Rotation sets and ergodic measures for torus homeomorphisms

by

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Abstract. We prove that for every homeomorphism f of the two-dimensional torus onto itself isotopic to the identity and a vector v from the interior of the rotation set of f there exists a closed non-empty invariant set whose each point has rotation vector v. It follows that there exists an ergodic invariant probability measure on the torus such that the expected value of the displacement by f is v. We also show examples that this is not necessarily true if v is from the boundary of the rotation set of f, even if the interior of this set is non-empty.

Introduction. There has been recently a rapid progress in the investigation of the rotation sets for torus homeomorphisms isotopic to the identity. To describe it, we need some definitions first.

Let $T^2 = R^2/Z^2$ be the two-dimensional torus and let $\pi\colon R^2 \to T^2$ be the natural projection. We shall denote by $\mathscr{H}$ the class of all liftings to R^2 of homeomorphisms of T^2 isotopic to the identity. Then for $F \in \mathscr{H}$:

$\varrho(F, x)$ denotes the set of all limit points of the sequence $\left(\frac{F^n(x)-x}{n}\right)_{n=1}^{\infty}$;

$\varrho(F)$ denotes the set of all limit points of all sequences $\left(\frac{F^n(x_n)-x_n}{n}\right)_{n=1}^{\infty}$, where $x_n \in R^2$; we call $\varrho(F)$ the *rotation set* of F;

$f_F\colon T^2 \to T^2$ denotes the map whose lifting is F;

$\mathscr{M}(F)$ denotes the set of all f_F-invariant probability measures on T^2;

$\mathscr{M}_E(F)$ denotes the set of all ergodic measures from $\mathscr{M}(F)$;

$\varphi_F\colon T^2 \to R^2$ is the function defined by $\varphi_F(x) = F(y)-y$ for $y \in \pi^{-1}(x)$ (this does not depend on the choice of y);

$\varrho_{\mathrm{mes}}(F) = \{\int \varphi_F\, d\mu\colon \mu \in \mathscr{M}_E(F)\}$;

$\mathrm{Conv}(P)$ denotes the convex hull of a set P.

Notice that by the ergodic theorem, if $\mu \in \mathscr{M}_E(F)$ then for μ-almost all x, if $y \in \pi^{-1}(x)$ then $\varrho(F, y) = \{\int \varphi_F\, d\mu\}$. Notice also that if a point x is f_F-periodic of period n then there exists $k \in Z^2$ such that for all $y \in \pi^{-1}(x)$ we have $F^n(y) = y+k$ and consequently $\varrho(F, y) = k/n$.

The main results of the papers [F] (see also [H2], where similar results are proved, although they are stated in a different way), [LM] and [MZ], concerning maps from $\mathscr{H}$, are the following.

THEOREM 1 ([F]). *If* $F\in\mathscr{H}$ *and* $v=(p/n,\ q/n)\in\operatorname{int}(\varrho(F))$ *with* $\mathrm{GCD}(p,q,n)=1$ *then* f_F *has a periodic point* x *with least period* n *such that* $F^n(y)=y+(p,q)$ *for* $y\in\pi^{-1}(x)$.

THEOREM 2 ([LM]+[F]). *If* $F\in\mathscr{H}$ *and* $\varrho(F)$ *has non-empty interior then the topological entropy of* f *is positive.*

THEOREM 3 ([LM]). *If* $F\in\mathscr{H}$, $\varrho(F)$ *has non-empty interior,* P *is a finite subset of* $\bigcup\varrho(F,x)$, *where the union is taken over those* x *for which* $\pi(x)$ *is periodic for* f_F, *and* C *is a compact connected subset of* $\operatorname{Conv}(P)$ *then there exists* $y\in\mathbf{R}^2$ *with* $\varrho(F,y)=C$. *In particular (since* C *can consist of one point), for every* $v\in\operatorname{int}(\varrho(F))$ *there exists* $y\in\mathbf{R}^2$ *with* $\varrho(F,y)=\{v\}$.

THEOREM 4 ([MZ]). *If* $F\in\mathscr{H}$ *then* $\varrho(F)$ *is compact and convex.*

THEOREM 5 ([MZ]). *If* $F\in\mathscr{H}$ *then*

$$\varrho(F)=\operatorname{Conv}(\varrho_{\mathrm{mes}}(F))=\{\textstyle\int\varphi_F\,d\mu\colon\ \mu\in\mathscr{M}(F)\}=\operatorname{Conv}\big(\bigcup_{y\in\mathbf{R}^2}\varrho(F,y)\big).$$

THEOREM 6 ([MZ]). *The function* ϱ *from* $\mathscr{H}$ *into the space of all subsets of* $\mathbf{R}^2$ *is upper semicontinuous, i.e. if* $F\in\mathscr{H}$ *and* U *is a neighborhood of* $\varrho(F)$ *in* $\mathbf{R}^2$ *then there exists a neighborhood of* F *in* $\mathscr{H}$ *with the topology of uniform convergence such that if* $G\in V$ *then* $\varrho(G)\subset U$.

The main result of the present paper is the following.

THEOREM A. *If* $F\in\mathscr{H}$ *and* $v\in\operatorname{int}(\varrho(F))$ *then*

(a) *there exists a non-empty closed* f_F*-invariant subset* X *of* $\mathbf{T}^2$ *such that* $\varrho(F,y)=\{v\}$ *for every* $y\in\pi^{-1}(X)$,

(b) *there exists a measure* $\mu\in\mathscr{M}_E(F)$ *with* $\int\varphi_F\,d\mu=v$.

The proof of the above theorem uses the technique developed by Llibre and MacKay in [LM]. From the construction we can also deduce the following.

THEOREM B. *The function* ϱ *from* $\mathscr{H}$ *into the space of all compact subsets of* $\mathbf{R}^2$ *with the Hausdorff metric is continuous at all* F *with* $\operatorname{int}(\varrho(F))\neq\emptyset$.

Moreover, we show examples of maps $F\in\mathscr{H}$ for which $\operatorname{int}(\varrho(F))\neq\emptyset$ but many points from the boundary of $\varrho(F)$ (among them there are points with both coordinates rational) are not of the form $\int\varphi_F\,d\mu$ for $\mu\in\mathscr{M}_E(F)$. However, notice that by Theorem 5 they are of the form $\int\varphi_F\,d\mu$ for some $\mu\in\mathscr{M}(F)$.

2. Proofs of Theorems A and B. In the proof of Theorem A we will need the following geometric lemma.

LEMMA 1. *Assume that* 0 *lies in the interior of the triangle with vertices* $b_1, b_2, b_3\in\mathbf{R}^2$. *Then for every* $K>0$ *there exists* $L>0$ *with the following property. If* $x\in\mathbf{R}^2$ *and* $\|x\|\leqslant L$ *then one can choose* $i\in\{1,2,3\}$ *and a non-negative integer* n *such that*

$$\|x+nb_i\|\leqslant L-K. \tag{1}$$

Proof. Let $S^1=\{y\in\mathbf{R}^2\colon\ \|y\|=1\}$ be the unit circle. The functions $t_i\colon S^1\to\mathbf{R}$ $(i=1,2,3)$ defined by

$$t_i(y)=\sup\Big\{t\geqslant 0\colon\ \Big\|y+t\frac{b_i}{\|b_i\|}\Big\|\leqslant 1\Big\}$$

are continuous. If $y\in S^1$ then the line tangent to S^1 at y divides $\mathbf{R}^2$ into two semiplanes. Denote the one containing 0 by $A(y)$. Since 0 lies in the interior of the triangle with vertices b_1, b_2, b_3, then at least one of the points $y+b_i$ $(i=1,2,3)$ belongs to the interior of $A(y)$. Then the corresponding $t_i(y)$ is positive. Consequently, the function $t_*=\max(t_1,t_2,t_3)$ is positive and continuous. Hence, there exists $\varepsilon>0$ such that for each $y\in S^1$ there is $i(y)\in\{1,2,3\}$ with $t_{i(y)}(y)\geqslant\varepsilon$.

Notice that the set

$$\Big\{y+t\frac{b_{i(y)}}{\|b_{i(y)}\|}\colon\ 0\leqslant t\leqslant t_{i(y)}(y)\Big\}$$

is a chord of the unit disc of length $t_{i(y)}(y)$. There exists $\delta>0$ such that if a chord C of the unit disc has length at least ε then the intersection of C with any disc with center 0 and radius at least $1-\delta$ has length at least $\varepsilon/3$. Therefore if $x\in\mathbf{R}^2$ and $\alpha/\|x\|\leqslant\varepsilon/3$, where $\alpha=\max(\|b_1\|,\|b_2\|,\|b_3\|)$, then $\|b_i/\|x\|\,\|\leqslant\varepsilon/3$ for each i and hence for some positive integer n and $i=i(x/\|x\|)$ we have

$$\Big\|\frac{x}{\|x\|}+n\frac{b_i}{\|x\|}\Big\|\leqslant 1-\delta. \tag{2}$$

Set $L=3\alpha/(\varepsilon(1-\delta))$ and $K=\delta L$. If $\|x\|\leqslant L-K$ then (1) holds with $n=0$. If $L-K\leqslant\|x\|\leqslant L$ then $\|x\|\geqslant L-K=3\alpha/\varepsilon$, so (2) holds for some positive integer n. Then from (2) we get

$$\|x+nb_i\|\leqslant(1-\delta)\|x\|\leqslant(1-\delta)L=L-K.$$

This completes the proof. ■

Remark 1. If $\|x\|\leqslant L$ and (1) is satisfied then $\|x+mb_i\|\leqslant L$ for all m with $0\leqslant m\leqslant n$.

As in [LM], we will have to work in the following situation. Let $G\in\mathscr{H}$. Let $W_1, W_2, W_3\subset\mathbf{T}^2$ be three open discs such that $f_G^{k_i}(W_i)=W_i$ for some $k_i\geqslant 1$, and the sets $W_1, f_G(W_1),\ldots,f_G^{k_1-1}(W_1), W_2, f_G(W_2),\ldots,f_G^{k_2-1}(W_2), W_3, f_G(W_3),\ldots,f_G^{k_3-1}(W_3)$ are disjoint. Denote the union of all these sets by W. Assume that $f_G|_{\mathbf{T}^2\setminus W}$ is pseudo-Anosov (see [FLP]). Then there exists a Markov partition $\mathscr{A}$ for $f_G|_{\mathbf{T}^2\setminus W}$. We shall say that *there is a transition from* $A\in\mathscr{A}$ *to* $B\in\mathscr{A}$ if $f_G(\operatorname{int}(A))$ intersects $\operatorname{int}(B)$. A sequence (finite or infinite) P of elements of $\mathscr{A}$ will be called *admissible* if there is a transition from each element of P (except the last one if P is finite) to the next one. We may assume that the elements of $\mathscr{A}$ are closed. Then we know that if a sequence $S=(B_i)_{i=0}^r$ (where r is finite or $r=\infty$) is admissible then the intersection $\bigcap_{i=0}^r f_G^{-i}(B_i)$ is non-empty. We shall denote this intersection by $T(S)$.

Any pseudo-Anosov diffeomorphism is transitive. Therefore for every $A, B\in\mathscr{A}$ there exists a finite sequence S such that ASB is admissible (by ASB we mean the sequence $(A, S_1,\ldots,S_s, B)$ where $S=(S_1,\ldots,S_s)$).

If $A\in\mathscr{A}$ then there is a natural partition of $\pi^{-1}(A)$ into sets with the same diameter as A such that for each such set C we have $\pi^{-1}(A) = \bigcup_{k\in Z^2}\{x+k\colon x\in C\}$. We shall call these sets *components* of $\pi^{-1}(A)$. We shall use the same terminology for subsets of $A\in\mathscr{A}$. Also, if $A, B\in\mathscr{A}$ are such that $f_G(A)$ intersects B then the F-image of any component of $\pi^{-1}(A)$ intersects exactly one component of $\pi^{-1}(B)$. From this by induction we get the following lemma.

LEMMA 2. *Let $S = (A_0, \ldots, A_n)$ be an admissible sequence. Then for any $k, m \in\{0, \ldots, n\}$ with $k\leqslant m$ and any component C of $\pi^{-1}(T(S))$, the set $G^k(C)$ is contained in some component of $\pi^{-1}(T(A_k, \ldots, A_m))$.*

From the properties of W_i it follows that there are points w_i $(i = 1, 2, 3)$ of Q^2 such that for every $i\in\{1, 2, 3\}$ and every component V of $\pi^{-1}(W_i)$ we have $G^{k_i}(V) = V+k_i w_i$. Notice that then for every $y\in\pi^{-1}\big(\bigcup_{n=0}^{k_i-1} f_G^n(W_i)\big)$ we have $\varrho(G, y) = \{w_i\}$.

For each $i\in\{1, 2, 3\}$ and $j\in Z$ denote by $\mathscr{A}_{i,j}$ the set of those elements of $\mathscr{A}$ which have a part of its boundary common to $f_G^j(W_i)$. We can also assume that any component of $\pi^{-1}(A)$ for $A\in\mathscr{A}_{i,j}$ has a part of its boundary common to only one component of $\pi^{-1}(f_G^j(W_i))$. We shall call these components of $\pi^{-1}(A)$ and $\pi^{-1}(f_G^j(W_i))$ *adjacent.*

For a sequence S we shall denote its length by $|S|$.

With all these assumptions and notations we can prove the next lemma.

LEMMA 3. *For each $v\in\operatorname{int}(\operatorname{Conv}\{w_1, w_2, w_3\})$ there exist $y\in R^2\setminus\pi^{-1}(W)$ and $M > 0$ such that*

$$\|G^n(y)-y-nv\| \leqslant M \quad \text{for all } n\geqslant 0. \tag{3}$$

Proof. We shall construct y as an element of $\pi^{-1}(T(P))$ for some infinite admissible sequence P. The sequence P will be built up from finite sequences which we shall call *bricks*. There will be two types of bricks. We start by describing the first type.

If $A\in\mathscr{A}_{i,j}$ then $f_G(A)$ has a part of its boundary common to $f_G^{j+1}(W_i)$, so there is a transition from A to some $B\in\mathscr{A}_{i,j+1}$. Starting from $A_0\in\mathscr{A}_{i,0}$ and repeating this procedure k_i times we get an admissible sequence $(A_0, A_1, \ldots, A_{k_i})$ such that $A_j\in\mathscr{A}_{i,j}$ for $j = 0, 1, \ldots, k_i$. However, $f_G^{k_i}(W_i) = W_i$, so $\mathscr{A}_{i,k_i} = \mathscr{A}_{i,0}$ and A_{k_i} belongs again to $\mathscr{A}_{i,0}$. In such a way we obtain for every $A_0\in\mathscr{A}_{i,0}$ a brick $S_i(A_0) = (A_0, A_1, \ldots, A_{k_i-1})$ and an element $\xi_i(A_0) = A_{k_i}\in\mathscr{A}_{i,0}$ such that the sequence $S_i(A_0)\,\xi_i(A_0)$ is admissible. If V is a component of $\pi^{-1}(W_i)$ and C is the component of $\pi^{-1}\big(T(A_0, A_1, \ldots, A_{k_i})\big)$ contained in the component of $\pi^{-1}(A_0)$ adjacent to V then by Lemma 2, $G^j(C)$ is contained in some component of $\pi^{-1}(A_j)$ for $j = 1, 2, \ldots, k_i$ and by induction we see that this component of $\pi^{-1}(A_j)$ is adjacent to $G^j(V)$. In particular, $G^{k_i}(C)$ is contained in the component of $\pi^{-1}(\xi_i(A_0))$ adjacent to $G^{k_i}(V) = V+k_i w_i$.

For every $A, B\in\mathscr{A}_{1,0}\cup\mathscr{A}_{2,0}\cup\mathscr{A}_{3,0}$ we fix a sequence $S(A, B)$ such that the first element of $S(A, B)$ is A and the sequence $S(A, B)B$ is admissible. These sequences $S(A, B)$ will be bricks of the second type.

We are again interested in what happens in the covering space. To measure this we fix some reference components V_i of $\pi^{-1}(W_i)$ for $i = 1, 2, 3$. Let $A\in\mathscr{A}_{i_1,0}$, $B\in\mathscr{A}_{i_2,0}$. If $V_{i_1}+u_1$ (where $u_1\in Z^2$) is a component of $\pi^{-1}(W_{i_1})$ and C is the component of $\pi^{-1}\big(T(S(A, B)B)\big)$ contained in the component of $\pi^{-1}(A)$ adjacent to $V_{i_1}+u_1$ then by Lemma 2, $G^{|S(A,B)|}(C)$ is contained in some component of $\pi^{-1}(B)$ adjacent to a component

$V_{i_2}+u_2$ of $\pi^{-1}(W_{i_2})$ for some $u_2\in Z^2$. Notice that although u_2 depends on u_1, the vector $u(S(A, B)) = u_2-u_1$ is already independent of the choice of u_1.

In our definition of bricks each brick R determined the next element A of a sequence such that RA is admissible. Therefore we can build longer admissible sequences from bricks, starting a new brick from an element determined by the preceding one.

Let $P = R_0 R_1 \ldots$ be such a sequence made up of bricks. Let A_j be the first element of R_j for $j = 0, 1, \ldots$ We have $A_j\in\mathscr{A}_{i_j,0}$ for some $i_j\in\{1, 2, 3\}$ for $j = 0, 1, \ldots$ Let C be a component of $\pi^{-1}(T(P))$. It is contained in a component of $\pi^{-1}(A_0)$ adjacent to some component of $\pi^{-1}(W_{i_0})$. This component of $\pi^{-1}(W_{i_0})$ is of the form $V_{i_0}+z_0$ for some $z_0\in Z^2$. Analogously, $G^{|R_0\ldots R_{j-1}|}(C)$ is contained in a component of $\pi^{-1}(A_j)$ adjacent to some component of $\pi^{-1}(W_{i_j})$ for $j = 1, 2, \ldots$ This component of $\pi^{-1}(W_{i_j})$ is of the form $V_{i_j}+z_j$ for some $z_j\in Z^2$. Using Lemma 2 and the description of bricks of both types we see that $z_{j+1}-z_j = k_{i_j} w_{i_j}$ if R_j is of the first type and $z_{j+1}-z_j = u(R_j)$ if R_j is of the second type.

Our aim is to construct a point y satisfying (3). Therefore we shall try to make z_j-z_0 close to $|R_0\ldots R_{j-1}|\,v$. We have

$$z_j-z_0-|R_0\ldots R_{j-1}|\,v = \sum_{m=0}^{j-1} z(R_m), \tag{4}$$

where $z(R_m) = k_{i_j}(w_{i_j}-v)$ if R_m is of the first type and $z(R_m) = u(R_m)-|R_m|\,v$ if R_m is of the second type.

We are going to use Lemma 1. Set $b_i = k_i(w_i-v)$. Since $v\in\operatorname{int}(\operatorname{Conv}\{w_1, w_2, w_3\})$, it follows that 0 is in the interior of the triangle with vertices b_1, b_2, b_3. Set

$$K = \max\{u(S(A, B))-|S(A, B)|\,v\colon\ A, B\in\mathscr{A}_{1,0}\cup\mathscr{A}_{2,0}\cup\mathscr{A}_{3,0}\}.$$

Now we construct P by induction. Assume that for some k the bricks $R_0, \ldots, R_{k-1}$ are already chosen and $\|x\|\leqslant L$, where

$$x = \sum_{m=0}^{k-1} z(R_m)$$

and L is the constant from Lemma 1. Then R_{k-1} determines the next element $A_k\in\mathscr{A}_{i_k,0}$ of the sequence. (We include the case $k = 0$, i.e. when no bricks are chosen yet, $x = 0$ and $A_0\in\mathscr{A}_{1,0}\cup\mathscr{A}_{2,0}\cup\mathscr{A}_{3,0}$ is chosen in an arbitrary way.) By Lemma 1 there exist $i\in\{1, 2, 3\}$ and a non-negative integer n such that (1) holds. Then we choose next $n+1$ bricks as follows: $R_k = S(A_k, B)$ for some (arbitrarily chosen) $B\in\mathscr{A}_{i,0}$ and $R_{k+1} = S_i(B)$, $R_{k+2} = S_i(\xi_i(B)), \ldots, R_{k+n} = S_i(\xi_i^{n-1}(B))$. This gives us

$$\|z(R_k)\|\leqslant K \tag{5}$$

and $z(R_{k+1}) = z(R_{k+2}) = \ldots = z(R_{k+n}) = b_i$. By (1) and (5), $\big\|\sum_{m=0}^{k+n} z(R_m)\big\|\leqslant L$, which enables us to continue induction (replacing k by $k+n+1$). By Remark 1 and (5), $\big\|\sum_{m=0}^{k+l} z(R_m)\big\|\leqslant L+K$ for $l = 0, 1, \ldots, n$. This shows that

$$\Big\|\sum_{m=0}^{j-1} z(R_m)\Big\|\leqslant L+K \quad \text{for all } j\geqslant 0. \tag{6}$$

Now we take $y \in \pi^{-1}(T(P))$ and try to estimate $\|G^n(y)-y-nv\|$ for $n \geqslant 0$. If $n = \sum_{m=0}^{j-1} |R_m|$ then by the definition of z_m we see that y lies in a component of some $A \in \mathscr{A}$, adjacent to $V_i + z_0$ for some $i \in \{1, 2, 3\}$ and $G^n(y)$ lies in a component of some $B \in \mathscr{A}$, adjacent to $V_{i'} + z_j$ for some $i' \in \{1, 2, 3\}$. By (4) we get

$$\|G^n(y)-y-nv\| \leqslant \sum_{m=0}^{j-1} z(R_m) + K_1, \tag{7}$$

where

$$K_1 = 2\max\{\operatorname{diam}(A)\colon A \in \mathscr{A}\} + \operatorname{diam}(V_1 \cup V_2 \cup V_3).$$

For a general n, we have to add an additional constant in the estimate, namely the number

$$K_2 = \operatorname{supp}\{\|G^j(x)-x-jv\|\colon x \in \boldsymbol{R}^2,\ 0 \leqslant j \leqslant \max\{|S|\colon S \text{ is a brick}\}\},$$

which is finite since G is a lifting of a torus map homotopic to the identity. Hence, from (7) and (6) we get (3) for all $n \geqslant 0$ with $M = L + K + K_1 + K_2$. ∎

Proof of Theorem A. Let $F \in \mathscr{H}$ and let $v \in \operatorname{int}(\varrho(F))$. There exist $w_1, w_2, w_3 \in \operatorname{int}(\varrho(F)) \cap \boldsymbol{Q}^2$ such that $v \in \operatorname{int}(\operatorname{Conv}\{w_1, w_2, w_3\})$. By the result of Franks (Theorem 1) there exist three periodic orbits Q_1, Q_2, Q_3 of f_F with $\varrho(F, y) = w_i$ for $y \in \pi^{-1}(Q_i)$. Now we perform the same construction as Llibre and MacKay did in the proof of Theorem 1 of [LM]. In such a way we obtain a map $G \in \mathscr{H}$ with all the properties used in this section up to now. Moreover, $G_{\boldsymbol{R}^2 \setminus \pi^{-1}(\operatorname{cl}(W))}$ is homotopic to $F_{\boldsymbol{R}^2 \setminus \pi^{-1}(Q)}$, where $Q = Q_1 \cup Q_2 \cup Q_3$. Since W was obtained by blowing up the elements of Q, we may identify in a natural way $\boldsymbol{T}^2 \setminus \operatorname{cl}(W)$ with $\boldsymbol{T}^2 \setminus Q$, so the above homotopy makes sense. Instead of repeating the details of the construction, we refer the reader to [LM].

By Lemma 3 there exist $M > 0$ and $z \in \boldsymbol{R}^2$ such that (3) holds. By Theorem 1(ii) of [H1] (as in [LM], we can use it on an open manifold) there exist $M_1 > 0$ and $z' \in \boldsymbol{R}^2$ such that

$$\|F^n(z') - G^n(z)\| \leqslant M_1 \quad \text{for all } n \geqslant 0.$$

Therefore we obtain

$$\|F^n(z') - z' - nv\| \leqslant M_2 \quad \text{for all } n \geqslant 0, \tag{8}$$

where $M_2 = 2M_1 + M$.

Let X be the set of ω-limit points of $\pi(z')$ for the map f_F. From (8) it follows that if $k, n \geqslant 0$ and $y \in \pi^{-1}(\pi(z'))$ then

$$\|F^n(F^k(y)) - F^k(y) - nv\| \leqslant 2M_2.$$

By passing to the limit over subsequences (k_i), we obtain

$$\|F^n(y) - y - nv\| \leqslant 2M_2 \quad \text{for every } y \in \pi^{-1}(X). \tag{9}$$

From this we get $\varrho(F, y) = \{v\}$ for every $y \in \pi^{-1}(X)$. Since the set X is non-empty, f_F-invariant and closed, this proves Theorem A(a).

By the Krylov–Bogolyubov Theorem, there exists an ergodic f_F-invariant probability measure μ on X. By (9) we have

$$\left\|\frac{1}{n}\sum_{k=0}^{n-1} \varphi_F(f_F^k(x)) - v\right\| \leqslant \frac{2M_2}{n}$$

for all $x \in \boldsymbol{T}^2$ and $n \geqslant 0$. Therefore by the ergodic theorem, $\int \varphi_F \, d\mu = v$. This proves Theorem A(b). ∎

Proof of Theorem B. Let $F \in \mathscr{H}$ and let $v \in \operatorname{int}(\varrho(F)) \cap \boldsymbol{Q}^2$. As in the proof of Theorem A, we construct w_i, Q_i $(i = 1, 2, 3)$ and G. Since $v \in \operatorname{int}(\operatorname{Conv}\{w_1, w_2, w_3\}) \subset \operatorname{int}(\varrho(G))$, by Theorem 1 there exists a periodic point x of f_G such that if $v = (p/n, q/n)$ with $\operatorname{GCD}(p, q, n) = 1$ then $G^n(y) = y + (p, q)$ for $y \in \pi^{-1}(x)$. Since $v \neq w_1, w_2, w_3$, we have $x \notin W$. It is known (see e.g. [H1]) that a fixed point index of a fixed point of an iterate of a pseudo-Anosov map is non-zero. Therefore by Theorem VI E3 of [B], the fixed point class of $f_G^n|_{\boldsymbol{T}^2\setminus W}$ to which x belongs is h-related to a fixed point class of $f_F^n|_{\boldsymbol{T}^2\setminus Q}$, where h is a homotopy between $f_G^n|_{\boldsymbol{T}^2\setminus W}$ and $f_F^n|_{\boldsymbol{T}^2\setminus Q}$ (we use here the terminology of [B]). For each element x' of this class, $(f_F^n|_{\boldsymbol{T}^2\setminus Q}, x')$ is Nielsen equivalent to $(f_G^n|_{\boldsymbol{T}^2\setminus W}, x)$. Thus, also (f_F^n, x') is Nielsen equivalent to (f_G^n, x). This means that for each $y \in \pi^{-1}(x')$ we have $F^n(y) = y + (p, q)$.

In such a way we obtain a bounded open set $U \subset \boldsymbol{R}^2$ without fixed points of $F^n - (p, q)$ on its boundary and with a non-zero index of the set of fixed points of $F^n - (p, q)$ contained in U. This situation will not change when we replace F by a map $H \in \mathscr{H}$ which is sufficiently close to F. Therefore for such H there is $y \in \boldsymbol{R}^2$ with $H^n(y) = y + (p, q)$. In particular, $\varrho(H, y) = v$.

Now, if $F \in \mathscr{H}$ and $\operatorname{int}(\varrho(F)) = \emptyset$ then we can approximate $\operatorname{int}(\varrho(F))$ from within by a polygon with vertices in $\boldsymbol{Q}^2$, with an arbitrary precision. By the properties proved already, if $H \in \mathscr{H}$ is sufficiently close to F then this polygon is contained in $\varrho(H)$. Along with Theorem 6, this proves Theorem B. ∎

Remark 2. Since $F \in \mathscr{H}$ is homotopic to a map without periodic points, from Theorem VI E3 of [B] it follows that index sum of the set of fixed points in any Nielsen class for f_F^n is zero. Therefore from the above proof of Theorem B it follows that Theorem 1 can be strengthened and one gets at least two periodic points x with $F^n(y) = y + (p, q)$ for $y \in \pi^{-1}(x)$.

3. Examples. Let $\psi_1, \psi_2\colon \boldsymbol{R} \to [0, 1]$ be continuous periodic functions with period 1 and such that for each $i \in \{1, 2\}$ we have $\psi_i(x) = 0$ if and only if $x \in \boldsymbol{Z}$ and $\psi_i(x) = 1$ if and only if $x - \frac{1}{2} \in \boldsymbol{Z}$. We define $F_1, F_2\colon \boldsymbol{R}^2 \to \boldsymbol{R}^2$ (in similar way to Example 2 of [LM]) by the formulae

$$F_1(x, y) = (x, y + \psi_1(x)), \qquad F_2(x, y) = (x + \psi_2(y), y).$$

Then we set $F = F_2 \circ F_1$. Clearly, $F \in \mathscr{H}$.

We have $F((0, 0)) = (0, 0)$, $F((\frac{1}{2}, 0)) = (\frac{1}{2}, 1)$, $F((0, \frac{1}{2})) = (1, \frac{1}{2})$, $F((\frac{1}{2}, \frac{1}{2})) = (\frac{3}{2}, \frac{3}{2})$. Therefore $\varrho(F, (0, 0)) = (0, 0)$, $\varrho(F, (\frac{1}{2}, 0)) = (0, 1)$, $\varrho(F, (0, \frac{1}{2})) = (1, 0)$ and $\varrho(F, (\frac{1}{2}, \frac{1}{2})) = (1, 1)$. Hence, by the convexity of $\varrho(F)$ (Theorem 4) we have $[0, 1]^2 \subset \varrho(F)$. On the other hand, if $F(x, y) = (x', y')$ then $0 \leqslant x' - x \leqslant 1$ and $0 \leqslant y' - y \leqslant 1$. Therefore $\varrho(F) \subset [0, 1]^2$. This proves that $\varrho(F) = [0, 1]^2$.

We shall show that although the interior of $\varrho(F)$ is non-empty, there is no measure $\mu\in\mathcal{M}_E(F)$ with $\int\varphi_F\,d\mu=v$ when v belongs to the boundary of $\varrho(F)$, unless v is an extreme point of $\varrho(F)$ (i.e. one of the vertices of the square $[0,1]^2$). In particular, if v belongs to the boundary of $\varrho(F)$ but is not a vertex of $[0,1]$ and both coordinates of v are rational then there is no point $z\in\mathbf{R}^2$ with $\pi(z)$ periodic and $\varrho(F,z)=v$.

Suppose that $\mu\in\mathcal{M}_E(F)$ and

$$\int\varphi_F\,d\mu=(\alpha,0). \tag{10}$$

Let $\varphi_F=(\varphi_1,\varphi_2)$. By the definition of F, we have $\varphi_2\geqslant 0$, which together with (10) gives $\varphi_2=0$ μ-almost everywhere. This means that

$$\pi^{-1}(\operatorname{supp}\mu)\subset\mathbf{Z}\times\mathbf{R}.$$

However, supp μ is f_F-invariant, so $\pi^{-1}(\operatorname{supp}\mu)$ is F-invariant. The only points of $\mathbf{Z}\times\mathbf{R}$ for which also their image belongs to $\mathbf{Z}\times\mathbf{R}$ are the points of the form p or $p+(0,\frac{1}{2})$ where $p\in\mathbf{Z}^2$. Therefore

$$\operatorname{supp}\mu\subset\{\pi((0,0)),\ \pi((0,\tfrac{1}{2}))\}.$$

The measure μ is ergodic, so μ is concentrated either at $\pi((0,0))$ or at $\pi((0,\frac{1}{2}))$. In the first case we have $\alpha=0$ and in the second case $\alpha=1$.

In a similar way we can show that if $\mu\in\mathcal{M}_E(F)$ and $\int\varphi_F\,d\mu$ is equal to $(\alpha,1)$, $(0,\alpha)$ or $(1,\alpha)$ then also $\alpha=0$ or $\alpha=1$ in each case. This completes the proof of described properties of F.

Remark 3. The functions ψ_1 and ψ_2 can be chosen even real analytic and then F is real analytic.

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The Axiom of Choice, the Löwenheim–Skolem Theorem and Borel models

by

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Abstract. We characterise those cardinal numbers $\mathfrak{m}$ for which one can prove without using the Axiom of Choice that if a countable theory has an infinite model, then it has a model of cardinality $\mathfrak{m}$. We prove also that if a countable theory has an infinite model, then it has a model whose universe is the real line and whose relations and functions are Borel sets.

§ 1. Introduction and results. A. Ehrenfeucht raised the following question. Does every countable first order theory T which has an infinite model have a Borel model, i.e., a model whose universe is the real line $\mathbf{R}$ and whose relations and functions are Borel subsets of the appropriate finite powers of $\mathbf{R}$? We shall see that the answer is yes (1). When thinking about this problem it occurred to us that if a Borel model exists then we should be able to prove this without using the Axiom of Choice (AC), and this in turn led us to the question of characterising those infinite cardinal numbers for which the Upward Löwenheim–Skolem Theorem can be proved without using AC. There exists a simple obstruction which was found by Vaught [4]. Namely if T_0 is the theory of one binary function based on the axiom

$$f(x,y)=f(u,v)\rightarrow(x=u\wedge y=v)$$

and M is a model of T_0 of cardinality $\mathfrak{m}$ then, of course, $\mathfrak{m}^2=\mathfrak{m}$. But, a well-known theorem of Tarski says that

$$(\forall\mathfrak{m}\geqslant\aleph_0)\,[m^2=m]\ \leftrightarrow\ \text{AC}.$$

Hence, in the absence of AC, the Upward Löwenheim–Skolem Theorem for $\mathfrak{m}$ cannot be proved without the assumption $\mathfrak{m}^2=\mathfrak{m}$. Another obstruction arises if we consider the theory T_1 of linear ordering relations. Namely if the Upward Löwenheim–Skolem Theorem holds for T_1 and $\mathfrak{m}$, then every set of cardinality $\mathfrak{m}$ can be linearly ordered. Our main result asserts that the above two obstructions are the only ones.

THEOREM 1. (Without AC) *For every cardinal number* $\mathfrak{m}>1$ *the following two conditions are equivalent*:

(i) $\mathfrak{m}^2=\mathfrak{m}$ *and every set of cardinality* $\mathfrak{m}$ *can be linearly ordered.*

(ii) *Every countable theory which has an infinite model has a model of cardinality* $\mathfrak{m}$.

(1) See Note added at the end of the paper.