^2$. Therefore

$$\operatorname{supp} \mu \subset \{\pi((0, 0)), \pi((0, \tfrac{1}{2}))\}.$$

The measure μ is ergodic, so μ is concentrated either at $\pi((0, 0))$ or at $\pi((0, \frac{1}{2}))$. In the first case we have $\alpha = 0$ and in the second case $\alpha = 1$.

In a similar way we can show that if $\mu \in \mathcal{M}_E(F)$ and $\int \varphi_F \, d\mu$ is equal to $(\alpha, 1)$, $(0, \alpha)$ or $(1, \alpha)$ then also $\alpha = 0$ or $\alpha = 1$ in each case. This completes the proof of described properties of F.

Remark 3. The functions ψ_1 and ψ_2 can be chosen even real analytic and then F is real analytic.

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INSTYTUT MATEMATYKI
UNIWERSYTET WARSZAWSKI

INSTITUTE OF MATHEMATICS
WARSAW UNIVERSITY
PKiN IX p.
00-901 Warsaw
Poland

Received 10 June 1989;
in revised form 3 November 1989

The Axiom of Choice, the Löwenheim–Skolem Theorem and Borel models

by

Jerome Malitz, Jan Mycielski and William Reinhardt (Boulder, Colo.)

Abstract. We characterise those cardinal numbers $\mathfrak{m}$ for which one can prove without using the Axiom of Choice that if a countable theory has an infinite model, then it has a model of cardinality $\mathfrak{m}$. We prove also that if a countable theory has an infinite model, then it has a model whose universe is the real line and whose relations and functions are Borel sets.

§ 1. Introduction and results. A. Ehrenfeucht raised the following question. Does every countable first order theory T which has an infinite model have a Borel model, i.e., a model whose universe is the real line $\mathbf{R}$ and whose relations and functions are Borel subsets of the appropriate finite powers of $\mathbf{R}$? We shall see that the answer is yes (1). When thinking about this problem it occurred to us that if a Borel model exists then we should be able to prove this without using the Axiom of Choice (AC), and this in turn led us to the question of characterising those infinite cardinal numbers for which the Upward Löwenheim–Skolem Theorem can be proved without using AC. There exists a simple obstruction which was found by Vaught [4]. Namely if T_0 is the theory of one binary function based on the axiom

$$f(x, y) = f(u, v) \rightarrow (x = u \wedge y = v)$$

and M is a model of T_0 of cardinality $\mathfrak{m}$ then, of course, $\mathfrak{m}^2 = \mathfrak{m}$. But, a well-known theorem of Tarski says that

$$(\forall \mathfrak{m} \geqslant \aleph_0)[m^2 = m] \leftrightarrow \text{AC}.$$

Hence, in the absence of AC, the Upward Löwenheim–Skolem Theorem for $\mathfrak{m}$ cannot be proved without the assumption $\mathfrak{m}^2 = \mathfrak{m}$. Another obstruction arises if we consider the theory T_1 of linear ordering relations. Namely if the Upward Löwenheim–Skolem Theorem holds for T_1 and $\mathfrak{m}$, then every set of cardinality $\mathfrak{m}$ can be linearly ordered. Our main result asserts that the above two obstructions are the only ones.

THEOREM 1. (Without AC) *For every cardinal number $\mathfrak{m} > 1$ the following two conditions are equivalent*:

(i) $\mathfrak{m}^2 = \mathfrak{m}$ *and every set of cardinality $\mathfrak{m}$ can be linearly ordered.*

(ii) *Every countable theory which has an infinite model has a model of cardinality $\mathfrak{m}$.*

(1) See Note added at the end of the paper.

Remark. In the absence of AC the two parts of condition (i) are independent of each other. Indeed, by the work of Halpern and Lévy [1], the statement "every set can be linearly ordered" does not imply AC, and hence, by the theorem of Tarski, the statement "there exists a set A such that A can be linearly ordered but $|A|^2 \neq |A|$" is consistent in the absence of AC.

To see that the converse implication also fails recall first that the set $PP(\omega)$ cannot be linearly ordered without using AC. If it could, then we could get without AC a nonprincipal ultrafilter in the Boolean algebra $P(\omega)$ (W. Sierpiński), which, as is well known, cannot be proved without AC (Feferman, Solovay). On the other hand $|PP(\omega)|^2 = |PP(\omega)|$ is immediate without using AC.

Our second theorem immediately gives a positive answer to Ehrenfeucht's question which was stated at the beginning.

THEOREM 2. (Without AC) *If T is a countable theory which has an infinite model, then T has a model of the form $({}^{\omega}\omega, R_0, R_1, \ldots)$, where ${}^{\omega}\omega$ is the space of irrational numbers with its natural topology and all R_i are Borel sets of class $\boldsymbol{F}_\sigma \cap \boldsymbol{G}_\delta$ in the appropriate finite powers of ${}^{\omega}\omega$.*

Remark. In Theorem 2 we have not distinguished between the relational and functional symbols of T since, in this case, without loss of generality, we can treat functions as relations. Let us recall also that if F is a Borel relation which happens to be a function, then F is a Borel measurable function. This follows from two facts: the projection of a function to its domain is continuous and injective, and continuous injective maps between Borel spaces preserve Borel sets (see [2]).

Theorems 1 and 2 were announced in [3]. As we shall see both theorems easily follow from a more technical theorem. To state this theorem we need some definitions.

Let L be any countable first order language without function symbols, whose relation symbols are $R_0, R_1, \ldots$ with $n(0), n(1), \ldots$ argument places respectively. Let $\mathfrak{A} = \langle A, < \rangle$ be a linear order. Then a model $\mathfrak{M}$ of type L is called an *$\mathfrak{A}$-model* if

$$\mathfrak{M} = \langle B, R_0, R_1, \ldots \rangle,$$

where B and R_i are of the following forms:

$$B = (K_0 \times A^0_<) \cup (K_1 \times A^1_<) \cup (K_2 \times A^2_<) \cup \ldots$$

where the K_i are disjoint sets of integers (some but not all of them can be empty) and

$$A^0_< = \{p\}, \quad \text{where } p \notin A,$$

$$A^m_< = \{(a_1, \ldots, a_m) \in A^m : a_1 < \ldots < a_m\}, \quad \text{for } m > 0.$$

And $R_i \subseteq B^{n(i)}$ has the following structure: $R_i = \bigcup_{j<\omega} R_{ij}$, and there are functions k: ${}^3\omega \to \omega$ and m: $\omega \to \omega$ such that $k(i, j, r) \in K_{m(k(i,j,r))}$ and

$$R_{ij} = \{((k(i, j, 1), \bar{a}_1), \ldots, (k(i, j, n(i)), \bar{a}_{n(i)})) : \bar{a}_r \in A^{m(k(i,j,r))}_<$$
$$\text{for } r = 1, \ldots, n(i) \text{ and } \mathfrak{A} \models \varphi_{ij}(\bar{a}_{r_1}, \ldots, \bar{a}_{r_s})\},$$

where $r_1 < \ldots < r_s$ is the sequence of all $r \in \{1, \ldots, n(i)\}$ for which $m(k(i, j, r)) > 0$, φ_{ij} are formulas in the language of $\mathfrak{A}$ such that φ_{ij} has $\sum_{t=1}^{s} m(k(i, j, r_t))$ free variables and φ_{ij} is the truth or falsehood symbol if $s = 0$.

THEOREM 3. (Without AC) *If T is a countable theory in L which has an infinite model and $\mathfrak{A}$ is a dense linear order without endpoints, then T has an $\mathfrak{A}$-model such that $K_1 \neq \emptyset$.*

Remark. Since a dense linear order $\mathfrak{A}$ without endpoints admits elimination of quantifiers, we can require also that the formulas φ_{ij} be quantifier-free. The theorem does not require that $K_0, K_2, K_3, \ldots$ be non-empty, but if wanted, this could also be secured.

§ 2. Derivations of Theorems 1 and 2 from Theorem 3. The proof of the implication (ii)$\to$(i) of Theorem 1 was already indicated prior to Theorem 1. The proof of (i)$\to$(ii) is the following: Let $\mathfrak{m}$ satisfy (i) and pick a set A with $|A| = \mathfrak{m}$. Let T be a countable theory with an infinite model. Then (i) yields $|A| \geqslant \aleph_0$ and hence $|A \times \boldsymbol{Q}| = |A|$, where $\boldsymbol{Q}$ is the set of rational numbers. Hence it follows easily from (i) that A has a dense linear ordering $<$ without first or last element. Form the $\langle A, < \rangle$-model for T given by Theorem 3. It is easy to derive from (i) that the universe B of this model, defined prior to Theorem 3, satisfies $|B| = |A|$. (Hint: use the fact that $K_1 \neq \emptyset$ and the Cantor–Bernstein theorem.) So (ii) follows.

To derive Theorem 2 let T satisfy its assumption and let $\langle A_0, < \rangle$ be the structure of irrational numbers with its natural ordering and order topology. (Recall that A_0 is homeomorphic to the product space ${}^{\omega}\omega$, where the basis ω is taken with the discrete topology.) Consider the $\langle A_0, < \rangle$-model of T given by Theorem 3, with the induced product topology and disjoint union topology (the sets K_m are assumed to be discrete). Then it is clear that B is homeomorphic to A_0 plus a closed discrete set of power $\leqslant \aleph_0$.

By the structure of the relations R_i described prior to Theorem 3 it is also clear that they are of class $\boldsymbol{F}_\sigma$ in B. To prove that R_i can be made simultaneously of class $\boldsymbol{G}_\delta$ in B it suffices to assume without loss of generality that T is definitionally closed. Then for each i there is a j such that $T \vdash R_i \leftrightarrow \neg R_j$.

Finally it is easy to define a bijection between B and ${}^{\omega}\omega$ which preserves the classes $\boldsymbol{F}_\sigma$ and $\boldsymbol{G}_\delta$. This completes the proof of Theorem 2.

§ 3. A lemma on linear orders. Let $\mathfrak{A}_0 = \langle \boldsymbol{Q}, < \rangle$, where $\boldsymbol{Q}$ is the set of rational numbers and $<$ is the natural ordering of $\boldsymbol{Q}$.

LEMMA. (i) *If $R \subseteq (\boldsymbol{Q}^n_<)^r$ is a relation which is invariant under all automorphisms of $\mathfrak{A}_0$ extended in the natural way to $\boldsymbol{Q}^n_<$, then there exists a formula φ of nr free variables in the language of $\mathfrak{A}_0$ such that $R \leftrightarrow \varphi$.*

(ii) *If $\equiv$ is an equivalence relation in $\boldsymbol{Q}^n_<$ which is invariant as above, then there exists a set $I \subseteq \{1, \ldots, n\}$ such that*

$$[(p_1, \ldots, p_n) \equiv (q_1, \ldots, q_n)] \leftrightarrow \bigwedge_{i \in I} p_i = q_i.$$

Proof. The part (i) is obvious.

We have worked out a proof of part (ii) in conversations with D. Ebbinghaus, R. Laver and J. Pawlikowski. Since it would be tedious to spell out all the details we shall only outline certain parts of the argument.

Let $I \subseteq \{1, \ldots, n\}$ be the maximal set such that for all elements of $Q^n_<$ we have

$$(\rightarrow) \qquad (p_1, \ldots, p_n) \equiv (q_1, \ldots, q_n) \rightarrow \bigwedge\!\!\!\bigwedge_{i \in I} p_i = q_i.$$

It suffices to prove the converse of $(\rightarrow)$. First we have to show that for every $(p_1, \ldots, p_n) \in Q^n_<$ there exists $(p_1^0, \ldots p_n^0) \in Q^n_<$ such that

$$(+) \qquad (p_1, \ldots, p_n) \equiv (p_1^0, \ldots, p_n^0) \quad \text{and} \quad (p_i = p_j^0 \rightarrow i = j \in I).$$

We will show that this follows from the following proposition: If $(p_1, \ldots, p_n) \equiv (q_1, \ldots, q_n)$, $(r_1, \ldots, r_n) \equiv (s_1, \ldots, s_n)$, $A = \{i\colon p_i \in \{q_1, \ldots, q_n\}\}$ and $B = \{i\colon r_i \in \{s_1, \ldots, s_n\}\}$, then there exists an automorphism α of $\mathfrak{A}_0$ such that $(q_1, \ldots, q_n) = (\alpha(r_1), \ldots, \alpha(r_n))$ and $\{i\colon p_i \in \{\alpha(s_1), \ldots, \alpha(s_n)\}\} = A \cap B$. In fact it is easy to check that this proposition is true for a generic α satisfying $(\alpha(r_1), \ldots, \alpha(r_n)) = (q_1, \ldots, q_n)$. Then, applying the proposition we obtain some $(p_1^0, \ldots, p_n^0)$ as required in $(+)$. Now we are going to prove the converse of $(\rightarrow)$. Let $\bigwedge\!\!\!\bigwedge_{i \in I} p_i = q_i$ and let $(p_1^0, \ldots, p_n^0)$ satisfy $(+)$. Consider the following diagram:

$$\begin{array}{ccc}
(p_1, \ldots, p_n) & \equiv & (p_1^0, \ldots, p_n^0) \\
\downarrow\alpha_0 & & \downarrow\alpha_0 \\
(p_1^1, \ldots, p_n^1) & \equiv & (p_1^0, \ldots, p_n^0) \\
\downarrow\alpha_1 & & \downarrow\alpha_1 \\
(p_1^1, \ldots, p_n^1) & \equiv & (p_1^2, \ldots, p_n^2) \\
\downarrow\alpha_2 & & \downarrow\alpha_2 \\
(p_1^3, \ldots, p_n^3) & \equiv & (p_1^2, \ldots, p_n^2) \\
 & \cdots\cdots\cdots & \\
\downarrow\alpha_{2k} & & \downarrow\alpha_{2k} \\
(p_1^k, \ldots, p_n^k) & \equiv & (p_1^{k-1}, \ldots, p_n^{k-1}) \\
\downarrow\alpha_{2k+1} & & \downarrow\alpha_{2k+1} \\
(p_1^k, \ldots, p_n^k) & \equiv & (q_1, \ldots, q_n).
\end{array}$$

This diagram is obtained by applying certain automorphisms α_s of $\mathfrak{A}_0$ to both sides of certain congruences, but the α_s are assumed to be such that only one side of the corresponding congruence changes. It is also assumed that no α_s moves the coordinates p_i with $i \in I$, thus $p_i = p_i^j = q_i$ for all $i \in I$ and $j = 0, \ldots, k$. It is easy to check that, under our assumptions, such a diagram exists. By the transitivity of $\equiv$ this diagram implies the desired conclusion, namely $(p_1, \ldots, p_n) \equiv (q_1, \ldots, q_n)$. So we have shown that the set I satisfies the converse of $(\rightarrow)$.

This concludes our proof of (ii).

§ 4. Proof of Theorem 3. We can assume without loss of generality that the language of T has a set of constants c_q, $q \in Q$, which do not appear in the axioms of T.

Since T has an infinite model, by the finite theorem of Ramsey, we can add to T, without inconsistency, the set of all axioms of the form

$$(*) \qquad \psi(c_{p_1}, \ldots, c_{p_n}) \leftrightarrow \psi(c_{q_1}, \ldots, c_{q_n}) \quad \text{and} \quad c_{p_1} \neq c_{p_2}, \qquad (n = 1, 2, \ldots)$$

where $\psi(x_1, \ldots, x_n)$ is any formula in the language of T with n free variables and without any constants c_q and $p_1 < \ldots < p_n$ and $q_1 < \ldots < q_n$. Let T^R be the resulting theory. Let T^{RS} denote the Skolemization of T^R. Now consider the theory

$$T^* = T^{RS} \cup (T^{RS})^{RS} \cup \ldots$$

Thus T^* is a consistent countable theory. The axioms of T^* are without quantifiers, $T^* \vdash T$, and all the sentences of the form $(*)$ in the language of T^* (which of course may involve quantifiers) are theorems of T^*. Since T^* is countable, without using AC, we can construct a complete consistent extension T^{*+} of T^*.

Let S be the set of all terms without variables in the language of T^*. Then the theory T^* has a premodel $\mathfrak{M}$ with universe S based on T^{*+}, which means that the function symbols are interpreted in S in the natural way, the equality symbol is interpreted by the congruence

$$\sigma \equiv \tau \leftrightarrow [T^{*+} \vdash \sigma = \tau]$$

and for any atomic formula α without free variables we have

$$\mathfrak{M} \models \alpha \leftrightarrow [T^{*+} \vdash \alpha].$$

Let $\tau_0, \ldots, \tau_\alpha, \ldots$ be a well ordering of all terms in the language of T^* which do not contain any of the constants c_q and are such that if τ_α has n variables, then it has precisely the variables $x_1, \ldots, x_n$. Assume moreover that if τ_α has m variables and τ_β has n variables and $m < n$, then $\alpha < \beta$. Now, for any equivalence class in $S/\equiv$ we choose one representative $\tau_\alpha(c_{q_1}, \ldots, c_{q_n})$ with $q_1 < \ldots < q_n$ and such that $\tau_\alpha(x_1, \ldots, x_n)$ is the first available term in the above well ordering. Let S_0 denote this selector.

Consider any $\tau(c_{q_1}, \ldots, c_{q_n}) \in S_0$. Of course S may contain more terms of the form $\tau(c_{p_1}, \ldots, c_{p_n})$ than S_0. Since the sentences $(*)$ are true in $\mathfrak{M}$, by Lemma (ii) it follows that for each such τ there exists a set $I \subseteq \{1, \ldots, n\}$ such that, for all elements of S_0,

$$\tau(c_{p_1}, \ldots, c_{p_n}) \equiv \tau(c_{q_1}, \ldots, c_{q_n}) \leftrightarrow \bigwedge\!\!\!\bigwedge_{i \in I} p_i = q_i,$$

and for every $(u_i\colon i \in I) \in Q^{|I|}_<$ there exists $(p_1, \ldots, p_n) \in Q^n_<$ such that

$$\tau(c_{p_1}, \ldots, c_{p_n}) \in S_0 \quad \text{and} \quad \bigwedge\!\!\!\bigwedge_{i \in I} p_i = u_i.$$

Let $\mathfrak{M}_0$ be the model obtained by restricting the universe and the relations of $\mathfrak{M}$ to S_0 and modifying the functions of $\mathfrak{M}$ in the obvious way. We claim that $\mathfrak{M}_0$ is (isomorphic to) an $\mathfrak{A}$-model of T with $K_1 \neq \emptyset$, as required in Theorem 3 for the case $\mathfrak{A} = \mathfrak{A}_0$. Of course $\mathfrak{M}_0$ is a model of T^* and hence of T. To see that it is a $\mathfrak{A}_0$-model notice first that, by the above discussion of $\equiv$, every element $\tau(c_{q_1}, \ldots, c_{q_n}) \in S_0$ is determined (within S_0) by the triple $(\tau, I, (q_i\colon i \in I))$ where τ and I are as above and $(q_i\colon i \in I) \in Q^{|I|}_<$. We can code the pair (τ, I) by a single integer k. Let K_m be the set of all codes k of those pairs (τ, I) for which $|I| = m$. We put $m(k) = |I|$ and $m(k) = 0$ for $k \notin \bigcup K_m$. (Of course $m(k) = 0$ also if $|I| = 0$.) So there is a natural bijection between S_0 and the set B defined prior to Theorem 3 with $\mathfrak{A} = \mathfrak{A}_0$. Let us identify S_0 with B via this bijection. So we can write $\mathfrak{M}_0 = \langle B, R_0, R_1, \ldots \rangle$.

Now we have to show that all the relations R_i have the form prescribed prior to Theorem 3. Let $\langle (k(i, j, 1), \ldots, k(i, j, n(i))) : j = 0, 1, \ldots \rangle$ be an enumeration of all $n(i)$-tuples of integers. Let

$$R_{ij} = R_i \cap (\{k(i, j, 1)\} \times A_<^{m(k(i,j,1))} \cup \ldots \cup \{k(i, j, n(i))\} \times A_<^{k(i,j,n(i))}).$$

Then, if we look again at the meaning of the codes $k \in \bigcup K_m$ and we use the fact that $\mathfrak{M}_0$ satisfies the axioms (*), by Lemma (i) it is clear that R_{ij} is of the form required prior to Theorem 3, with a formula φ_{ij} in the language of $\mathfrak{A}_0$ with $\sum_{r=1}^{n(i)} m(k(i, j, r))$ variables.

This concludes our proof that $\mathfrak{M}_0$ is an $\mathfrak{A}_0$-model of T. The inequality $K_1 \neq \emptyset$ follows from the fact that K_1 must contain a code of the pair $(x_1, \{1\})$.

Now let $\mathfrak{A}$ be an arbitrary dense linear order without endpoints. The same functions $k(i, j, r)$, $m(k)$ and formulas φ_{ij} which we found for $\mathfrak{M}_0$ yield a certain $\mathfrak{A}$-model $\mathfrak{N}$ with $K_1 \neq \emptyset$. It remains to check that $\mathfrak{N}$ satisfies T. But it is clear that every finite part of $\mathfrak{N}$ is isomorphic to some finite part of $\mathfrak{M}_0$. Since $\mathfrak{M}_0 \models T^*$ and the axioms of T^* are universal, $\mathfrak{N} \models T^*$. Since $T^* \vdash T$, the proof is complete.

Note added in July 1989. After this paper was written the authors learned that the problem of existence of Borel models was independently posed and solved by H. Friedman, see

[a] C. I. Steinhorn, *Borel structures for first order and extended logics*, in: *Harvey Friedman Research in the Foundations of Mathematics*, L. A. Harrington et al. (eds.), Elsevier Science Publishers B. V. (North-Holland), 1985, 161–178.

[b] — *Borel structures and measure and category logics*, in: *Model-Theoretic Logics*, J. Barwise and S. Feferman (eds.), Springer, New York 1985, 579–596.

The proofs of the existence of Borel models presented in those papers are closely related to ours, but we decided to keep Theorem 2 and its proof because it gives a sharper estimate of the Borel classes of the relations and because the concrete structure of the model described in Theorem 3 may be of independent interest. The papers [a] and [b] discuss additional aspects and extensions of Theorem 2 but its proofs given there are not as detailed as ours.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF COLORADO
Boulder, CO 80309-0426
U.S.A.

Received 31 July 1989

The shrinking property of products of cardinals

by

Nobuyuki Kemoto (Oita)

Abstract. It is known that for cardinals $\kappa > \omega$ and $\lambda > 1$, κ^λ is normal if and only if κ is regular and $\lambda < \kappa$. We show that normality can be replaced by the shrinking property in this result.

Ordinals and cardinals are considered as sets of smaller ordinals. In particular, $n = \{0, 1, \ldots, n-1\}$ for each $n \in \omega$. Let $\{\chi_\alpha : \alpha \in \lambda\}$ be a collection of spaces. $\prod_{\alpha\in\lambda} X_\alpha$ denotes the usual Tikhonov product space of X_α's. Each element f of $\prod_{\alpha\in\lambda} X_\alpha$ is considered as a function whose domain is λ and $f(\alpha)$ is in X_α for each $\alpha \in \lambda$. Whenever X_α is a single space X for each $\alpha \in \lambda$, $\prod_{\alpha\in\lambda} X_\alpha$ is denoted by X^λ.

Let X be a space and let κ be a cardinal. Assume $\mathscr{U}$ is an open cover of X. A cover $\mathscr{V} = \{V(U) : U \in \mathscr{U}\}$ is said to be a *shrinking* of $\mathscr{U}$ if $\operatorname{cl} V(U) \subset U$ for each $U \in \mathscr{U}$. In particular, $\mathscr{V}$ is said to be an *open* (*closed*) *shrinking* of $\mathscr{U}$ if each member of $\mathscr{V}$ is open (closed, respectively). X is said to have the *κ-shrinking property* if every open cover of size $\leqslant \kappa$ has an open shrinking. A space has the *shrinking property* if it has the κ-shrinking property for every infinite cardinal κ. Note that 2-shrinking property is normality and that ω-shrinking property is countable paracompactness plus normality. It is easy to show that a normal space which has the property that every open cover of size $\leqslant \kappa$ has a closed shrinking has the κ-shrinking property. Note that paracompact spaces, in particular compact Hausdorff spaces and regular Lindelöf spaces, have the shrinking property. On the other hand, ω_1 with the order topology has the shrinking property but is not paracompact. In general, ordered spaces have the shrinking property, see [Ke]. But the product space $\omega_1 \times (\omega_1 + 1)$ does not have the shrinking property, in fact it is not normal, see [Pr, 2.2]. But note that it is countably paracompact since it is a perfect preimage of the countably paracompact space ω_1. Note that κ-shrinking property implies normality if $\kappa \geqslant 2$. It is strangely difficult to find an example of a normal space without the κ-shrinking property for $\kappa \geqslant \omega$. For each $\kappa \geqslant \omega$, we know of essentially one real such example, namely the κ-Dowker space, see [Ru1], [Ru2].

1980 *Mathematics Subject Classification* (1985 *Revision*): 54B10, 54D20, 54F05, 54D15.
Key words and phrases: shrinking property, product, normal.