But then $\{\beta < \lambda_0 \colon L(f, \varkappa, \beta, \lambda_0)\}$ is cofinal below λ_0. By closure considerations yet again, $\{\beta < \lambda_0 \colon M_U \models L(f, \varkappa, \beta, \lambda_0)\}$ is cofinal below λ_0. Then $\{\beta < \lambda_0 \colon \{\gamma < \lambda_0 \colon L(f, \varkappa, \beta, \gamma)\}$ is cofinal below $\lambda_0\}$ is cofinal below λ_0. It is then easy to find a set A such that A is cofinal below λ_0, A contains none of its limit points, and, if $\{\alpha_\delta \colon \delta < \lambda_0\}$ is an increasing enumeration of the elements of A, then, for any $\delta < \lambda_0$, $L(f, \varkappa, \alpha_\delta, \alpha_{\delta+1})$ holds. This A satisfies the premises of the theorem, and hence its conclusion. ■

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On the l-equivalence of metric spaces

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Abstract. In this paper we present topological properties of metric spaces which are preserved by l-equivalence. Furthermore, we present an isomorphic classification of the function spaces $C_p(X)$ where X is any countable metric space with scattered height less than or equal to ω.

0. Introduction. *By a space we mean a Tikhonov space.* For a space X we define $C(X)$ ($C^*(X)$) to be the set of all continuous (bounded continuous) real valued functions on X. We can topologize these function spaces in several natural ways. Whenever we endow $C(X)$ ($C^*(X)$) with the compact-open topology we denote it by $C_0(X)$ ($C_0^*(X)$), and if we endow $C(X)$ ($C^*(X)$) with the topology of pointwise convergence we denote it by $C_p(X)$ ($C_p^*(X)$).

In [10] van Mill proved that for a countable metric space X which is not locally compact we have $C_p^*(X) \approx \sigma_\omega$, where $\sigma_\omega = (l_f^2)^\infty$ and $l_f^2 = \{x \in l^2 \mid x_i = 0$ for all but finitely many $i\}$ (l^2 denotes separable Hilbert space). Furthermore in [5] it was proved that under the same conditions $C_p(X) \approx \sigma_\omega$. It is easily seen that for an infinite countable discrete space X, $C_p(X) \approx \boldsymbol{R}^\infty$. The gap between "discrete" and "not locally compact" was filled in by Dobrowolski, Gul'ko and Mogilski in [7]. They proved that for every countable metric nondiscrete space X, $C_p^*(X) \approx C_p(X) \approx \sigma_\omega$. After these results it is interesting to study linear homeomorphism between the function spaces $C_p(X)$ ($C_p^*(X)$), for countable metric spaces X.

In [12] Pelant gives an example of two countable metric spaces X and Y, which are both not locally compact, such that $C_p^*(X)$ and $C_p^*(Y)$ are not linearly homeomorphic. In [5], Baars, de Groot, van Mill and Pelant gave an example of two countable metric spaces X and Y, which are both not locally compact, such that $C_p(X)$ and $C_p(Y)$ are not linearly homeomorphic. In [3] Baars and de Groot presented an isomorphic classification of the function spaces $C_p(X)$, for zero-dimensional locally compact separable metric spaces X. These classification results depend strongly on results in [1] of Arkhangel'skiĭ and on the isomorphic classification of the function spaces $C_0(X)$, where

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X is any countable compact metric space, which was proved by Bessaga and Pełczyński in [6].

In this paper we present topological properties of metric spaces which are preserved by l-equivalence. We show that these properties are sufficient to give an isomorphic classification for the function spaces $C_p(X)$, where X is any countable metric space with scattered height less than or equal to ω. In addition we give an example which shows that these properties are not sufficient outside this class of spaces.

1. Preliminaries. In this section we present some definitions and results (some old and some new) which we use in the next sections. First we state some definitions and a proposition from [1].

Let $\varphi: C(X) \to C(Y)$ be a linear function, where X and Y are Tikhonov spaces. For every $y \in Y$, the *support* of y in X with respect to φ is defined to be the set $\operatorname{supp}(y)$ of all $x \in X$ satisfying the condition that for every neighborhood U of x, there is an $f \in C(X)$ such that $f(X \backslash U) \subset \{0\}$ and $\varphi(f)(y) \neq 0$. For a subset A of Y, we denote $\bigcup\{\operatorname{supp}(y) \mid y \in A\}$ by $\operatorname{supp} A$. Whenever φ is a linear bijection, we can consider the support of a point in Y with respect to φ and the support of a point in X with respect to φ^{-1}. It will always be clear which support we mean. Finally, a subset A of X is said to be *bounded* if for every $f \in C(X)$, $f(A)$ is bounded in $\boldsymbol{R}$.

1.1. Proposition ([1], Arkhangel'skiĭ). *Let X and Y be spaces, and let φ: $C_p(X) \to C_p(Y)$ be a continuous linear mapping. If A is a bounded subset of Y, then $\overline{\operatorname{supp} A}$ is bounded in X (in particular, if X is metric, then $\overline{\operatorname{supp} A}$ is compact).*

Another result in [1] is that if X and Y are metric and φ: $C_p(X) \to C_p(Y)$ is a continuous linear mapping, then φ considered as a map from $C_0(X)$ to $C_0(Y)$ is also a continuous linear mapping. Hence every linear homeomorphism between $C_p(X)$ and $C_p(Y)$ can also be considered as a linear homeomorphism between $C_0(X)$ and $C_0(Y)$. In the sequel we shall use this result without explicit reference.

Whenever spaces X and Y are homeomorphic, we denote that $X \approx Y$, and whenever linear spaces E and F are linearly homeomorphic, we denote that by $E \sim F$.

When dealing with function spaces endowed with the topology of pointwise convergence, it is possible to give a precise description of supports (cf. Lemma 1.2). We would like to thank J. Pelant for providing us with this description of supports.

Let X and Y be Tikhonov spaces, φ: $C_p(X) \to C_p(Y)$ a continuous linear map and $y \in Y$ fixed. Notice that the map ψ_y: $C_p(X) \to \boldsymbol{R}$ defined by $\psi_y(f) = \varphi(f)(y)$ is continuous and linear. So $\psi_y \in L(X)$, the dual of $C_p(X)$. Since the evaluation mappings ξ_x $(x \in X)$ defined by $\xi_x(f) = f(x)$ for $f \in C_p(X)$ form a Hamel basis for $L(X)$ (cf. [11]), for $\psi_y \neq 0$ there are $x_1, \ldots, x_n \in X$ and $\lambda_1, \ldots, \lambda_n \in \boldsymbol{R} \backslash \{0\}$ such that $\psi_y = \sum_{i=1}^n \lambda_i \xi_{x_i}$ (notice that whenever φ is a bijection, $\psi_y \neq 0$ for every $y \in Y$). This means that for every $f \in C_p(X)$, $\varphi(f)(y) = \sum_{i=1}^n \lambda_i f(x_i)$. Then

1.2. Lemma. $\operatorname{supp}(y) = \{x_1, \ldots, x_n\}$.

Proof. Let $x \in \operatorname{supp}(y)$ and suppose that $x \notin \{x_1, \ldots, x_n\}$. Since $X \backslash \{x_1, \ldots, x_n\}$ is open, there is $f \in C_p(X)$ such that $f(x_i) = 0$ for every $i \leqslant n$ and $\varphi(f)(y) \neq 0$. But $\varphi(f)(y) = \sum_{i=1}^n \lambda_i f(x_i) = 0$. Contradiction.

Now let $i \leqslant n$ be fixed and U an open neighborhood of x_i. Let $V = U \backslash \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}$. Notice that V is an open neighborhood of x_i. Let $f \in C_p(X)$ be a Urysohn function with $f(X \backslash V) = 0$ and $f(x_i) = 1$. Then $\varphi(f)(y) = \sum_{i=1}^n \lambda_i f(x_i) = \lambda_i \neq 0$. Since $X \backslash U \subset X \backslash V$ we also have $f(X \backslash U) = 0$ and we are done. ■

1.3. Corollary. *Let X and Y be Tikhonov spaces, $y \in Y$, φ: $C_p(X) \to C_p(Y)$ a continuous linear map and $f, g \in C_p(X)$. If f and g are equal on $\operatorname{supp}(y)$, then $\varphi(f)(y) = \varphi(g)(y)$.* ■

Another useful property of supports with respect to the topology of pointwise convergence is given in the following

1.4. Proposition. *Let X and Y be Tikhonov spaces and φ: $C_p(X) \to C_p(Y)$ be a linear homeomorphism. Then for every $x \in X$ we have $x \in \operatorname{supp}\operatorname{supp}(x)$. (In other words, for every $x \in X$ there is $y \in \operatorname{supp}(x)$ such that $x \in \operatorname{supp}(y)$.)*

Proof. Let $x \in X$ and suppose $x \notin \operatorname{supp}\operatorname{supp}(x)$. Since $\operatorname{supp}\operatorname{supp}(x)$ is finite (Lemma 1.2), there is a Urysohn function $f \in C_p(X)$ such that $f(x) = 1$ and $f(\operatorname{supp}\operatorname{supp}(x)) = 0$. By Corollary 1.3 it follows that $\varphi(f) = 0$ on $\operatorname{supp}(x)$ and again by Corollary 1.3 it then follows that $f(x) = \varphi^{-1}(\varphi(f))(x) = 0$. Contradiction. ■

We now come to another subject. Let X be a space. For every ordinal α we define $X^{(\alpha)}$, the *αth derivative*, by transfinite induction as follows:

(a) $X^{(0)} = X$.

(b) If α is a successor, say $\alpha = \beta + 1$, then $x \in X^{(\alpha)}$ iff x is an accumulation point of $X^{(\beta)}$.

(c) If α is a limit ordinal, then $X^{(\alpha)} = \bigcap_{\beta < \alpha} X^{(\beta)}$.

We say that X is *scattered* if there is an ordinal α with $X^{(\alpha)} = \emptyset$ and we define the *scattered height* $\varkappa(X)$ to be the smallest ordinal α such that $X^{(\alpha)} = \emptyset$.

For every pair of ordinals $\alpha \leqslant \beta$ let $[\alpha, \beta] = \{\gamma \mid \alpha \leqslant \gamma \leqslant \beta\}$ with the order-topology.

We have the well-known

1.5. Theorem (Sierpiński–Mazurkiewicz [9]). *Let X be a countable compact metric space and m finite. Then $X \approx [1, \omega^\alpha \cdot m]$ if and only if $\varkappa(X) = \alpha + 1$ and $X^{(\alpha)}$ contains m points.*

In the following theorem we state classification results from Bessaga and Pełczyński (see Theorem 1 of [6]) and from [3].

1.6. Theorem. *Let $\omega \leqslant \alpha \leqslant \beta < \omega_1$. Then the following statements are equivalent:*

(i) $C_0([1, \alpha]) \sim C_0([1, \beta])$.

(ii) $C_p([1, \alpha]) \sim C_p([1, \beta])$.

(iii) $\beta < \alpha^\omega$.

An ordinal α is a *prime component* if the following holds: whenever $\alpha = \beta + \delta$ for ordinals β and δ, then $\delta = 0$ or $\delta = \alpha$.

Another useful lemma which can easily be obtained from results in [6] is (cf. Lemma 3.7 in [3]):

1.7. LEMMA. *Let ψ: $C_0([1, \omega^\mu]) \to C_0([1, \omega^\nu])$ be a linear embedding with $\mu, \nu \geqslant 1$ and μ a prime component. Then $\mu \leqslant \nu$.*

Finally, whenever (X, d) is a metric space, $x \in X$ and $\varepsilon > 0$, we put $B(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$.

2. Some topological properties preserved by *l*-equivalence. Two spaces X and Y are said to be *l-equivalent* whenever $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic (cf. Arkhangel'skiĭ [2]). In this section we present topological properties of metric spaces which are preserved by *l*-equivalence, i.e. properties such that if X and Y are *l*-equivalent and X has this property, then Y has this property.

Let X be a space and $X_0 \subset X$. For every ordinal we define the set $X^{(\alpha)}$ with respect to the pair (X, X_0) by transfinite induction as follows:

(1) $X^{(0)} = X_0$.

(2) If α is a successor, say $\alpha = \beta + 1$, then $x \in X^{(\alpha)}$ iff for every neighborhood U of x, $U \cap X^{(\beta)}$ is not compact.

(3) If α is a limit ordinal, then $X^{(\alpha)} = \bigcap_{\beta < \alpha} X^{(\beta)}$.

The following lemma will be used frequently, but will not always be mentioned.

2.1. LEMMA. *Let X be a space and X_0 a closed subspace of X. Then for every ordinal α, $X^{(\alpha)}$ is closed.*

Proof. We prove this by transfinite induction on α. The case $\alpha = 0$ is a triviality. First suppose that α is a successor, say $\alpha = \beta + 1$. Let $x \in X \backslash X^{(\alpha)}$. Then there is an open neighborhood U of x such that $\bar{U} \cap X^{(\beta)}$ is compact. Then $U \cap X^{(\alpha)} = \emptyset$. Therefore $X^{(\alpha)}$ is closed. Secondly, if α is a limit ordinal, then $X^{(\alpha)} = \bigcap_{\beta < \alpha} X^{(\beta)}$, so by our inductive hypothesis, $X^{(\alpha)}$ is closed. ∎

2.2. LEMMA. *Let X be a paracompact space, X_0 closed in X and $\alpha \geqslant 1$ an ordinal. Let $V \subset X$ be open such that $\bar{V} \cap X^{(\alpha)} = \emptyset$. Then there is a locally finite family $\{V_s \mid s \in S\}$ consisting of open sets such that $V = \bigcup_{s \in S} V_s$ and for every $s \in S$, there is $\beta < \alpha$ with $\bar{V}_s \cap X^{(\beta)}$ compact.*

Proof. *Case* 1: α is a successor, say $\alpha = \beta + 1$.

Since $\bar{V} \cap X^{(\alpha)} = \emptyset$, for every $x \in \bar{V}$, there is a neighborhood U_x of x such that $U_x \cap X^{(\beta)}$ is compact. For every $x \in \bar{V}$ find W_x open with $x \in W_x \subset \bar{W}_x \subset U_x$. Since $\{W_x \mid x \in \bar{V}\} \cup \{X \backslash \bar{V}\}$ is an open cover of X, there is a locally finite open refinement $\{O_s \mid s \in S\}$. For every $s \in S$, let $V_s = O_s \cap V$. Then $\{V_s \mid s \in S\}$ is a locally finite family consisting of open sets such that $V = \bigcup_{s \in S} V_s$. In addition, if $s \in S$ and $V_s \neq \emptyset$ there is $x \in \bar{V}$ with $V_s \subset W_x$. Then $\bar{V}_s \cap X^{(\beta)} \subset U_x \cap X^{(\beta)}$. So $\bar{V}_s \cap X^{(\beta)}$ is compact.

Case 2: α is a limit ordinal.

Then $\mathscr{U} = \{X \backslash X^{(\beta)} \mid \beta < \alpha\} \cup \{X \backslash \bar{V}\}$ is an open cover of X. So there is a locally finite open family $\{O_s \mid s \in S\}$ such that $\{\bar{O}_s \mid s \in S\}$ refines $\mathscr{U}$. For every $s \in S$ let $V_s = V \cap O_s$. Then $\{V_s \mid s \in S\}$ is a locally finite family of open sets such that $V = \bigcup_{s \in S} V_s$. Now fix $s \in S$ and suppose $V_s \neq \emptyset$. Then there is $\beta < \alpha$ such that $\bar{V}_s \subset X \backslash X^{(\beta)}$, which implies $\bar{V}_s \cap X^{(\beta)} = \emptyset$. ∎

We now define the following notions. Let X and Y be spaces. Let X_0 be closed in X and Y_0 be closed in Y. Let φ: $C_p(X) \to C_p(Y)$ be a linear bijection and α an ordinal. We define the pair (X, X_0) to be (φ, α)*-relative* to the pair (Y, Y_0) if the following holds:

If U and V are open in X and W is open in Y such that $(\operatorname{supp} U) \cap W = \emptyset$ and $\operatorname{supp} W \subset U \cup V$, then $W \cap Y^{(\alpha)} \neq \emptyset$ implies $\bar{V} \cap X^{(\alpha)} \neq \emptyset$.

We define (X, X_0) and (Y, Y_0) to be *l-equivalent pairs* if there is a linear homeomorphism φ: $C_p(X) \to C_p(Y)$ such that (X, X_0) is $(\varphi, 0)$-relative to (Y, Y_0) and (Y, Y_0) is $(\varphi^{-1}, 0)$-relative to (X, X_0). Note that two spaces X and Y are *l*-equivalent spaces if and only if $(X, \emptyset)$ and $(Y, \emptyset)$ are *l*-equivalent pairs.

Before we prove an important lemma which uses the notion of φ-relativeness, we first prove the following

2.3. LEMMA. *Let X and Y be normal spaces. Let K be compact and nonempty in Y and suppose $\{V_n \mid n \subset N\}$ is a decreasing base for K in Y. Let $\{A_s \mid s \in S\}$ be a locally finite family in X. Furthemore, let φ: $C_p(X) \to C_p(Y)$ be a continuous linear mapping. Then there are $m \in N$ and $s_1, \ldots, s_m \in S$ such that $(\operatorname{supp} V_m) \cap \bigcup_{s \notin \{s_1, \ldots, s_m\}} A_s = \emptyset$.*

Proof. Suppose the contrary. Then there are distinct $s_i \in S$ $(i \in N)$ and points $x_i \in \operatorname{supp} V_i \cap A_{s_i}$. Suppose $x_i \in \operatorname{supp} y_i$ with $y_i \in V_i$. Notice that $L = \{y_i \mid i \in N\} \cup K$ is compact, so by Proposition 1.1, $\operatorname{supp} L$ is bounded. It follows that $\{x_i \mid i \in N\}$ is also bounded. However, since $\{A_{s_i} \mid i \in N\}$ is locally finite, $\{x_i \mid i \in N\}$ is a closed and discrete set. Contradiction. ∎

Remark. In a metric space, every nonempty compact subset has a countable decreasing open base.

2.4. LEMMA. *Let X and Y be metric spaces, X_0 closed in X and Y_0 closed in Y. Let φ: $C_p(X) \to C_p(Y)$ be a continuous linear bijection such that (X, X_0) is $(\varphi, 0)$-relative to (Y, Y_0). Then for every ordinal α, (X, X_0) is (φ, α)-relative to (Y, Y_0).*

Proof. We prove the lemma by transfinite induction on α. Since (X, X_0) is $(\varphi, 0)$-relative to (Y, Y_0), the case $\alpha = 0$ is established. So assume the lemma is true for every ordinal $\beta < \alpha$ with $\alpha \geqslant 1$. Suppose the lemma is false for α. Then there are U and V open in X and W open in Y such that $(\operatorname{supp} U) \cap W = \emptyset$, $\operatorname{supp} W \subset U \cup V$, $W \cap Y^{(\alpha)} \neq \emptyset$ and $\bar{V} \cap X^{(\alpha)} = \emptyset$. By Lemma 2.2, there is a locally finite family $\{V_s \mid s \in S\}$ consisting of open sets such that $V = \bigcup_{s \in S} V_s$ and for every $s \in S$ there is $\beta < \alpha$ such that $\bar{V}_s \cap X^{(\beta)}$ is compact. Let $y \in W \cap Y^{(\alpha)}$ and $\{W_m \mid m \in N\}$ be a base for y in W such that for every $m \in N$, $\bar{W}_{m+1} \subset W_m$. By Lemma 2.3, there are $m \in N$ and $s_1, \ldots, s_m \in S$ with

$$\operatorname{supp} W_m \cap \bigcup_{s \notin \{s_1, \ldots, s_m\}} V_s = \emptyset. \tag{1}$$

Now let $A = \bigcup_{i=1}^m V_{s_i}$. There is $\beta < \alpha$ such that $\bar{A} \cap X^{(\beta)}$ is compact. Also, notice the following: A and U are open in X, W_m is open in Y, $(\operatorname{supp} U) \cap W_m = \emptyset$ (because $W_m \subset W$ and $(\operatorname{supp} U) \cap W = \emptyset$) and $\operatorname{supp} W_m \subset U \cup A$ (by (1) and the fact that $\operatorname{supp} W \subset U \cup V$). Since $y \in W_m \cap Y^{(\beta)}$, our inductive hypothesis implies that $\bar{A} \cap X^{(\beta)} \neq \emptyset$. By the remark following Lemma 2.3, there is an open base $\{A_s \mid s \in N\}$ for $\bar{A} \cap X^{(\beta)}$ such that $\bar{A}_{s+1} \subset A_s$. Since $y \in Y^{(\alpha)}$ and $\bar{W}_{m+1}$ is a neighborhood of y, $\bar{W}_{m+1} \cap Y^{(\beta)}$ is not compact, so in Y there is a closed discrete subset $\{y_s \mid s \in N\}$ contained in $W_m \cap Y^{(\beta)}$. Let $\{O_s \mid s \in N\}$ be an open discrete family in W_m such that $y_s \in O_s$. Then by Lemma 2.3, there is $s \in N$ with

(2) $$\operatorname{supp} A_s \cap \bigcup_{i \geq s} O_i = \emptyset.$$

Now let $U' = U \cup A_s$, $V' = A \backslash \bar{A}_{s+1}$ and $W' = O_s$. Then U' and V' are open in X and W' is open in Y. We also have

$$(\operatorname{supp} U') \cap W' = (\operatorname{supp} U \cup \operatorname{supp} A_s) \cap O_s = \emptyset \quad \text{(by 2)}$$

and

$$\operatorname{supp} W' \subset \operatorname{supp} W_m \subset U \cup A \subset U' \cup V'.$$

Furthermore, $y_s \in W' \cap Y^{(\beta)}$ and

$$\overline{V'} \cap X^{(\beta)} = \overline{(A \backslash \bar{A}_{s+1})} \cap X^{(\beta)} \subset (\bar{A} \backslash A_{s+1}) \cap X^{(\beta)} = \emptyset.$$

This contradicts our inductive assumption. ■

2.5. Theorem. *Let X and Y be metric spaces, X_0 closed in X and Y_0 closed in Y. Suppose that (X, X_0) and (Y, Y_0) are l-equivalent pairs. Then for every ordinal α we have*

(a) *$X^{(\alpha)} = \emptyset$ if and only if $Y^{(\alpha)} = \emptyset$,*

(b) *$X^{(\alpha)}$ is compact if and only if $Y^{(\alpha)}$ is compact.*

Proof. Let $\varphi\colon C_p(X) \to C_p(Y)$ be a linear homeomorphism such that (X, X_0) is $(\varphi, 0)$-relative to (Y, Y_0) and (Y, Y_0) is $(\varphi^{-1}, 0)$-relative to (X, X_0). For (a), by applying Lemma 2.4 and the definition of (φ, α)-relativeness to $U = \emptyset$, $V = X$ and $W = Y$, we get $X^{(\alpha)} = \emptyset$ if $Y^{(\alpha)} = \emptyset$.

For (b) suppose that $Y^{(\alpha)}$ is compact and $X^{(\alpha)}$ is not. Since $X^{(\alpha)} \neq \emptyset$, by (a) we have $Y^{(\alpha)} \neq \emptyset$. Let $\{W_m \mid m \in N\}$ be an open decreasing base in Y for $Y^{(\alpha)}$ such that for every $m \in N$, $\bar{W}_{m+1} \subset W_m$. Furthermore, let $\{x_m \mid m \in N\}$ be closed and discrete in $X^{(\alpha)}$. Let $\{O_m \mid m \in N\}$ be an open discrete family in X such that $x_m \in O_m$. Then by Lemma 2.3 there is $m \in N$ such that $(\operatorname{supp} W_m) \cap \bigcap_{i \geq m} O_i = \emptyset$.

Now let $U = W_m$, $V = Y \backslash \bar{W}_{m+1}$ and $W = O_m$. Then U and V are open, W is open, $(\operatorname{supp} U) \cap W = \emptyset$ and $\operatorname{supp} W \subset Y = U \cup V$. In addition

$$\bar{V} \cap Y^{(\alpha)} = \overline{Y \backslash \bar{W}_{m+1}} \cap Y^{(\alpha)} = \emptyset \quad \text{and} \quad W \cap X^{(\alpha)} = O_m \cap X^{(\alpha)} \neq \emptyset.$$

Contradiction with Lemma 2.4. ■

2.6. Corollary. *Let X and Y be metric spaces, X_0 closed in X and Y_0 closed in Y. Suppose (X, X_0) and (Y, Y_0) are l-equivalent pairs. Let α be an ordinal. Then $X^{(\alpha)}$ is locally compact if and only if $Y^{(\alpha)}$ is locally compact.*

Proof. Notice that $X^{(\alpha)}$ is locally compact if and only if $X^{(\alpha+1)} = \emptyset$. So the Corollary is a direct consequence of Theorem 2.5 (a). ■

We now give examples of l-equivalent pairs. Therefore let α be an ordinal, and X and Y l-equivalent metric spaces. We prove that if α is a prime component, then $(X, X^{(\alpha)})$ and $(Y, Y^{(\alpha)})$ are l-equivalent pairs. For that we first need the following result which undoubtedly is known.

2.7. Lemma. *Let X be a first countable space and $\alpha < \omega_1$ an ordinal such that $X^{(\alpha)} \neq \emptyset$. Then there is $K \subset X$ such that $K \approx [1, \omega^\alpha]$.*

Proof. We prove the Lemma by transfinite induction on α. For $\alpha = 0$, it is a triviality. Now suppose the lemma is true for every ordinal $\beta < \alpha$, with $\alpha \geq 1$. Let $x \in X^{(\alpha)}$.

Case 1: α is a successor, say $\alpha = \beta + 1$.

Then there is a sequence $(x_n)_n$ in $X^{(\beta)}$ such that $x_n \to x$. Let $\{U_n \mid n \in N\}$ be a decreasing open base at x such that $x_n \in V_n = U_n \backslash \bar{U}_{n+1}$. Notice that V_n is open, so $V_n^{(\beta)} \supset V_n \cap X^{(\beta)}$. Therefore, $x_n \in V_n^{(\beta)}$. So by the induction hypothesis, there is $K_n \subset V_n$ such that $K_n \approx [1, \omega^\beta]$. Notice that for every $n \neq m$, $K_n \cap K_m = \emptyset$. Let $K = \bigcup_{n=1}^{\infty} K_n \cup \{x\}$. Then by Theorem 1.5, $K \approx [1, \omega^\alpha]$.

Case 2: α is a limit ordinal.

Let $(\beta_n)_n$ be an increasing sequence converging to α. Since $x \in X^{(\alpha)}$, there is a decreasing open base $\{U_n \mid n \in N\}$ for x such that if $V_n = U_n \backslash \bar{U}_{n+1}$, then $V_n^{(\beta)} \neq \emptyset$. By the induction hypothesis there is $K_n \subset V_n$ such that $K_n \approx [1, \omega^{\beta_n}]$. Then by Theorem 1.5, $K = \bigcup_{n=1}^{\infty} K_n \cup \{x\}$ is as required. ■

2.8. Proposition. *Let $\alpha \in \{0, 1\}$ and let X and Y be l-equivalent metric spaces. Then $(X, X^{(\alpha)})$ and $(Y, Y^{(\alpha)})$ are l-equivalent pairs.*

Proof. Notice that $X^{(\alpha)}$ is closed in X. Let $\varphi\colon C_p(X) \to C_p(Y)$ be a linear homeomorphism. It suffices to prove that $(X, X^{(\alpha)})$ is $(\varphi, 0)$-relative to $(Y, Y^{(\alpha)})$. Therefore let U and V be open disjoint in X and W open in Y such that $(\operatorname{supp} U) \cap W = \emptyset$ and $\operatorname{supp} W \subset U \cup V$. Suppose that $W \cap Y^{(\alpha)} = \emptyset$ and $\bar{V} \cap X^{(\alpha)} = \emptyset$.

Case 1: $\alpha = 0$.

Since $\bar{V} = \emptyset$, we have $\operatorname{supp} W \subset U$. Therefore by Proposition 1.4,

$$W \subset \operatorname{supp} \operatorname{supp} W \subset \operatorname{supp} U.$$

Since $(\operatorname{supp} U) \cap W = \emptyset$ this gives $W = \emptyset$. Contradiction.

Case 2: $\alpha = 1$.

Since $\bar{V} \cap X^{(1)} = \emptyset$, $V = \bar{V}$ consists of isolated points, say $V = \{x_s \mid s \in S\}$. Let $y \in W \cap Y^{(1)}$ and $\{W_m \mid m \in N\}$ a decreasing open base for y in W. By Lemma 2.3, there is $m \in N$ and $s_1, \ldots, s_m \in S$ such that $\operatorname{supp} W_m \cap \{x_s \mid s \notin \{s_1, \ldots, s_m\}\} = \emptyset$.

Now let $V' = \{x_{s_1}, \ldots, x_{s_m}\}$. Since $\operatorname{supp} W_m \subset U \cup V'$, it follows that

$$W_m \subset \operatorname{supp} \operatorname{supp} W_m \subset \operatorname{supp}(U \cup V') = \operatorname{supp} U \cup \operatorname{supp} V'.$$

Since $W_m \cap \operatorname{supp} U = \emptyset$, we have $W_m \subset \operatorname{supp} V'$. Because V' is finite, W_m is finite. Contradiction. ■

2.9. Proposition. *Let $\alpha < \omega_1$ be a prime component and let X and Y be l-equivalent separable metric zero-dimensional spaces. Then $(X, X^{(\alpha)})$ and $(Y, Y^{(\alpha)})$ are l-equivalent pairs.*

Proof. Notice that $X^{(\alpha)}$ is closed in X and that for every U clopen in X, $U^{(\alpha)} = U \cap X^{(\alpha)}$. Let $\varphi\colon C_p(X) \to C_p(Y)$ be a linear homeomorphism. It suffices to prove that $(X, X^{(\alpha)})$ is $(\varphi, 0)$-relative to $(Y, Y^{(\alpha)})$. Therefore let U and V be open in X and W open in Y such that $(\operatorname{supp} U) \cap W = \emptyset$ and $\operatorname{supp} W \subset U \cup V$. Suppose that $W \cap Y^{(\alpha)} \neq \emptyset$ and $\bar{V} \cap X^{(\alpha)} = \emptyset$. By Proposition 2.8 we assume $\alpha \geq \omega$. Let $y \in W \cap X^{(\alpha)}$ and let $\{W_m \mid m \in N\}$ be a decreasing clopen base for y in W.

Claim. *There is a discrete clopen family $\{V_m \mid m \in N\}$ such that $V \subset \bigcup_{m \in N} V_m$ and for every $m \in N$ there is $\beta < \alpha$ with $(V_m)^{(\beta)} = \emptyset$.*

Indeed, since $\bar{V} \cap X^{(\alpha)} = \emptyset$, $\{X \backslash X^{(\beta)} \mid \beta < \alpha\} \cup \{X \backslash \bar{V}\}$ is an open cover of X. Since X is separable metric zero-dimensional, this cover has a clopen disjoint refinement $\{V_i \mid i \in N\}$. Let $\mathscr{U} = \{V_i \mid V_i \cap V \neq \emptyset\}$. Notice that $\mathscr{U}$ is discrete and that $V \subset \bigcup \mathscr{U}$. Now let $V_i \in \mathscr{U}$. Since $V_i \cap V \neq \emptyset$, $V_i \not\subset X \backslash \bar{V}$, so there is $\beta < \alpha$ such that $V_i \subset X \backslash X^{(\beta)}$. Now $V_i^{(\beta)} = V_i \cap X^{(\beta)} \subset X \backslash X^{(\beta)} \cap X^{(\beta)} = \emptyset$ and the claim is proved.

Again by Lemma 2.3, there is $m \in N$ such that

$$\operatorname{supp} W_m \cap \bigcup_{i > m} V_m = \emptyset.$$

Let $V' = \bigcup_{i=1}^{m} V_i$. Notice that V' is clopen, $V' \cap X^{(\beta)} = \emptyset$ for some $\beta < \alpha$ and $\operatorname{supp} W_m \subset U \cup V'$. Since $W_m^{(\alpha)} = W_m \cap Y^{(\alpha)} \neq \emptyset$, by Lemma 2.7 there is a set $K \subset W_m$ such that $K \approx [1, \omega^{\alpha}]$. Let $L = \overline{\operatorname{supp} K} \cap V'$. By Proposition 1.1, L is compact. Also, L is nonempty. Indeed, if $(\operatorname{supp} K) \cap V' = \emptyset$, then $\operatorname{supp} K \subset U$, and therefore by Proposition 1.4, $K \subset \operatorname{supp} \operatorname{supp} K \subset \operatorname{supp} U$. Since $(\operatorname{supp} U) \cap K = \emptyset$, $K = \emptyset$. Contradiction.

For every $f \in C_0(V')$, define $f^+ \in C_0(X)$ by $f^+ \mid V' = f$ and $f^+ \mid X \backslash V' \equiv 0$, and for every $f \in C_0(W_m)$, define $f^* \in C_0(Y)$ by $f^* \mid W_m = f$ and $f^* \mid Y \backslash W_m \equiv 0$. Furthermore, let $r\colon V' \to L$ and $s\colon W_m \to K$ be retractions (see [8]).

Define $\psi\colon C_0(K) \to C_0(L)$ by $\psi(f) = \varphi^{-1}((f \circ s)^*) \mid L$ and $\theta\colon C_0(L) \to C_0(K)$ by $\theta(f) = \varphi((f \circ r)^+) \mid K$. Observe that ψ and θ are linear.

Claim. *$\theta(\psi(f)) = f$ for every $f \in C_0(K)$.*

Indeed, suppose not, say $\varphi((\psi(f) \circ r)^+) \mid K \neq f = (f \circ s)^* \mid K$. Then by Proposition 1.3, $(\psi(f) \circ r)^+ \mid \operatorname{supp} K \neq \varphi^{-1}((f \circ s)^*) \mid \operatorname{supp} K$. Since $(f \circ s)^* \mid Y \backslash W_m \equiv 0$ and $\operatorname{supp} U \subset Y \backslash W_m$, it follows that $\varphi^{-1}((f \circ s)^*) \mid U \equiv 0$. Furthermore, since $U \backslash V' \subset X \backslash V'$, we have $((\psi(f) \circ r)^+) \mid U \backslash V' \equiv 0$. So it follows that

$$\psi(f) = ((\psi(f) \circ r)^+) \mid L \neq \varphi^{-1}((f \circ s)^*) \mid L = \psi(f).$$

Contradiction and the claim is proved.

From the claim we conclude that ψ is a linear embedding. Since $L \subset V'$, we have $L^{(\beta)} = \emptyset$, so by the Cantor–Bendixson theorem L is countable metric and therefore by Theorem 1.5, there is $\gamma < \beta$ and $n \in N$ such that $L \approx [1, \omega^{\gamma} \circ n]$. Since by Theorem 1.6 $C_0([1, \omega^{\gamma} \circ n]) \sim C_0([1, \omega^{\gamma}])$, we have a linear embedding $\psi\colon C_0([1, \omega^{\alpha}]) \to C_0([1, \omega^{\gamma}])$. By Lemma 1.7 and the fact that α is a prime component it follows that $\alpha \leqslant \gamma$. But this gives a contradiction because $\gamma < \beta < \alpha$. ■

Remark. (a) For $\alpha < \omega_1$ not a prime component, there are l-equivalent countable metric spaces X and Y such that $X^{(\alpha)} = \emptyset$, $Y^{(\alpha)} \neq \emptyset$ (see Example 2.4 of [4]). So $(X, X^{(\alpha)})$ and $(Y, Y^{(\alpha)})$ are not l-equivalent pairs.

(b) From the proof of Proposition 2.9 it follows that any linear homeomorphism between $C_p(X)$ and $C_p(Y)$ gives that $(X, X^{(\alpha)})$ and $(Y, Y^{(\alpha)})$ are l-equivalent pairs. The question arises whether "being l-equivalent pairs" is independent of the choice of linear homeomorphism.

Let $\alpha, \beta < \omega_1$ be ordinals with α a prime component. By $X^{(\alpha, \beta)}$ we denote the set $X^{(\beta)}$ with respect to the pair $(X, X^{(\alpha)})$. Notice that if β is a successor, say $\beta = \gamma + 1$, then we have $X^{(\alpha, \beta)} = (X^{(\alpha, \gamma)})^{(0, 1)}$.

2.10. Corollary. *Let X and Y be l-equivalent separable metric zero-dimensional spaces and let $\alpha, \beta < \omega_1$ be ordinals with α a prime component. Then*

(a) $X^{(\alpha, \beta)} = \emptyset$ if and only if $Y^{(\alpha, \beta)} = \emptyset$,

(b) $X^{(\alpha, \beta)}$ is compact if and only if $Y^{(\alpha, \beta)}$ is compact.

If $\alpha \in \{0, 1\}$, then the above is also true for X and Y metric.

Proof. This follows directly from Propositions 2.8, 2.9 and Theorem 2.5. ■

2.11. Corollary. *Let X and Y be l-equivalent separable metric zero-dimensional spaces and let $\alpha < \omega_1$ be a prime component. Then*

(a) $X^{(\alpha)} = \emptyset$ if and only if $Y^{(\alpha)} = \emptyset$,

(b) $X^{(\alpha)}$ is compact if and only if $Y^{(\alpha)}$ is compact.

If $\alpha \in \{0, 1\}$, then the above is also true for X and Y metric.

Proof. This is an application of Corollary 2.10; take $\beta = 0$. ■

Remark. Corollary 2.11 partially answers a question in [4].

3. An isomorphic classification. In this section we give an isomorphic classification of function spaces of countable metric spaces which have scattered height less than or equal to ω. Since the case of finite spaces is easy, we deal with infinite spaces only.

Let X be a space. For ordinals, $\alpha, \beta < \omega_1$, we define the following:

$X(\alpha, \beta) = 0$ iff $X^{(\alpha, \beta)} = \emptyset$,

$X(\alpha, \beta) = 1$ iff $X^{(\alpha, \beta)}$ is nonempty and compact, and

$X(\alpha, \beta) = 2$ iff $X^{(\alpha, \beta)}$ is not compact.

3.1. Lemma. *For every $n \in N$, $X^{(0, n)} \subset X^{(1, n-1)} \subset X^{(0, n-1)}$.*

Proof. We prove the lemma by induction on n. For $n = 1$ we have

$$X^{(0,1)} \subset X^{(1)} = X^{(1,0)} \subset X = X^{(0,0)}.$$

Suppose the lemma is true for every $n < m$ with $m > 1$. Then

$$X^{(0,m)} = (X^{(0,m-1)})^{(0,1)} \subset (X^{(1,m-2)})^{(0,1)} = X^{(1,m-1)},$$

$$X^{(1,m-1)} = (X^{(1,m-2)})^{(0,1)} \subset (X^{(0,m-2)})^{(0,1)} = X^{(0,m-1)}. \ ■$$

Before we are going to deal with function spaces of countable metric spaces, we first deal with the countable metric spaces itself.

3.2. Corollary. *Let X be a countable metric space such that there is $n \in N$ with $X(0, n) = 0$. Let $n_0 = \min\{n \mid X(0, n) = 0\}$. Then $n_1 = \min\{n \mid X(1, n) = 0\}$ is well defined and $n_0 = n_1$ or $n_0 = n_1 + 1$.* ■

Proof. By Lemma 3.1, $X^{(1, n_0)} \subset X^{(0, n_0)}$, so that $n_1 \leqslant n_0$. Again by Lemma 3.1, $X^{(0, n_1 + 1)} \subset X^{(1, n_1)}$, so that $n_0 \leqslant n_1 + 1$. ■

3.3. Lemma. *Let A and B be closed in X with $A \subset B$ and suppose that $A(0, 1) = B(0, 1) = 1$. Then there is a decreasing clopen base $\{U_n \mid n \in N\}$ for $B^{(0,1)}$ in X such that $U_1 = X$ and $(U_n \backslash U_{n+1}) \cap A$ is not compact for every $n \in N$.*

Proof. Since $B^{(0,1)}$ is compact, there is a decreasing clopen base $\{V_n \mid n \in N\}$ for $B^{(0,1)}$ in X. We now inductively find the U_n. Let $U_1 = X$ and suppose we have found

$U_1, \ldots, U_n$ for some $n \in N$. Since $A^{(0,1)} \subset B^{(0,1)}$, U_n is a neighborhood of $A^{(0,1)}$. But then $U_n \cap A$ is not compact, from which it follows that there is an infinite closed discrete set E in $U_n \cap A$. Since $B^{(0,1)}$ is compact, without loss of generality we may assume that $E \cap B^{(0,1)} = \emptyset$, so there is $i > n$ such that $V_i \subset X \setminus E$. If we now let $U_{n+1} = V_i$, then $E \subset (U_n \setminus U_{n+1}) \cap A$. ■

3.4. Corollary. *Let* $m \in N$.

(a) *If* $X(0, m) = X(1, m) = 1$, *there is a clopen decreasing base* $\{U_n \mid n \in N\}$ *for* $X^{(0,m)}$ *in* X, *such that* $U_1 = X$ *and* $(U_n \setminus U_{n+1})(1, m-1) = 2$ *for every* $n \in N$.

(b) *If* $X(1, m) = X(0, m+1) = 1$, *there is a clopen decreasing base* $\{U_n \mid n \in N\}$ *for* $X^{(1,m)}$ *in* X *such that* $U_1 = X$ *and* $(U_n \setminus U_{n+1})(0, m) = 2$ *for every* $n \in N$.

Proof. This is a direct consequence of Lemmas 3.1 and 3.3. ■

3.5. Lemma. *Let* A *and* B *be closed in* X *with* $A \subset B$. *If* A *and* B *are locally compact but not compact, then* X *can be written as* $X = \oplus_{i=1}^{\infty} X_i$ *such that for each* i, $X_i \cap A$ *and* $X_i \cap B$ *are compact and nonempty.*

Proof. Since B is locally compact but not compact and X is zero-dimensional, we can write $X = \oplus_{i=1}^{\infty} K_i$ with $K_i \cap B$ compact for every $i \in N$. Since $A \subset B$, $A \cap K_i$ is compact for every i. Since A is not compact, there are infinitely many i's such that $A \cap K_i$ is not empty. By taking finite unions of the K_i's in the right order, we obtain the desired decomposition of X. ■

3.6. Corollary. *Let* $m \in N$.

(a) *If* $X(0, m) = 0$ *and* $X(1, m-1) = 2$, *then* $X = \oplus_{i=1}^{\infty} A_i$ *with* $A_i(0, m-1) = A_i(1, m-1) = 1$ *for every* $i \in N$.

(b) *If* $X(1, m) = 0$ *and* $X(0, m) = 2$, *then* $X = \oplus_{i=1}^{\infty} A_i$ *with* $A_i(0, m) = A_i(1, m-1) = 1$ *for every* $i \in N$.

Proof. This a direct consequence of Lemmas 3.1 and 3.5. ■

3.7. Lemma. *Let* A *and* B *be closed in* X *with* $A \subset B$. *If* A *is compact and nonempty and* B *is locally compact but not compact, then* $X = X_1 \oplus X_2$ *with*

(1) $X_1 \cap B$ *compact and nonempty and*

(2) $X_2 \cap A = \emptyset$.

Proof. As in the proof of Lemma 3.5, $X = \oplus_{i=1}^{\infty} K_i$ with $K_i \cap B$ compact and nonempty for each i. Since A is compact, there is i_0 such that $A \cap \oplus_{i > i_0} K_i = \emptyset$. Now let $X_1 = K_1 \oplus \ldots \oplus K_{i_0}$ and $X_2 = \oplus_{i > i_0} K_i$. ■

3.8. Corollary. *Let* $m \in N \cup \{0\}$.

(a) *If* $X(0, m) = 2$, $X(0, m+1) = 0$ *and* $X(1, m) = 1$, *then* $X = A \oplus B$ *with* $A(0, m) = 1$ *and* $B(1, m) = 0$.

(b) *If* $X(1, m) = 2$, $X(1, m+1) = 0$ *and* $X(0, m+1) = 1$, *then* $X = A \oplus B$ *with* $A(1, m) = 1$ *and* $B(0, m+1) = 0$.

Proof. This is a direct consequence of Lemmas 3.1 and 3.7. ■

Remark. Notice that in Corollary 3.8 (a) we also have $A(1, m) = 1$, $B(0, m+1) = 0$, $B(0, m) = 2$ and if $m \neq 0$, $B(1, m-1) = 2$, cf. Lemma 3.1.

In addition, in Corollary 3.8 (b) we have $A(0, m+1) = 1$, $B(1, m+1) = 0$, $B(1, m) = 2$ and $B(0, m) = 2$, cf. Lemma 3.1.

We now come to the subject function spaces. Let X_i $(i = 1, \ldots, 4)$ be spaces and let E_i be a linear subspace of $C_p(X_i)$ $(i = 1, \ldots, 4)$. Let $\varphi = (\varphi_3, \varphi_4)$: $E_1 \times E_2 \to E_3 \times E_4$ be a linear mapping. We define φ to be a *linear k-mapping* whenever the following holds:

For all $f_i \in E_i$ such that $f_i(X_i) \subset (-1/k, 1/k)$ $(i = 1, 2)$ we have

$$\varphi_i(f_1, f_2)(X_i) \subset (-1, 1) \quad (i = 3, 4).$$

We define φ to be a *linear k-homeomorphism* whenever φ is a linear homeomorphism such that both φ and φ^{-1} are linear k-mappings. Whenever there is a linear k-homeomorphism between $E_1 \times E_2$ and $E_3 \times E_4$ we write $E_1 \times E_2 \overset{k}{\sim} E_3 \times E_4$. By this definition we have also defined linear k-homeomorphism between E_1 and E_2 and between E_1 and $E_3 \times E_4$, by identifying E_i with $E_i \times \{0\}$ $(i = 1, 2)$. It is easily seen that the composition of a linear k-homeomorphism and a linear l-homeomorphism is a linear kl-homeomorphism.

For a space X and a subspace A of X we define $C_{p,A}(X)$ to be the linear subspace of $C_p(X)$ consisting of those functions which vanish on A. Whenever $A = \{a\}$ for some point $a \in X$, we write $C_{p,a}(X)$ instead of $C_{p,\{a\}}(X)$.

3.9. Lemma. *Let* X *be a countable metric space and* A *a closed subspace of* X. *Then*

$$C_p(X) \overset{2}{\sim} C_{p,A}(X) \times C_p(A).$$

Proof. Define ϱ: $C_p(X) \to C_p(A)$ by $\varrho(f) = f|A$. Let r: $X \to A$ be a retraction (see [8]) and define ξ: $C_p(A) \to C_p(X)$ by $\xi(f) = f \circ r$. Define φ: $C_p(X) \to C_{p,A}(X) \times C_p(A)$ by $\varphi(f) = (f - \xi\varrho(f), \varrho(f))$ and ψ: $C_{p,A}(X) \times C_p(A) \to C_p(X)$ by $\psi(f, g) = f + \xi(g)$. Then φ is a linear homeomorphism with $\varphi^{-1} = \psi$ (see [3]). It is easily seen that both φ and ψ are linear 2-mappings. ■

For a space X and a compact subspace A of X, let $Z_{X,A}$ be the space obtained from X by identifying A to a single point a.

3.10. Lemma. *Let* X *be a countable space and* A *a compact subspace of* X. *Then*

$$C_{p,A}(X) \overset{1}{\sim} C_{p,a}(Z_{X,A}).$$

Proof. For every $f \in C_{p,A}(X)$ there is a unique $\tilde{f}$ such that $\tilde{f} \circ p = f$, where p is the quotient mapping between X and $Z_{X,A}$. Then φ: $C_{p,A}(X) \to C_{p,a}(Z_{X,A})$ defined by $\varphi(f) = \tilde{f}$ is a linear homeomorphism. (cf. [3]). It is easily seen that φ is a linear 1-homeomorphism. ■

The next three lemmas are useful in the sequel. The proofs are left to the reader.

3.11. Lemma. *If* X *and* Y *are homeomorphic spaces, then* $C_p(X) \overset{1}{\sim} C_p(Y)$. ■

3.12. Lemma. *If* X *and* Y *are spaces and* A *is a subspace of* X, *then*

$$C_{p,A}(X) \times C_p(Y) \overset{1}{\sim} C_{p,A}(X \oplus Y). \ ■$$

3.13. Lemma. *If* $X = \oplus_{i=1}^{\infty} X_i$ *and* $Y = \oplus_{i=1}^{\infty} Y_i$ *such that* $C_p(X_i) \overset{k}{\sim} C_p(Y_i)$ *for every* $i \in N$, *then* $C_p(X) \overset{k}{\sim} C_p(Y)$. ■

3.14. Lemma. *Let* X *be a metric space and let* A *be a nonempty compact subspace of* X. *Let* $\{U_n \mid n \in N\}$ *be a clopen decreasing base for* A *in* X *such that* $U_1 = X$. *Let* Y *be a metric space and let* B *be a nonempty compact subspace of* Y. *Let* $\{V_n \mid n \in N\}$ *be a clopen* *decreasing base for B in Y such that $V_1 = X$. Suppose that $C_p(U_n \backslash U_{n+1}) \overset{k}{\sim} C_p(V_n \backslash V_{n+1})$ for every $n \in N$. Then*

$$C_{p,A}(X) \overset{k}{\sim} C_{p,B}(Y).$$

Proof. For every $n \in N$, let φ_n: $C_p(U_n \backslash U_{n+1}) \to C_p(V_n \backslash V_{n+1})$ be a linear k-homeomorphism. Define φ: $C_{p,A}(X) \to C_{p,B}(Y)$ by

$$\varphi(f) | V_n \backslash V_{n+1} = \varphi_n(f | U_n \backslash U_{n+1}) \quad \text{and} \quad \varphi(f) | B = 0.$$

To prove that φ is well defined it suffices to prove that $\varphi(f)$ is continuous at points of B. Therefore let $\varepsilon > 0$. Since $f(A) = 0$, there is an open neighborhood W of A with $f(W) \subset \subset (-\varepsilon/k, \varepsilon/k)$. There is $n_0 \in N$ such that $A \subset U_{n_0} \subset W$, so

$$f(U_{n_0}) \subset (-\varepsilon/k, \varepsilon/k).$$

Then it easily follows by k-linearity of φ_n for every n that $\varphi(f)(V_{n_0}) \subset (-\varepsilon, \varepsilon)$, so that $\varphi(f)$ is continuous at points of B. That φ is continuous is easily seen.

Define ψ: $C_{p,B}(Y) \to C_{p,A}(X)$ by

$$\psi(f) | U_n \backslash U_{n+1} = \varphi_n^{-1}(f | V_n \backslash V_{n+1}) \quad \text{and} \quad \psi(f) | A = 0.$$

As above we can show that ψ is a well-defined linear continuous mapping. In addition, it is easily seen that $\psi = \varphi^{-1}$ and that φ is a linear k-homeomorphism. ■

We are now in a position to prove an isomorphic classification of function spaces of countable spaces which have scattered height less than or equal to ω. First we consider the case of countable spaces which have scattered height less than ω. Therefore we first have to deal with some special cases in the following two lemmas.

3.15. Lemma. *Let $p \in N$. There is $k_p \in N$ such that if X and Y are infinite countable compact spaces with $\varkappa(X)$, $\varkappa(Y) \leqslant p$, then $C_p(X) \overset{k_p}{\sim} C_p(Y)$.*

Proof. Let X be an infinite countable compact metric space with $\varkappa(X) \leqslant p$. By Theorem 1.5, there are $1 \leqslant m \leqslant p$ and $n \in N$ such that $X \approx [1, \omega^m \cdot n]$. Let $a = \omega^m$ and $A = X^{(m)}$. Notice that A is finite. Then

$$\begin{aligned} C_p(X) &\overset{2}{\sim} C_{p,a}([1, \omega^m]) \times C_p(A) && \text{(Lemmas 3.9, 3.10 and 3.11)} \\ &\overset{1}{\sim} C_{p,a}(A \oplus [1, \omega^m]) && \text{(Lemma 3.12)} \\ &\overset{1}{\sim} C_{p,a}([1, \omega^m]) && \text{(Lemma 3.11)} \\ &\overset{2}{\sim} C_p([1, \omega^m]) && \text{(Lemmas 3.9, 3.10 and 3.11).} \end{aligned}$$

So that $C_p(X) \overset{4}{\sim} C_p([1, \omega^m])$.

To finish the lemma it suffices to prove the following

Claim. *There is $l \in N$ such that for every $1 \leqslant r \leqslant p$ we have*

$$C_p([1, \omega^r]) \overset{l}{\sim} C_p([1, \omega]).$$

Let $1 \leqslant r \leqslant p$. By Lemma 1.8, there is a linear homeomorphism φ: $C_p([1, \omega^r]) \to C_p([1, \omega])$. Then by the remark following Proposition 1.1, it follows that φ: $C_0([1, \omega^r]) \to C_0([1, \omega])$ is also a linear homeomorphism. Since these two function spaces are Banach spaces, there is $l(r) \in N$ such that for every $f \in C_0([1, \omega^r])$ we have

$$\frac{1}{l(r)} \|f\| < \|\varphi(f)\| < l(r) \|f\|.$$

Then $l = \max\{l(r) \mid r \leqslant p\}$ suffices. ■

Let $T = N^2 \cup \{\infty\}$, with the following base for its topology: every point in N^2 is isolated and a basic neighborhood for ∞ is $(\{n, n+1, \ldots\} \times N) \cup \{\infty\}$ $(n \in N)$.

3.16. Lemma. *Let $p \in N$. There is $l_p \geqslant k_p$ such that if X and Y are countable metric spaces with $\varkappa(X)$, $\varkappa(Y) \leqslant p$, $X(0, 1) = Y(0, 1) = 1$ and $X(1, 0) = Y(1, 0) = 1$, then*

$$C_p(X) \overset{l_p}{\sim} C_p(Y).$$

Proof. Let X be a countable metric space with $\varkappa(X) \leqslant p$ and $X(0, 1) = X(1, 0) = 1$. Let $A = X^{(1)}$. Then by assumption A is compact. It is easily seen that $Z_{X,A} \approx T$, so that by Lemmas 3.9, 3.10 and 3.11, $C_p(X) \overset{2}{\sim} C_{p,\infty}(T) \times C_p(A)$.

If A is finite, then $T \oplus A \approx T$, so $C_p(X) \overset{2}{\sim} C_{p,\infty}(T)$. If A is infinite, we have by Lemma 3.15, $C_p(A) \overset{k}{\sim} C_p([1, \omega])$. Note that by the above argument $C_p(T) \overset{2}{\sim} C_{p,\infty}(T)$, so that $C_p(X) \overset{4 \cdot k_p}{\sim} C_p(T \oplus [1, \omega])$. Since $(T \oplus [1, \omega])^{(1)}$ is finite, the same argument gives $C_p(T \oplus [1, \omega]) \overset{2}{\sim} C_{p,\infty}(T)$. We conclude that $C_p(X) \overset{8 \cdot k_p}{\sim} C_p(T)$. ■

3.17. Lemma. *Let $p \in N$. Then there are $r_1, \ldots, r_p \in N$ such that for $1 \leqslant n \leqslant p$ the following holds:*

If X and Y are infinite countable metric spaces with $\varkappa(X) \leqslant p$ and $\varkappa(Y) \leqslant P$ satisfying

$$(1)_n \qquad X(0, n) = Y(0, n) = 0, \quad X(0, n-1) = Y(0, n-1) \neq 0, \quad \textit{and} \quad X(1, n-1) = Y(1, n-1) \neq 0, \quad \textit{or}$$

$$(2)_n \qquad X(1, n) = Y(1, n) = 0, \quad X(0, n) = Y(0, n) \neq 0, \quad \textit{and} \quad X(1, n-1) = Y(1, n-1) \neq 0,$$

then $C_p(X) \overset{r_n}{\sim} C_p(Y)$.

Proof. Let $r_1 = l_p$ and for $1 < n \leqslant p$, $r_n = 4r_{n-1}$. We prove by induction on n that $r_1, \ldots, r_p$ are sufficient. For that, suppose we have proved the lemma for every $n < m$ with $m \geqslant 1$.

We prove $(1)_m$: $X(0, m) = Y(0, m) = 0$,

$$X(0, m-1) = Y(0, m-1) \neq 0 \quad \text{and} \quad X(1, m-1) = Y(1, m-1) \neq 0.$$

(Notice that then also $X(1, m) = Y(1, m) = 0$.)

Case 1: $X(0, m-1) = Y(0, m-1) = 1$.

Notice that in this case we also have $X(1, m-1) = Y(1, m-1) = 1$.

For $m = 1$ we have by Lemma 3.15, $C_p(X) \overset{k_p}{\sim} C_p(Y)$. Since $k_p \leqslant l_p = r_1$, this case is done. For $m > 1$, let $A = X^{(0,m-1)}$ and $B = Y^{(0,m-1)}$. By Lemma 3.9,

$$(1) \qquad C_p(X) \overset{2}{\sim} C_{p,A}(X) \times C_p(A) \quad \text{and} \quad C_p(Y) \overset{2}{\sim} C_{p,B}(Y) \times C_p(B).$$

Let $Z_1 = X \oplus A$ and $Z_2 = Y \oplus B$. Notice that since $m > 1$, $Z_1^{(0,m-1)} = X^{(0,m-1)}$,

$Z_2^{(0,m-1)} = Y^{(0,m-1)}$, $Z_1^{(1,m-1)} = X^{(1,m-1)}$ and $Z_2^{(1,m-1)} = Y^{(1,m-1)}$. Let $C = Z_1^{(0,m-1)}$ and $D = Z_2^{(0,m-1)}$. Then by (1) and by Lemma 3.12,

$$C_p(X) \overset{2}{\sim} C_{p,C}(Z_1) \quad \text{and} \quad C_p(Y) \overset{2}{\sim} C_{p,D}(Z_2).$$

By Corollary 3.4 (a), there are clopen decreasing bases $\{U_n \mid n \in N\}$ and $\{V_n \mid n \in N\}$ for C resp. D such that $U_1 = Z_1$, $V_1 = Z_2$,

$$(U_n \setminus U_{n+1})(1, m-2) = 2 \quad \text{and} \quad (V_n \setminus V_{n+1})(1, m-2) = 2.$$

Notice that then also $(U_n \setminus U_{n+1})(0, m-2) = (V_n \setminus V_{n+1})(0, m-2) = 2$. It is easily seen that

$$(U_n \setminus U_{n+1})(0, m-1) = 0 \quad \text{and} \quad (V_n \setminus V_{n+1})(0, m-1) = 0.$$

Then $(1)_{m-1}$ gives $C_p(U_n \setminus U_{n+1}) \overset{r_{m-1}}{\sim} C_p(V_n \setminus V_{n+1})$ for every $n \in N$. So that by Lemma 3.14, $C_{p,C}(Z_1) \overset{r_{m-1}}{\sim} C_{p,D}(Z_2)$. In conclusion we have $C_p(X) \overset{r_m}{\sim} C_p(Y)$. This completes the proof in case 1.

Case 2: $X(0, m-1) = Y(0, m-1) = 2$.

First assume that $X(1, m-1) = Y(1, m-1) = 1$. Then by Corollary 3.8 (a), $X = A \oplus B$ and $Y = C \oplus D$ with $A(0, m-1) = C(0, m-1) = 1$ and $B(1, m-1) = D(1, m-1) = 0$. By the remark following Corollary 3.8 we now have by case 1, $C_p(A) \overset{4\cdot r_{m-1}}{\sim} C_p(C)$ and for $m > 1$, by $(2)_{m-1}$, $C_p(B) \overset{r_{m-1}}{\sim} C_p(D)$. If $m = 1$ then B and D are infinite discrete and so $C_p(B) \overset{1}{\sim} C_p(D)$. With Lemma 3.13 it now follows that $C_p(X) \overset{r_m}{\sim} C_p(Y)$.

Secondly, if $X(1, m-1) = Y(1, m-1) = 2$, we have by Corollary 3.6 (a), $X = \bigoplus_{i=1}^{\infty} A_i$ and $Y = \bigoplus_{i=1}^{\infty} B_i$ with $A_i(0, m-1) = B_i(0, m-1) = A_i(1, m-1) = B_i(1, m-1) = 1$. By case 1, we then have $C_p(A_i) \overset{4\cdot r_{m-1}}{\sim} C_p(B_i)$, so that by Lemma 3.13, $C_p(X) \overset{r_m}{\sim} C_p(Y)$. This completes the proof of case 2.

The proof for the situation that X and Y satisfy the conditions of $(2)_m$ is almost the same as the one given above. Instead of Lemma 3.15 we use Lemma 3.16 and instead of the (a)-parts of Corollaries 3.4, 3.6 and 3.8 we use their (b)-parts. ■

3.18. THEOREM. *Let X and Y be infinite countable metric spaces with $\varkappa(X), \varkappa(Y) < \omega$ such that for every $n \in N$, $X(0, n) = Y(0, n)$ and $X(1, n) = Y(1, n)$. Then $C_p(X) \sim C_p(Y)$.*

Proof. Let $p = \max(\varkappa(X), \varkappa(Y))$. Notice that there is $n \in N$ such that $X(0, n) = 0$, so let n_0 and n_1 be as in Corollary 3.2. Notice than $n_0 > 0$ and that the respective values for X and Y are the same. If $n_1 = 0$ then X and Y are infinite discrete and therefore $C_p(X) \sim C_p(Y)$. If $n_1 > 0$ then X and Y satisfy $(1)_{n_1}$ or $(2)_{n_1}$ of Lemma 3.17 and so $C_p(X) \sim C_p(Y)$. ■

We have completed the case of countable metric spaces with scattered height less than ω, so from now on we have to consider spaces with scattered height equal to ω. Therefore let X be a countable metric space with $\varkappa(X) = \omega$. There are two cases to consider:

(a) there is $n \in N$ such that $X(0, n) = 0$,

(b) for every $n \in N$, $X(0, n) = 2$.

We will first deal with the first case.

3.19. LEMMA. *Let X and Y be countable metric spaces such that $\varkappa(X) = \omega$, $\varkappa(Y) \leqslant \omega$, $X(0, n) = Y(0, n)$ and $X(1, n) = Y(1, n)$ for every $n \in N \cup \{0\}$ and such that case* (a) *holds for X. Then $X = \bigoplus_{i=1}^{\infty} X_i$ and $Y = \bigoplus_{i=1}^{\infty} Y_i$ such that $\varkappa(X_i), \varkappa(Y_i) < \omega$ and for every $i, n \in N$, $X_i(0, n) = Y_i(0, n)$ and $X_i(1, n) = Y_i(1, n)$.*

Proof. Since X satisfies (a), there is $k \in N$ such that

$(1)_k$ $$X(0, k) = Y(0, k) = 0, \quad X(0, k-1) = Y(0, k-1) \neq 0, \quad \text{and}$$ $$X(1, k-1) = Y(1, k-1) \neq 0, \quad \text{or}$$

$(2)_k$ $$X(1, k) = Y(1, k) = 0, \quad X(0, k) = Y(0, k) \neq 0,$$ $$X(1, k-1) = Y(1, k-1) \neq 0.$$

We prove the lemma by induction on k. Suppose we have proved the lemma for every $k < m$ with $m \geqslant 1$. First let X and Y satisfy $(1)_m$.

Case 1: $X(0, m-1) = Y(0, m-1) = 1$.

Since $\varkappa(X) = \omega$, X is not compact. This implies $m > 1$. By Corollary 3.4 (a), there are clopen decreasing bases $\{U_n \mid n \in N\}$ and $\{U^n \mid n \in N\}$ for $X^{(0,m-1)}$ and $Y^{(0,m-1)}$ respectively, such that $U_1 = X$ and $V_1 = Y$,

$$(U_n \setminus U_{n+1})(1, m-2) = 2 \quad \text{and} \quad (V_n \setminus V_{n+1})(1, m-2) = 2.$$

CLAIM. *There is $l \in N$ such that $\varkappa(U_l) < \omega$.*

Since $\varkappa(X) = \omega$, $\mathscr{U} = \{X \setminus X^{(n)} \mid n \in N\}$ is an open cover of X without finite subcover. Since X is zero-dimensional, there is a disjoint clopen refinement $\{A_i \mid i \in N\}$ of $\mathscr{U}$. Since $X^{(0,m-1)}$ is compact, there is n such that $X^{(0,m-1)} \subset A_1 \oplus \ldots \oplus A_n$. There is $l \in N$ such that $U_l \subset A_1 \oplus \ldots \oplus A_n$, and this l satisfies the claim.

Without loss of generality we may assume that $\varkappa(V_l) < \omega$. Now let $X_1 = U_l$ and $Y_1 = V_l$. Notice that

$$X_1(0, n) = Y_1(0, n) \quad \text{and} \quad X_1(1, n) = Y_1(1, n) \quad \text{for every } n \in N,$$

$$(X \setminus U_k)(1, m-2) = (Y \setminus V_k)(1, m-2) = 2, \quad (X \setminus U_k)(0, m-1) = (Y \setminus V_k)(0, m-1) = 0.$$

So by $(1)_{m-1}$ we have $X \setminus U_k = \bigoplus_{i=2}^{\infty} X_i$ and $Y \setminus V_k = \bigoplus_{i=2}^{\infty} Y_i$ with $X_i(0, n) = Y_i(0, n)$ and $X_i(1, n) = Y_i(1, n)$ for every $i \geqslant 2$ and $n \in N$, and the lemma is proved in this case.

Case 2: $X(0, m-1) = Y(0, m-1) = 2$.

First, assume that $X(1, m-1) = Y(1, m-1) = 1$. Then by Corollary 3.8 (a), $X = A \oplus B$ and $Y = C \oplus D$ with $A(0, m-1) = C(0, m-1) = 1$ and $B(1, m-1) = D(1, m-1) = 0$. By the remark following Corollary 3.8 we now have in cases of scattered height ω, by case 1 or by $(2)_{m-1}$, the desired decomposition of X and Y.

Second, if $X(1, m-1) = Y(1, m-1) = 2$, we have by Corollary 3.6 (a), $X = \bigoplus_{i=1}^{\infty} A_i$ and $Y = \bigoplus_{i=1}^{\infty} B_i$ with $A_i(0, m-1) = B_i(0, m-1) = A_i(1, m-1) = B_i(1, m-1) = 1$. By case 1 (applied in cases where A_i or B_i has scattered height ω), we have the desired decomposition of X and Y. This completes $(1)_m$.

Whenever X and Y satisfy $(2)_m$, the proof is similar to the proof of $(1)_m$.

Instead of Corollaries 3.4 (a), 3.6 (a) and 3.8 (a) we now use their (b)-parts and instead of $(2)_{m-1}$ we use $(1)_m$. ∎

3.20. THEOREM. *Let X and Y be countable metric spaces such that $\varkappa(X) = \omega$, $\varkappa(Y) \leqslant \omega$ and for every $n \in N \cup \{0\}$, $X(0, n) = Y(0, n)$ and $X(1, n) = Y(1, n)$. If X is a space satisfying* (a), *then X and Y are l-equivalent.*

Proof. This follows directly from Theorem 3.18 and Lemmas 3.13 and 3.19. ∎

3.21. THEOREM. *Let X and Y be countable metric spaces such that $\varkappa(X) = \varkappa(Y) = \omega$ and both satisfying* (b). *Then X and Y are l-equivalent.*

Proof. We begin with the following.

CLAIM. *We can write $X = \bigoplus_{i=1}^{\infty} X_i$ and $Y = \bigoplus_{i=1}^{\infty} Y_i$ so that there are sequences $(n_i)_{i \in N}$ and $(m_i)_{i \in N}$ such that $n_i + 1 < m_i$, $m_i + 1 < n_{i+1}$, $X_i(1, n_i) \neq 0$, $X_i(1, n_{i+1}) = 0$, $Y_i(1, m_i) \neq 0$ and $Y_i(1, m_{i+1}) = 0$.*

It is easily seen that $\{X \backslash X^{(1,n)} \mid n \in N\}$ is an open cover of X without finite subcover. Since X is countable, there is a clopen disjoint refinement $\{A_i \mid i \in N\}$ of this cover. This means that for every $i \in N$, there is $k_i \in N$ such that $A_i(1, k_i) \neq 0$ and $A_i(1, k_i + 1) = 0$. The set $\{k_i \mid i \in N\}$ is not bounded (!), so we may assume $k_1 < k_2 \ldots$ (take unions of the A_i's). In the same way $Y = \bigoplus_{i=1}^{\infty} B_i$, and there are $l_1 < l_2 \ldots$ such that $B_i(1, l_i) \neq 0$ and $B_i(1, l_i + 1) = 0$. (Notice that Y satisfies (c) as well and therefore $\varkappa(Y) = \omega$.) Now let $(n_i)_{i \in N}$ and $(m_i)_{i \in N}$ be subsequences of $(k_i)_{i \in N}$ and $(l_i)_{i \in N}$, respectively, such that $n_i + 1 < m_i$, $m_i + 1 < n_{i+1}$. By letting X_i be a finite union of A_j's in the right order and the same for the Y_i's, we are done.

Let $Z = X_1 \oplus Y_1 \oplus X_2 \oplus Y_2 \oplus \ldots$

Because $n_i + 1 < m_i$, $(X_i \oplus Y_i)(0, n) = Y_i(0, n)$ and $(X_i \oplus Y_i)(1, n) = Y_i(1, n)$ for every $n \in N \cup \{0\}$. Both $X_i \oplus Y_i$ and Y_i satisfy (a), so by Theorem 3.18 or Theorem 3.20, $C_p(X_i \oplus Y_i) \sim C_p(Y_i)$, so that $C_p(Z) \sim C_p(Y)$. By interchanging the role of X and Y we also have $C_p(Z) \sim C_p(X)$. We conclude that $C_p(X) \sim C_p(Y)$. ∎

3.22. THEOREM. *Let X and Y be infinite countable metric spaces such that $\varkappa(X)$, $\varkappa(Y) \leqslant \omega$. Then X and Y are l-equivalent iff for every $n \in N$, $X(0, n) = Y(0, n)$ and $X(1, n) = Y(1, n)$.*

Proof. This follows immediately from Theorems 3.18, 3.20, 3.21 and Corollary 2.10. ∎

4. Remarks. The question naturally arises whether Theorem 3.22 can be generalized to all countable metric spaces. One is tempted to conjecture the following:

Let X and Y be countable metric spaces. Then X and Y are l-equivalent iff for every prime component α and ordinal β we have $X(\alpha, \beta) = Y(\alpha, \beta)$.

We will show that this conjecture is false. To this end, let X be a space and put $X^* = \bigcap_{n \in N} (X^{(n)})^{(0,1)}$.

4.1. LEMMA. *Let X and Y be separable metric zero-dimensional l-equivalent spaces. Then (X, X^*) and (Y, Y^*) are l-equivalent pairs.*

Proof. It is easily seen that X^* is closed in X. Now let $\varphi\colon C_p(X) \to C_p(Y)$ be a linear homeomorphism. Notice that also $\varphi\colon C_0(X) \to C_0(Y)$ is a linear homeomorphism.

By Lemma 2.4 it suffices to prove that (X, X^*) is $(\varphi, 0)$-relative to (Y, Y^*). Let U and V be open subsets of X and W an open subset of Y such that

$$(\operatorname{supp} U) \cap W = \emptyset \quad \text{and} \quad \operatorname{supp} W \subset U \cup V,$$

and suppose $W \cap Y^* \neq \emptyset$ and $\overline{V} \cap X^* = \emptyset$.

Let $y \in W \cap Y^*$ and let $\{W_n \mid n \in N\}$ be a clopen decreasing base at y in W. It is easily seen that $V \subset \bigcup_{i=1}^{\infty} A_i$, where $\{A_i \mid i \in N\}$ is a clopen discrete family such that for every $i \in N$, there is $n_i \in N$ with $(A_i^{(n_i)})^{(0,1)} = \emptyset$. By Lemma 2.3 there is $n \in N$ such that

$$(1) \qquad \operatorname{supp} W_n \cap \bigcup_{i > n} A_i = \emptyset.$$

Let $A = A_1 \cup \ldots \cup A_n$. Then there is $m \in N$ such that $(A^{(m)})^{(0,1)} = \emptyset$. This means $A = \bigcup_{i=1}^{\infty} B_i$ where for every $i \in N$, B_i is clopen, $B_i^{(m)}$ is compact, and for $i \neq j$ we have $B_i \cap B_j = \emptyset$. Again by Lemma 2.3 there is $k > n$ such that

$$(2) \qquad \operatorname{supp} W_k \cap \bigcup_{i > k} B_i = \emptyset.$$

Let $B = B_1 \cup \ldots \cup B_k$. Then $B^{(m)}$ is compact. Since $\operatorname{supp} W_k \subset U \cup B$, $(\operatorname{supp} U) \cap W_k = \emptyset$ and $W_k^{(\omega)} \neq \emptyset$, we see by Proposition 2.9 that $B^{(\omega)} \neq \emptyset$, so that $B^{(m)} \neq \emptyset$. This implies that $\overline{\operatorname{supp} B^{(m)}}$ is compact. Since for every $n > k$, $y \in (W^{(n)})^{(0,1)}$, we can find a topological copy L_n of $[1, \omega^n]$ in $W_n \backslash \overline{\operatorname{supp} B^{(m)}}$ (cf. Lemma 2.7). Let $M = (\overline{\operatorname{supp} B^{(m)}} \cap W_k) \cup \{y\}$, $L_w = \bigcup_{n > k} L_n \cup \{y\}$ and $L = L_\omega \cup M$. Let $K = (\overline{\operatorname{supp} L} \cap B) \cup B^{(m)}$. Since K and L are compact, there are retractions $r\colon W_k \to L$ and $s\colon B \to K$ (cf. [8]). For every $f \in C_0(W_k)$ let $f^+ \in C_0(Y)$ be the extension of f which is 0 outside W_k. Similarly we define for every $g \in C_0(B)$, $g^* \in C_0(X)$.

Define $\theta\colon C_{0,M}(L) \to C_{0,B^{(m)}}(K)$ by

$$\theta(f) = \varphi^{-1}((f \circ r)^+) \mid K$$

and $\psi\colon C_0(K) \to C_0(L)$ by

$$\psi(g) = \varphi((g \circ s)^*) \mid L$$

(we let $C_{0,M}(L)$ be the linear subspace of $C_0(L)$ consisting of functions vanishing on M). It is easily seen that both θ and ψ are well-defined continuous linear mappings (use Proposition 1.4).

CLAIM 3. *For every $f \in C_{0,M}(L)$, $\psi(\theta(f)) = f$.*

Suppose to the contrary that $\psi(\theta(f)) \neq f$. This means $\varphi((\theta(f) \circ s)^*) \mid L \neq (f \circ r)^+ \mid L$. Then we have $(\theta(f) \circ s)^* \mid \operatorname{supp} L \neq \varphi^{-1}((f \circ r)^+) \mid \operatorname{supp} L$. Now $(\theta(f) \circ s)^* = 0$ outside B, $(f \circ r)^+ = 0$ outside W_k and $\operatorname{supp} U \subset Y \backslash W_k$, so that $\varphi^{-1}((f \circ r)^+) = 0$ on U. Since by Claims 1 and 2, $\operatorname{supp} L \subset U \cup B$, we have

$$(\theta(f) \circ s)^* \mid (\operatorname{supp} L \cap B) \neq \varphi^{-1}((f \circ r)^+) \mid (\operatorname{supp} L \cap B),$$

so that

$$(\theta(f) \circ s)^* \mid K \neq \varphi^{-1}((f \circ r)^+) \mid K = \theta(f).$$

Contradiction.

From the last claim we conclude that θ is a linear embedding. Since $L_\omega \approx [1, \omega^\omega]$, $C_{0,M}(L) \sim C_{0,y}(L_\omega)$ (since $L_\omega \cap M = \{y\}$) and $C_{0,y}(L_\omega) \sim C_0(L_\omega)$, we have $C_{0,M}(L) \sim C_0([1, \omega^\omega])$. However,

$$C_{0,B^{(m)}}(K) \sim C_{0,a}(Z_{K,B^{(m)}}) \sim C_0(Z_{K,B^{(m)}}),$$

where $Z_{K,B^{(m)}}$ is the space obtained from K by identifying $B^{(m)}$ to one point a (here we need that Lemma 3.10 is also valid for the compact-open topology). By Theorem 1.5, we have $Z_{K,B^{(m)}} \approx [1, \omega^m]$, so that we finally have a linear embedding of $C_0([1, \omega^\omega])$ into $C_0([1, \omega^m])$. This is a contradiction with Lemma 1.7. ∎

Let T be the space described in Section 3. Let Z be the space obtained from T by replacing every isolated point by $[1, \omega]$, and let $X \approx [1, \omega^\omega] \oplus Z$. Let Y be the space obtained from T by replacing (n, m) by $[1, \omega^n]$. Then it is easily seen that for every prime component α and ordinal β, $X(\alpha, \beta) = Y(\alpha, \beta)$. However, $X^* = \emptyset$ and $Y^* \neq \emptyset$. So by Lemma 4.1 and Corollary 2.11, X is not l-equivalent with Y. This shows that the above conjecture is false.

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