where $\alpha = \{p, p_2\}$ is a bipath which has p as a selection. We homotope p with end points fixed to a path q so that $q(I) = Q$ is an arc in $\operatorname{Int} M$ with $\operatorname{Fix} \varphi = \{x_1, x_2\}$. Then $\{q, p_2\} \sim \{p, p_2\} \sim \varphi \circ p \sim \varphi \circ q$. For any $\varepsilon > 0$ let

$$p_\varepsilon(s) = q(s - \delta \sin \pi s) \quad \text{for } 0 \leqslant s \leqslant 1,$$

where $0 < \delta = \delta(\varepsilon) < 1$ is selected so that $\bar{d}(p_\varepsilon, q) < \varepsilon$. It follows from an easy general position argument that the path p_2 is homotopic to a path $p_2'\colon M \to \operatorname{Int} M$ with $p_2'(s) \notin Q$ for $0 < s < 1$. Then $\alpha_0 = \varphi \circ q$ and $\alpha_1 = \{p_\varepsilon, p_2'\}$ are two bipaths which are special with respect to q, and as

$$\alpha_0 \sim \{q, p_2\} \sim \{p_\varepsilon, p_2'\} = \alpha_1,$$

they are bihomotopic. According to Lemma 3.5 they are specially bihomotopic, and hence the bimaps $\varphi | Q$ and $\alpha_1 \circ q^{-1}\colon Q \to M$ are specially bihomotopic. Therefore Lemma 3.6 states that we can choose $\varepsilon > 0$ so that φ is bihomotopic to a bimap φ' with the necessary properties.

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Received 8 December 1988;
in revised form 22 August 1989

Making the hugeness of $\varkappa$ resurrectable after $\varkappa$-directed closed forcing

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Abstract. We consider generalizations, to the context of huge cardinals, of Laver's result ([7]) on the indestructibility of supercompactness.

§0. Introduction. Questions of which large cardinals are preserved by which forcing notions often arise. Most large cardinal properties are preserved by small forcing notions. In particular, the inaccessibility, weak compactness, measurability, supercompactness, or hugeness, of a cardinal $\varkappa$ is preserved by any forcing notion of cardinality less than $\varkappa$ (see [8]).

In one sense, large forcing notions trivially preserve large cardinals, where, by large, we do not mean large in cardinality, but large in closure. In particular, the inaccessibility (respectively weak compactness, measurability, λ-supercompactness, hugeness with target λ) of a cardinal $\varkappa$ is preserved by any forcing notion which is $\varkappa$-closed (respectively $\varkappa^+$-closed, $\varkappa^+$-closed, $(\lambda^{\underset{\sim}{\varkappa}})^+$-closed, λ^+-closed). This is so since a forcing notion adds no new subsets of the ground model of cardinality less than its degree of closure.

It is easy to see that all of the large cardinal properties we have mentioned can be destroyed by a forcing notion which is γ-closed, for any $\gamma < \varkappa$ we choose, and has cardinality $\varkappa$. Simply consider the standard forcing notion for adding a function from γ to $\varkappa$. In the extension, $\varkappa$ is not even a cardinal.

More interesting questions arise when we consider preservation of large cardinal properties of $\varkappa$, by forcing notions which are $\varkappa$-closed. These are the types of forcing notions that allow us to manipulate the value of $2^\varkappa$.

Clearly, it is consistent that $\varkappa$-closed forcing can destroy the measurability of $\varkappa$. Consider a model in which $\varkappa$ is measurable, and the GCH holds (the standard model for this is $L[U]$, the collection of all sets constructible from U, where U is any normal ultrafilter on $\varkappa$). The standard forcing notion that makes $2^\varkappa = \varkappa^{++}$ is $\varkappa$-closed. Hence, the GCH below $\varkappa$ is not affected. It follows that, in the

This paper was written while the author was visiting The Hebrew University, Jerusalem, Israel. We thank the Department of Mathematics at The Hebrew University for their wonderful hospitality. We thank the referee for a number of perceptive comments that enabled us to simplify and clarify in several places.

generic extension, $\varkappa$ is the first cardinal to violate the GCH. Hence, $\varkappa$ is not measurable in the generic extension. We may also reach this conclusion, that the standard forcing notion that makes $2^\varkappa = \varkappa^{++}$ can destroy the measurability of $\varkappa$, by noting Kunen's result in [5]: Con(ZFC+$\varkappa$ measurable $+2^\varkappa > \varkappa^+$) is strictly stronger than Con(ZFC+$\varkappa$ measurable).

As Laver points out in [7], it is also easy to see that it is consistent that $\varkappa$-closed forcing can destroy the supercompactness of $\varkappa$. Silver (see [3] or [4]) showed that it is possible to preserve the supercompactness of $\varkappa$ while, for example, making $2^\varkappa = \varkappa^{++}$. His method, now generally known as reverse Easton forcing, consists of a $\varkappa+1$ stage iterated forcing construction. At almost every $\gamma < \varkappa$ (with respect to some normal ultrafilter on $\varkappa$), we force over the model obtained so far, using the standard forcing notion from that model, for making $2^\gamma = \gamma^{++}$. The last stage uses the standard forcing notion, in the model obtained after the $\varkappa$-stage iteration, for making $2^\varkappa = \varkappa^{++}$. Silver's method shows that for any $\lambda \geqslant \varkappa^{++}$, any embedding witnessing that $\varkappa$ is λ-supercompact in the ground model, can be extended to an embedding witnessing that $\varkappa$ is λ-supercompact in the final generic extension. The basic principle here is that a specific $\varkappa$-closed forcing notion can preserve the supercompactness of $\varkappa$, as long as we do some "preparation" below $\varkappa$. This preparation involves doing "the same type" of forcing almost everywhere below $\varkappa$ (with respect to some normal ultrafilter on $\varkappa$) that is done at $\varkappa$.

Laver ([7]) generalized Silver's method to show that this preparation below $\varkappa$ can be made canonical, and thus the supercompactness of $\varkappa$ can be made indestructible under certain types of forcing. In particular, he showed the following: If $\varkappa$ is supercompact, then there is a partial order P such that after forcing with P, $\varkappa$ is supercompact, and remains supercompact upon any further forcing with a $\varkappa$-directed closed partial order.

In this paper, we consider generalizations of Laver's result to the context of huge cardinals. We will show that, although hugeness cannot be made indestructible, it can, under certain assumptions, be made resurrectable.

§ 1. Preliminaries. We work in ZFC throughout. Our set-theoretic notation is quite standard. V denotes the universe of all sets. Greek letters α, β, γ, δ, σ, $\varkappa$ and λ, refer to ordinals. V_α denotes the collection of all sets of rank less than α. For any set x, $|x|$ denotes the cardinality of x and, if x is a set of ordinals, $\bar{x}$ denotes its order type. By the term "inner model", we shall always mean a transitive class which satisfies ZFC. If M is an inner model and λ is an infinite cardinal, we say that M is closed under λ-sequences iff for any $x \subseteq M$, if $|x| \leqslant \lambda$, then $x \in M$. By the term "inaccessible", we shall always mean "strongly inaccessible".

If U is some countably complete ultrafilter, $\prod V/U$ denotes the associated ultrapower of V, M_U denotes the transitive collapse of the ultrapower $\prod V/U$, and i_U: $V \to M_U$ is the canonical embedding.

For $\varkappa \leqslant \lambda$, $P_\varkappa(\lambda) = \{x \subseteq \lambda\colon |x| < \varkappa\}$ and $P_{=\varkappa}(\lambda) = \{x \subseteq \lambda\colon |x| = \varkappa\}$. $\varkappa$ is *λ-supercompact* iff there exists a normal ultrafilter on $P_\varkappa(\lambda)$, and $\varkappa$ is *huge with target* $\lambda > \varkappa$ iff there exists a normal ultrafilter on $P_{=\varkappa}(\lambda)$. We assume familiarity with the basic techniques involving supercompact and huge cardinals, and, in particular, with the equivalent characterizations in terms of elementary embeddings (see, for example [10]). We write $\varkappa \to (\lambda)$ to denote the fact that $\varkappa$ is huge with target λ.

We will need certain generalizations of hugeness. A cardinal $\varkappa$ is *2-huge with targets λ_0 and λ_1* iff there exists an inner model M and an elementary embedding i: $V \to M$ such that M is closed under λ_1 sequences, $\varkappa$ is the critical point of i, $i(\varkappa) = \lambda_0$, and $i(\lambda_0) = \lambda_1$. We write $\varkappa \to (\lambda_0, \lambda_1)$ to denote this property. Equivalently, $\varkappa \to (\lambda_0, \lambda_1)$ if and only if there exists a normal ultrafilter U on $P_{=\lambda_0}(\lambda_1)$ such that

$$\{x \in P_{=\lambda_0}(\lambda_1)\colon |x \cap \lambda_0| = \varkappa\} \in U.$$

In [2], we considered certain notions that are strictly between huge and 2-huge. For $\varkappa \leqslant \varkappa'$, we write $\varkappa \to (\lambda; \varkappa', \lambda')$ to denote the existence of an inner model M and an elementary embedding i: $V \to M$ such that M is closed under λ' sequences, $\varkappa$ is the critical point of i, $i(\varkappa) = \lambda$, and $i(\varkappa') = \lambda'$. Equivalently, $\varkappa \to (\lambda; \varkappa', \lambda')$ if and only if there exists a normal ultrafilter U on $P_{=\varkappa'}(\lambda')$ such that

$$\{x \in P_{=\varkappa'}(\lambda')\colon |x \cap \lambda| = \varkappa\} \in U.$$

In [1], we studied many-times huge cardinals. A cardinal $\varkappa$ is *α-times huge* iff there is a set A with $\bar{A} = \alpha$ and, for every $\beta \in A$, $\varkappa \to (\beta)$.

If $\varkappa \leqslant \alpha < \beta$ and U is a normal ultrafilter witnessing that $\varkappa$ is β-supercompact, we define $U \upharpoonright \alpha$, the restriction of U to α, as follows:

For $A \subseteq P_\varkappa(\alpha)$, $A \in U \upharpoonright \alpha$ iff $\{x \in P_\varkappa(\beta)\colon x \cap \alpha \in A\} \in U$. Then, $U \upharpoonright \alpha$ witnesses that $\varkappa$ is α-supercompact. There exists a canonical elementary embedding k: $M_{U \upharpoonright \alpha} \to M_U$ which fixes all sets of rank less than or equal to α. Similarly, we can restrict a normal ultrafilter witnessing one of the various types of hugeness, and obtain a canonical elementary embedding between the corresponding inner models.

Our forcing notation generally follows that of [3] or [6]. If P is a partial order and $p, q \in P$, $p \leqslant q$ means that p extends q, or p is stronger than q. We assume familiarity with closure and chain conditions. We use the term "$\varkappa$-closed" to mean γ-closed for every $\gamma < \varkappa$. P is *$\varkappa$-directed closed* iff whenever $A \subseteq P$, A is directed and $|A| < \varkappa$, then, for some $p \in P$, $p \leqslant q$ for every $q \in A$. 1_P is weakest element of P.

If G is P-generic over V, $V[G]$ is the corresponding extension. If π is a P-name, π_G is its realization in $V[G]$. For any $x \in V$, $\hat{x}$ is its canonical name. Thus, for $x \in V$, $\hat{x}_G = x$. We shall also use the fact that there is a canonical name $\Gamma \in V$ for a generic subset of P. Hence, $\Gamma_G = G$ for any G which is P-generic over V.

If P is a partial order and π is a P-name for a partial order, $P * \pi = \{\langle p, \tau\rangle\colon p \in P$ and $1_P \Vdash \tau \in \pi\}$. If $\langle p, \tau\rangle, \langle p', \tau'\rangle \in P * \pi$, we define $\langle p, \tau\rangle \leqslant \langle p', \tau'\rangle$ if and only if $p \leqslant p'$ and $p \Vdash \tau \leqslant \tau'$. Then, forcing with $P * \pi$ is equivalent to forcing with P and then forcing with the realization of π. If G is P-generic over V and G' is π_G-generic over $V[G]$, then $G * G' = \{\langle p, \tau\rangle\colon p \in G$ and $\tau_G \in G'\}$ is $P * \pi$-generic over V, and $V[G * G'] = V[G][G']$.

Suppose P is an α stage iterated forcing construction. If $p \in P$, then p is an α-sequence, or, equivalently, a function with domain α. If $\beta < \alpha$, let $P_\beta = \{p \upharpoonright \beta\colon p \in P\}$. Then P_β is the partial order which accomplishes the first β stages of the iteration given by P. If G is P-generic over V, let $G_\beta = \{p \upharpoonright \beta\colon p \in G\}$. Then, G_β is P_β-generic over V.

For $p \in P$, let $p^\beta = p \upharpoonright \{\gamma: \beta \leqslant \gamma < \alpha\}$, and define $P \backslash P_\beta = \{p^\beta: p \in P\}$. In the presence of any P_β-generic set G_β, we can define an ordering on $P \backslash P_\beta$ as follows: For $q, r \in P \backslash P_\beta$, $q \leqslant r$ if and only if for some $s \in G_\beta$, $s \cup q \leqslant s \cup r$. Forcing over $V[G_\beta]$ with $P \backslash P_\beta$ results in a generic extension of V by P.

We assume some familiarity with Silver's method of reverse Easton forcing (see [3] or [4]). Familiarity with Laver's work in [7] would be helpful, but not necessary.

The following two lemmas will be used numerous times in our work.

Chain Condition Lemma. *Suppose α is a cardinal, P_α is the direct limit of $\langle P_\beta: \beta < \alpha\rangle$, $|P_\beta| < \alpha$ for all $\beta < \alpha$, and $\{\beta < \alpha: P_\beta$ is the direct limit of $\langle P_\gamma: \gamma < \beta\rangle\}$ is stationary in α. Then P_α has the α-chain condition.*

Proof. This follows immediately from Theorem 2.2 of [3]. ∎

Closure Lemma. *Suppose M is an inner model of V which is closed under λ-sequences with respect to V, $P \in M$ is a partial order which has the $\varkappa^+$-chain condition for some $\varkappa \leqslant \lambda$, and G is P-generic over V. Then $M[G]$ is closed under λ-sequences with respect to $V[G]$.*

Proof. The proof is as in Lemma 6.4 of [3]. First, suppose $x \subseteq M[G]$, $x \in V[G]$, $V[G] \models |x| \leqslant \lambda$, and x is a set of ordinals. Let π be a P-name for x. Using the fact that P has the $\varkappa^+$-chain condition, we can find a set of ordinals $y \in V$ such that $x \subseteq y$ and $V \models |y| \leqslant \lambda$. For any $\alpha \in y$, let B_α be a maximal incompatible subset of $\{p \in P: p \Vdash \hat{\alpha} \in \pi\}$. Then, for each $\alpha \in y$, $|B_\alpha| \leqslant \varkappa$. Hence, by our closure assumption, for each $\alpha \in y$, $B_\alpha \in M$. Then, since $V \models |y| \leqslant \lambda$, our closure assumption also implies that $\{B_\alpha: \alpha \in y\} \in M$. Then $x \in M[G]$, since $x = \{\alpha \in y: B_\alpha \cap G \neq 0\}$.

We must show that given any set x (not necessarily a set of ordinals), if $x \subseteq M[G]$, $x \in V[G]$, and $V[G] \models |x| \leqslant \lambda$, then $x \in M[G]$. Fix some such x, and let β be such that $x \subseteq [V_\beta]_{M[G]}$. Let $\gamma = |V_\beta|_{M[G]}$, and pick a bijection $f: \gamma \to [V_\beta]_{M[G]}$ with $f \in M[G]$. Then $f \in V[G]$ and, since $x \in V[G]$, $f^{-1}[x] \in V[G]$. In $V[G]$, $|x| \leqslant \lambda$ and so $|f^{-1}[x]| \leqslant \lambda$. Hence, by the previous paragraph, $f^{-1}[x] \in M[G]$. It follows that $x \in M[G]$, since $x = f[f^{-1}[x]]$. ∎

§2. Preservation of hugeness: An example. In this section, we give an example to show how hugeness may be preserved for one specific forcing notion. The techniques used in this example will be central to the remainder of our work.

We consider the problem of getting a model in which $\varkappa$ is huge and $2^\varkappa > \varkappa^+$. Let us assume that $V \models \varkappa \to (\lambda)$ and $2^\varkappa = \varkappa^+$. It seems reasonable to try to generalize Silver's method directly, and do a $\varkappa+1$ stage reverse Easton iteration, making $2^\gamma > \gamma^+$ for, let us say, every inaccessible cardinal $\gamma \leqslant \varkappa$. The problem is that if $V[G]$ is the resulting generic extension and $i: V[G] \to M'$ witnesses, in $V[G]$, that $\varkappa \to (\lambda)$, then, by elementarity, $M' \models \{\gamma < \lambda: 2^\gamma > \gamma^+\}$ is cofinal below λ. By closure considerations, it is true in $V[G]$ that $\{\gamma < \lambda: 2^\gamma > \gamma^+\}$ is cofinal below λ. Hence, if we want to use Silver's method to obtain an extension of V satisfying $\varkappa \to (\lambda)$ and $2^\varkappa > \varkappa^+$, we must force above $\varkappa$.

The next reasonable idea would be to do a $\lambda+1$ stage reverse Easton iteration, making $2^\gamma > \gamma^+$ for every inaccessible cardinal $\gamma \leqslant \lambda$. Those familiar with Silver's method will quickly recognize that, if $i: V \to M$ is a witness to $\varkappa \to (\lambda)$ in V that we wish to extend, M does not possess enough closure to guarantee the existence of the appropriate mastercondition. Since the cardinality of our partial order is λ^{++}, we need that M be closed under λ^{++}-sequences, instead of just λ-sequences. This is precisely what we assume.

Theorem 1. *Assume $V \models \mathrm{ZFC} + \varkappa \to (\lambda;\ \varkappa^{++},\ \lambda^{++})$. Then there exists a generic extension $V[G]$ of V such that $V[G] \models \mathrm{ZFC} + \varkappa \to (\lambda;\ \varkappa^{++},\ \lambda^{++}) + 2^\varkappa > \varkappa^+$.*

Proof. Assume $V \models \mathrm{ZFC} + \varkappa \to (\lambda;\ \varkappa^{++},\ \lambda^{++})$, and also that $2^\varkappa = \varkappa^+$ (or else we are done). We shall also need to assume that $2^{\lambda^{++}} = \lambda^{+++}$. This can be arranged by an easy forcing argument which does not destroy the fact that $\varkappa \to (\lambda;\ \varkappa^{++},\ \lambda^{++})$.

We wish to define a length $\lambda+1$ iterated forcing construction. For each $\gamma \leqslant \lambda$, let P_γ be the partial order corresponding to the first γ stages of the iteration. We note that there will be no conflict between this notation and that given in the preliminaries. As we shall soon see, P will be defined in such a way that the P_γ we define will be the same as the P_γ obtained from P, as given in the preliminaries.

Assume that P_γ has been defined for some $\gamma \leqslant \lambda$. To define $P_{\gamma+1}$ from P_γ, we let $P_{\gamma+1} = P_\gamma * Q_\gamma$ where Q_γ is defined as follows: If γ is not an inaccessible cardinal, then we let Q_γ be a P_γ-name for the trivial partial order $\{0\}$. If γ is an inaccessible cardinal, we let Q_γ be a P_γ-name for the standard partial order (in a model obtained after forcing with P_γ) for making $2^\gamma = \gamma^{++}$.

Assume now that $\langle P_\alpha: \alpha < \gamma\rangle$ has been defined, where $\gamma \leqslant \lambda$ is a limit ordinal. If γ is not an inaccessible cardinal, then let P_γ be the inverse limit of $\langle P_\alpha: \alpha < \gamma\rangle$. If γ is an inaccessible cardinal, let P_γ be the direct limit of $\langle P_\alpha: \alpha < \gamma\rangle$. Finally, let $P = P_{\lambda+1}$.

It is straightforward to show, using standard methods involving closure and chain conditions, that forcing with P preserves all cardinals, preserves inaccessibility, and, if G is P-generic over V, then $V[G] \models 2^\gamma > \gamma^+$ for every inaccessible cardinal $\gamma \leqslant \lambda$. We note that we cannot claim that $2^\gamma = \gamma^{++}$ for every such γ, since there may well have been many $\gamma < \lambda$ with $2^\gamma > \gamma^{++}$ in V. However, it is certainly true that for almost every $\gamma < \lambda$, with respect to some normal ultrafilter on $\varkappa$ that is in V, $2^\gamma = \gamma^{++}$ holds in $V[G]$.

In order to employ Silver's method, we cannot choose an arbitrary G which is P-generic over V, and expect to show that $V[G] \models \varkappa \to (\lambda;\ \varkappa^{++},\ \lambda^{++})$. We must be somewhat more careful in our selection. Let G' be any set which is P_λ-generic over V. We will define $G = G' * G''$, where G'' will be defined shortly.

Let U be a normal ultrafilter witnessing that $\varkappa \to (\lambda;\ \varkappa^{++},\ \lambda^{++})$, and let i: $V \to M$ be the associated elementary embedding and inner model. Then, by elementarity, $M \models i(P)$ is a length $i(\lambda)+1$ iterated forcing construction. By elementarity and closure considerations, for $\gamma \leqslant \lambda+1$, $P_\gamma = [i(P)]_\gamma$.

Clearly, G' is $P_\lambda = [i(P)]_\lambda$-generic over M. We shall consider $[i(P)]_{\lambda+1} \backslash [i(P)]_\lambda$. Let

$$A = \{q(\lambda) \in [i(P)]_{\lambda+1} \backslash [i(P)]_\lambda:\ q = i(p) \text{ for some } p \in G'\}.$$

We note that

$$A \subseteq \{(i(p))(\lambda):\ p \in P_{\varkappa+1}\},$$

and hence $|A| \leqslant |P_{\varkappa+1}| = \varkappa^{++}$.

By the chain condition lemma, P_λ has the λ-chain condition. Clearly $A \in V[G']$ and $A \subseteq M[G']$. It follows by the closure lemma that $A \in M[G']$. Also, $M[G'] \models |A| \leqslant \varkappa^{++}$,

$A \subseteq [i(P)]_{\lambda+1} \setminus [i(P)]_\lambda$, and $[i(P)]_{\lambda+1} \setminus [i(P)]_\lambda$ is a λ-directed closed partial order. Clearly, A is directed. It follows that some $q' \in [i(P)]_{\lambda+1} \setminus [i(P)]_\lambda$ extends every element of A.

Of course, $[i(P)]_{\lambda+1} \setminus [i(P)]_\lambda$ is also a partial order in $V[G']$. Let G'' be any set such that $q' \in G''$ and G'' is $[i(P)]_{\lambda+1} \setminus [i(P)]_\lambda$-generic over $V[G']$. Let $G = G' * G''$.

We will show that $V[G] \models \varkappa \to (\lambda;\ \varkappa^{++},\ \lambda^{++})$. We must construct, within $V[G]$, a set H which is $i(P)$-generic over M, and is such that $p \in G$ if and only if $i(p) \in H$. We will define $H = G * G'''$, where G''' will be defined shortly.

Clearly, G is $P(= [i(P)]_{\lambda+1})$-generic over M. Consider $i(P) \setminus [i(P)]_{\lambda+1}$. We proceed to find an appropriate mastercondition, as before.

Let $B = \{q^{\lambda+1} \in i(P) \setminus [i(P)]_{\lambda+1}\colon q = i(p) \text{ for some } p \in G\}$. It is not hard to see that $|P| = \lambda^{++}$. Hence $|B| \leqslant \lambda^{++}$.

Standard methods establish that P has the λ^+-chain condition. Clearly, $B \in V[G]$ and $B \subseteq M[G]$. It follows by the closure lemma that $B \in M[G]$. Also, $M[G] \models |B| \leqslant \lambda^{++}$, $B \subseteq i(P) \setminus [i(P)]_{\lambda+1}$, and $i(P) \setminus [i(P)]_{\lambda+1}$ is a γ-directed closed partial order, where γ is the first inaccessible cardinal above λ. Clearly, $\lambda^{++} < \gamma$ and hence, since B is directed, it follows that some $q'' \in i(P) \setminus [i(P)]_{\lambda+1}$ extends every element of B.

We cannot proceed now, as we did previously, and simply pick any G''' which is $i(P) \setminus [i(P)]_{\lambda+1}$-generic over $M[G]$ and includes q''. We must construct G''' within $V[G]$.

Let us count the number of dense subsets of $i(P) \setminus [i(P)]_{\lambda+1}$ which are in $M[G]$. The following cardinality computation takes place in $V[G]$:

$$|\{D \in M[G]\colon D \text{ is dense in } i(P) \setminus [i(P)]_{\lambda+1}\}| \leqslant |\boldsymbol{P}(i(P) \setminus [i(P)]_{\lambda+1})_{M[G]}|$$
$$= |(2^{i(\lambda^{++})})_{M[G]}| = |(2^{i(\lambda^{++})})_M| = |i(2^{\lambda^{++}})| = |i(\lambda^{+++})| = \lambda^{+++}.$$

The inequality and the first equality are clear. The second equality follows from standard forcing techniques using the fact that $|P| = \lambda^{++} < i(\lambda^{++})$ (see, e.g. [6]). The third equality follows from elementarity. The fourth equality follows from our assumption that $2^{\lambda^{++}} = \lambda^{+++}$. The last equality follows from a straightforward computation in the ultrapower $\prod V/U$.

Let $\langle D_\alpha\colon \alpha < \lambda^{+++}\rangle$ be an enumeration, in $V[G]$, of all the elements of $M[G]$ which are dense subsets of $i(P) \setminus [i(P)]_{\lambda+1}$. Since any proper initial segment of this enumeration has cardinality at most λ^{++}, any such initial segment is in $M[G]$. Using this, and the fact that $M[G] \models i(P) \setminus [i(P)]_{\lambda+1}$ is a γ-directed closed partial order, where γ is the first inaccessible cardinal above λ (note that $\gamma > \lambda^{++}$), we can inductively define a sequence $\langle q_\alpha\colon \alpha < \lambda^{+++}\rangle$ of elements of $i(P) \setminus [i(P)]_{\lambda+1}$ such that:

a. For each $\alpha < \lambda^{+++}$, $q_\alpha \leqslant q''$.
b. For each $\alpha < \lambda^{+++}$, $q_\alpha \in D_\alpha$.
c. For $\alpha < \beta < \lambda^{+++}$, $q_\beta \leqslant q_\alpha$.

Now, define $G''' = \{r \in i(P) \setminus [i(P)]_{\lambda+1}\colon q_\alpha \leqslant r \text{ for some } \alpha < \lambda^{+++}\}$. It is straightforward to verify that G''' is $i(P) \setminus [i(P)]_{\lambda+1}$-generic over $M[G]$. Let $H = G * G'''$. Then H is $i(P)$-generic over M.

It follows from our construction that for any $p \in P$, $p \in G$ if and only if $i(p) \in H$.

Silver's method now allows us to extend $i\colon V \to M$ to $i^*\colon V[G] \to M[H]$ as follows: For any $x \in V[G]$, let $\pi \in V$ be a P-name for x. Then, in M, $i(\pi)$ is an $i(P)$-name. Define $i^*(x) = [i(\pi)]_H$. As in [3] or [4], the fact that $p \in G$ if and only if $i(p) \in H$ implies that i^* is well defined and elementary.

It is important to note that $M[H]$ and i^* have been defined within $V[G]$. Since, by the closure lemma, $M[G]$ is closed under λ^{++}-sequences with respect to $V[G]$, it follows that $i^*[\lambda^{++}] \in M[G]$, and so $i^*[\lambda^{++}] \in M[H]$. We define $W = \{x \subseteq P_{=\varkappa^{++}}(\lambda^{++})\colon i^*[\lambda^{++}] \in i^*(x)\}$. Then $W \in V[G]$, and it is straightforward to verify that, in $V[G]$, W witnesses $\varkappa \to (\lambda;\ \varkappa^{++},\ \lambda^{++})$. ■

We close this section by noting that our extra closure assumption on M is not necessary. We were informed by Moti Gitik, and by Jean-Pierre Levinski, that it is possible to show, using a technique developed by Hugh Woodin, that if $V \models \mathrm{ZFC} + \varkappa \to (\lambda)$, then there exists a generic extension $V[G]$ of V such that $V[G] \models \mathrm{ZFC} + \varkappa \to (\lambda) + 2^\varkappa > \varkappa^+$. We did not present the proof here, since the technique does not generalize to the settings we consider in the remainder of the paper.

§ 3. Making the hugeness of $\varkappa$ resurrectable. In this section, we consider how Laver's theorem (which we stated in the introduction) on making the supercompactness of $\varkappa$ indestructible under $\varkappa$-directed closed forcing may be adapted to the huge case.

It is not hard to see that Laver's result is not true if "huge with target λ" is substituted for "supercompact". There are two central problems that we run into in trying to obtain something like Laver's theorem for huge cardinals.

Firstly, as we saw in the previous section, some forcing above $\varkappa$ will generally be necessary. Thus, the best we can hope for is to obtain a model such that, following some forcing at $\varkappa$ which, in general, will kill the hugeness of $\varkappa$, we can further force above $\varkappa$ and resurrect the hugeness of $\varkappa$.

Secondly, it is easy to see that $\varkappa$-directed closed forcing can kill the fact that $\varkappa \to (\lambda)$ and make it non-resurrectable. For example, using a $\varkappa$-directed closed partial order, we can collapse λ to $\varkappa$. Clearly, we cannot have $\varkappa \to (\lambda)$, since λ is no longer a cardinal. Since forcing cannot create cardinals, $\varkappa \to (\lambda)$ cannot be true in any further extension. Hence, we will have to consider only a certain restricted family of $\varkappa$-directed closed partial orders.

THEOREM 2. *Suppose* $V \models \mathrm{ZFC} + \varkappa \to (\lambda_0,\ \lambda_1)$. *Then there is a $\varkappa$-chain condition partial order P such that if G is P-generic over V, and $P' \in V[G]$ is such that $V[G] \models P'$ is a $\varkappa$-directed closed partial order and $|P'| < \lambda_0$, then the following holds: For any G' which is P'-generic over $V[G]$, there exists a $P'' \in V[G*G']$ such that $V[G*G'] \models P''$ is a $|P'|^+$-directed closed partial order, and there exists a set G'' which is P''-generic over $V[G*G']$ and is such that $V[G*G'*G''] \models \varkappa \to (\lambda_0)$.*

We first note that the fact that $V[G*G'] \models P''$ is $|P'|^+$-directed closed is important, because it will imply that no new sequences of size less than or equal to $|P'|$ are added by forcing with P'', and hence this forcing will not undo what forcing with P' did.

We do not know whether assuming $\varkappa \to (\lambda_0,\ \lambda_1)$, rather than just $\varkappa \to (\lambda_0)$, is necessary. However, we strongly conjecture that some such stronger assumption is necessary. The reason we make this conjecture is the following: As we will see, the normal ultrafilters in $V[G*G'*G'']$ that we will use to witness $\varkappa \to (\lambda_0)$ are extensions of normal ultrafilters witnessing $\varkappa \to (\lambda_0)$ in V. By using different partial orders P', we can, for example, obtain various different values for $2^\varkappa$. This implies that the corresponding normal ultrafilters that ultimately witness $\varkappa \to (\lambda_0)$ in the extensions by the appropriate P'''s have restrictions to $\varkappa$ which differ. It follows that there must be many normal ultrafilters witnessing $\varkappa \to (\lambda_0)$ in V. Hence, we need some assumption which implies that many such normal ultrafilters exist. $\varkappa \to (\lambda_0, \lambda_1)$ implies (by directly generalizing the proof of the corresponding result for supercompactness in [10]) that there are many normal ultrafilters witnessing that $\varkappa \to (\lambda_0)$. We conjecture that $\varkappa \to (\lambda_0)$ does not imply the existence of many such normal ultrafilters.

In order to prove Theorem 2, we prove a lemma which is an adaptation of the central lemma of [7]. First, we give a definition.

Definition. We write $L(f, \alpha, \beta, \gamma)$ to denote the fact that f is a function, α, β, and γ are ordinals, $f\colon \alpha \to V_\alpha$, and, given any $x \in V_\beta$ with $|x| \geqslant \alpha$, there exists a normal ultrafilter U witnessing $\alpha \to (\beta;\ |x|,\ \beta')$ for some $\beta' < \gamma$ where $[i_U(f)](\alpha) = x$.

Lemma. *Assume $\varkappa \to (\lambda_0, \lambda_1)$. Then, for some $f\colon \varkappa \to V_\varkappa$, $L(f, \varkappa, \lambda_0, \lambda_1)$ holds.*

Proof. Assume that W witnesses $\varkappa \to (\lambda_0, \lambda_1)$, and suppose, by way of contradiction, that for every $f\colon \varkappa \to V_\varkappa$, $\sim L(f, \varkappa, \lambda_0, \lambda_1)$. Let $A = \{\delta < \varkappa\colon$ for every $f\colon \delta \to V_\delta$, $\sim L(f, \delta, \varkappa, \lambda_0)\}$. Then $A \in W \restriction \varkappa$.

We define, by induction, a function $f\colon \varkappa \to V_\varkappa$. Suppose $\delta < \varkappa$ and we have defined $f \restriction \delta$. If $\delta \notin A$ or it is not the case that $f \restriction \delta\colon \delta \to V_\delta$, let $f(\delta) = 0$. If $\delta \in A$ and $f \restriction \delta\colon \delta \to V_\delta$, then, by the definition of A, $\sim L(f \restriction \delta, \delta, \varkappa, \lambda_0)$. Hence, we may define $f(\delta)$ to be some $x \in V_\varkappa$ with $|x| \geqslant \delta$ such that for no normal ultrafilter U witnessing $\delta \to (\varkappa;\ |x|,\ \beta')$, where $\beta' < \lambda_0$, do we have $[i_U(f \restriction \delta)](\delta) = x$. In this manner, we obtain $f\colon \varkappa \to V_\varkappa$.

Clearly, $i_W(f) \restriction \varkappa = f$. By elementarity and the definition of f, $M_W \models [i_W(f)](\varkappa) = x$, where $x \in V_{\lambda_0}$, $|x| \geqslant \varkappa$, and for no normal ultrafilter U witnessing $\varkappa \to (\lambda_0;\ |x|,\ \beta')$ for $\beta' < \lambda_1$ do we have $[i_U(f)](\varkappa) = x$. By closure considerations, this must also be the case in V.

Let $\beta' = i_W|x|$. Note that $\varkappa < \beta' < \lambda_1$. Then, $W \restriction \beta'$ witnesses $\varkappa \to (\lambda_0;\ |x|,\ \beta')$. Recall that there is a canonical elementary embedding $k\colon M_{W\restriction\beta'} \to M_W$ which fixes all sets of rank less than or equal to β', and is such that $k \circ i_{W\restriction\beta'} = i_W$. It follows that $[i_{W\restriction\beta'}(f)](\varkappa) = x$. But it was shown in the previous paragraph that there is no such $W \restriction \beta'$. This is a contradiction. ∎

Proof of Theorem 2. Assume that $V \models \varkappa \to (\lambda_0, \lambda_1)$ and f is as in the lemma. We will define P as a length $\varkappa$ iterated forcing construction. For each $\gamma \leqslant \varkappa$, let P_γ be the partial order corresponding to the first γ stages of the iteration.

As we define each P_γ for $\gamma < \varkappa$, we shall also choose an ordinal σ_γ. For each such γ, we shall have $|P_\gamma| < \varkappa$, and $\sigma_\gamma < \varkappa$.

We define the P_γ's by induction. To define $P_{\gamma+1}$ from P_γ, we let $P_{\gamma+1} = P_\gamma * Q_\gamma$, where Q_γ is defined as follows:

1. If $f(\gamma)$ is not a P_γ-name for a γ-directed closed partial order, or if $\gamma < \sigma_\gamma$, then let Q_γ be a P_γ-name for the trivial partial order $\{0\}$, and let $\sigma_{\gamma+1} = \sigma_\gamma$.

2. If $f(\gamma)$ is a P_γ-name for a γ-directed closed partial order, and $\gamma \geqslant \sigma_\gamma$, let $Q_\gamma = f(\gamma)$ and let $\sigma_{\gamma+1}$, be the least inaccessible cardinal $\sigma < \varkappa$ such that $\sigma > \sigma_\gamma$ and $\sigma > |f(\gamma)|$. We note that since $\varkappa$ is inaccessible and $f(\gamma) \in V_\varkappa$, we have $|f(\gamma)| < \varkappa$. Hence our definition of $\sigma_{\gamma+1}$ makes sense. We also note that $\sigma_{\gamma+1} > |f(\gamma)|$ implies that $1_{P_\gamma} \Vdash \check{\sigma}_{\gamma+1} > |f(\gamma)|$.

Clearly we have, in either case, $|P_{\gamma+1}| < \varkappa$.

Assume now that $\langle P_\alpha\colon \alpha < \gamma\rangle$ has been defined, where γ is a limit ordinal. If γ is not an inaccessible cardinal, then let P_γ be the inverse limit of $\langle P_\alpha\colon \alpha < \gamma\rangle$. If γ is an inaccessible cardinal then let P_γ be the direct limit of $\langle P_\alpha\colon \alpha < \gamma\rangle$. In either case, let $\sigma_\gamma = \sup_{\alpha<\gamma}\sigma_\alpha$. It is clear that if, for $\alpha < \gamma$, $|P_\alpha| < \varkappa$ and $\sigma_\alpha < \varkappa$, then, if $\gamma < \varkappa$, the inaccessibility of $\varkappa$ implies that $|P_\gamma| < \varkappa$ and $\sigma_\gamma < \varkappa$. Finally, let $P = P_\varkappa$. Then $|P| = \varkappa$.

The idea behind our use of the σ_γ's in the definition is that if we do some non-trivial forcing in forming $P_{\gamma+1}$ at stage γ, then, if γ' is the next stage at which we do some non-trivial forcing, γ' is at least as big as the first inaccessible cardinal above $|P_{\gamma+1}|$. Hence, any generic extension obtained by forcing with $P_{\gamma+1}$ satisfies that $P \backslash P_{\gamma+1}$ is a δ-directed closed partial order, where δ is the first inaccessible cardinal above $|P_{\gamma+1}|$. We will use this fact frequently.

Clearly, we have taken direct limits on a measure one set (with respect to any normal ultrafilter on $\varkappa$), and hence, on a stationary set. If follows from the chain condition lemma that P has the $\varkappa$-chain condition.

Let G be P-generic over V, and suppose $P' \in V[G]$ and $V[G] \models P'$ is a $\varkappa$-directed closed partial order, and $|P'| < \lambda_0$. Clearly, we may assume, without loss of generality, that, in $V[G]$, $P' \in V_{\lambda_0}$, since all that matters about P' is its ordering, and not its actual elements.

Let π be a P-name for P'. Clearly we can choose $\pi \in V_{\lambda_0}$. Also, let us pick π such that $|\pi| > \varkappa$ and $|\pi|$ is a singular strong limit cardinal. This presents no difficulty, since a P-name can always be made bigger and, since λ_0 is inaccessible, there are unboundedly many singular strong limit cardinals below λ_0. It follows from work of Solovay (see [9]) that $2^{|\pi|} = |\pi|^+$.

By the lemma, let U be a normal ultrafilter witnessing $\varkappa \to (\lambda_0;\ |\pi|,\ \beta')$ for some $\beta' < \lambda_1$, with $[i_U(f)](\varkappa) = \pi$. Then, by elementarity, $i_U(P)$ is a length λ_0 iterated forcing construction, which is defined in M_U from $i_U(f)$ in precisely the same manner that P was defined in V from f. Since i_U fixes all sets of rank less than $\varkappa$, $P = [i_U(P)]_\varkappa$. It follows from the way P was defined, and from our choice of U, that $[i_U(P*\pi)]_{\varkappa+1} = [i_U(P)]_{\varkappa+1} = P*\pi$.

Let G' be P'-generic over $V[G]$. Then clearly $G*G'$ is $P*\pi = [i_U(P*\pi)]_{\varkappa+1}$-generic over M_U. We shall consider the partial order $i_U(P*\pi) \backslash [i_U(P*\pi)]_{\varkappa+1} = i_U(P*\pi) \backslash (P*\pi)$.

Note that $|P*\pi| = \sup\{|P|, |\pi|\} < \lambda_0 < \beta'$. Hence $P*\pi$ trivially has the $(\beta')^+$ chain condition. Then, the closure lemma implies that $M_U[G*G']$ is closed under β'-sequences with respect to $V[G*G']$. In $V[G*G']$, $|G*G'| \leqslant |P*\pi| < \beta'$. It follows that if we define

$$A = \{q^{\varkappa+1} \in i_U(P*\pi) \backslash (P*\pi)\colon\ q = i_U(p) \text{ for some } p \in G*G'\},$$

then $A \in M_U[G*G']$.

Clearly, A is directed. Hence, since $|A| \leqslant |P*\pi|$ and $M_U[G*G'] \models i_U(P*\pi)\backslash(P*\pi)$ is $|P*\pi|^+$-directed closed, we can find a $q' \in i_U(P*\pi)\backslash(P*\pi)$ which extends every element of A. Let G'' be any set which is $i_U(P*\pi)\backslash(P*\pi)$-generic over $V[G*G']$ with $q' \in G''$.

We must show that $V[G*G'*G''] \models \varkappa \to (\lambda_0)$. We first note that $i_U^2(P*\pi) = i_U(i_U(P*\pi))$ is, in M_U, a length $i_U(\lambda)+1$ iterated forcing construction, and $[i_U^2(P*\pi)]_{\lambda+1} = i_U(P*\pi)$. Clearly, $G*G'*G''$ is $i_U(P*\pi)$-generic over M_U. We must consider $i_U^2(P*\pi)\backslash[i_U^2(P*\pi)]_{\lambda+1} = i_U^2(P*\pi)\backslash i_U(P*\pi)$ in $M[G*G'*G'']$.

We proceed to find an appropriate mastercondition, as before. Let

$$B = \{q^{\lambda+1} \in i_U^2(P*\pi)\backslash i_U(P*\pi):\ q = i_U(p) \text{ for some } p \in G*G'*G''\}.$$

The following cardinality argument takes place in $V[G*G'*G'']$:

$$|B| \leqslant |G*G'*G''| \leqslant |i_U(P*\pi)| = |i_U(\pi)| = i_U(|\pi|) = \beta'.$$

Next we wish to apply the closure lemma. In order to do so, we first note that $i_U(P*\pi)$ has the $(\beta')^+$-chain condition. This is trivial, since the above computation shows that $|i_U(P*\pi)| = \beta'$. Then, since M_U is closed under β'-sequences with respect to V, and $G*G'*G''$ is $i_U(P*\pi)$-generic over V, the closure lemma implies that $M_U[G*G'*G'']$ is closed under β'-sequences with respect to $V[G*G'*G'']$. Hence $B \in M_U[G*G'*G'']$.

Clearly, B is directed, and we have shown that $|B| \leqslant |i_U(P*\pi)|$. Hence, since $M_U[G*G'*G''] \models i_U^2(P*\pi)\backslash i_U(P*\pi)$ is $|i_U(P*\pi)|^+$-directed closed, we can find a $q'' \in i_U^2(P*\pi)\backslash i_U(P*\pi)$ which extends every element of B.

The rest of the proof is very similar to the end of the proof in the last section. We wish to construct, within $V[G*G'*G'']$, a set G''' which is $i_U^2(P*\pi)\backslash i_U(P*\pi)$-generic over $M_U[G*G'*G'']$, with $q'' \in G'''$.

We first claim that we can construct, within $V[G*G'*G'']$, an enumeration, of length at most $(\beta')^+$, of all the dense subsets of $i_U^2(P*\pi)\backslash i_U(P*\pi)$ which are in $M_U[G*G'*G'']$.

To show this, we first consider the following cardinality computation in $M_U[G*G'*G'']$:

$$\left|\boldsymbol{P}\big(i_U^2(P*\pi)\backslash i_U(P*\pi)\big)\right| \leqslant \left|\boldsymbol{P}\big(i_U^2(P*\pi)\big)\right| = 2^{|i_U^2(P*\pi)|} = 2^{i_U(\beta')}.$$

The inequality and the first equality are clear and the second equality follows from the fact that $|i_U(P*\pi)| = \beta'$. We have also used the fact that, by standard chain condition arguments, β' and $i_U(\beta')$ are cardinals in $M_U[G*G'*G'']$.

As in the previous section, standard forcing techniques, using the fact that $|i_U(P*\pi)| = \beta' < i_U(\beta')$, tell us that $(2^{i_U(\beta)})_{M_U[G*G'*G'']} = (2^{i_U(\beta')})_M$. By elementarity $(2^{i_U(\beta')})_M = i_U(2^{\beta'})$. Putting this all together, we have shown that, in $V[G*G'*G'']$,

$$\left|\boldsymbol{P}\big(i_U^2(P*\pi)\backslash i_U(P*\pi)\big)_{M_U[G*G'*G'']}\right| \leqslant |i_U(2^{\beta'})|.$$

We note that by standard chain condition arguments $2^{\beta'} = (2^{\beta'})_{V[G*G'*G'']}$.

Next, we wish to show that $2^{(\beta')} = (\beta')^+$. We have

$$2^{(\beta')} = |2^{(\beta')}_M| = |(2^{i_U(|\pi|)})_M| = \left|\big(i_U(2^{(|\pi|)})\big)\right| = \left|i_U(|\pi|^+)\right| = \left|\big(i_U(|\pi|)\big)^+_M\right| = |(\beta')^+_M| = (\beta')^+.$$

The first equality follows from the fact that M is closed under β'-sequences, and hence $\boldsymbol{P}(\beta) = (\boldsymbol{P}(\beta'))_M$. The second and sixth equalities are immediate, since $i_U(|\pi|) = \beta'$. The third and fifth equalities follow by elementarity. The fourth equality follows from the fact that, by our choice of π (as discussed previously), $2^{|\pi|} = |\pi|^+$. The last equality follows from the fact that M is closed under β'-sequences, and hence $(\beta')^+_M = (\beta')^+$.

Putting together the previous two paragraphs, we have that in $V[G*G'*G'']$,

$$\left|\boldsymbol{P}\big(i_U^2(P*\pi)\backslash i_U(P*\pi)\big)_{M[G*G'*G'']}\right| \leqslant \left|i_U((\beta')^+)\right|.$$

We have used the fact that, by standard chain condition arguments, $(\beta')^+ = (\beta')^+_{V[G*G'*G'']}$. A straightforward calculation in the ultrapower $\prod V/U$ tells us that $\left|i_U((\beta')^+)\right| = (\beta')^+$. Hence, in $V[G*G'*G'']$,

$$\left|\boldsymbol{P}\big(i_U^2(P*\pi)\backslash i_U(P*\pi)\big)_{M[G*G'*G'']}\right| \leqslant (\beta')^+.$$

Next, consider the following cardinality computation in $V[G*G'*G'']$:

$$|\{D \in M_U[G*G'*G'']:\ D \text{ is dense in } i_U^2(P*\pi)\backslash i_U(P*\pi)\}|$$
$$\leqslant \left|\boldsymbol{P}\big(i_U^2(P*\pi)\backslash i_U(P*\pi)\big)_{M_U[G*G'*G'']}\right| \leqslant (\beta')^+.$$

The first inequality is obvious, and the second equality follows from the previous paragraph.

Hence, an enumeration within $V[G*G'*G'']$, of length at most $(\beta')^+$, of all the dense subsets of $i_U^2(P*\pi)\backslash i_U(P*\pi)$ which are in $M_U[G*G'*G'']$, is possible.

Any proper initial segment of this enumeration of dense sets must be in $M_U[G*G'*G'']$, by the closure lemma. Using this, and the fact that $M_U[G*G'*G''] \models i_U^2(P*\pi)\backslash i_U(P*\pi)$ is an $|i_U(P*\pi)|^{++} = (\beta')^+$-directed closed partial order, we can obtain the desired G'''.

We have been careful to choose our generic sets so that, for any p, $p \in G*G'*G''$ if and only if $i_U(p) \in G*G'*G''*G'''$. As in the previous section, Silver's method allows us to extend $i_U\colon V \to M$ to $i_U^*\colon V[G*G'*G''] \to M_U[G*G'*G''*G''']$.

Since, by the closure lemma, $M_U[G*G'*G'']$ is closed under β'-sequences with respect to $V[G*G'*G'']$, we have $i_U^*[\beta'] \in M_U[G*G'*G'']$. Thus, $i_U^*[\beta'] \in M_U[G*G'*G''*G''']$. We define $W = \{x \subseteq P_{=|\pi|}(\beta'):\ i_U^*[\beta'] \in i_U^*(x)\}$. Then, since $M_U[G*G'*G''*G''']$ and i_U^* have been defined within $V[G*G'*G'']$, it follows that $W \in V[G*G'*G'']$. It is straightforward to verify that, in $V[G*G'*G'']$, W witnesses $\varkappa \to (\lambda_0;\ |\pi|,\ \beta')$. This is strictly stronger than $\varkappa \to (\lambda_0)$, which we needed to show. ∎

§ 4. Making the many-times hugeness of $\varkappa$ resurrectable. Suppose that A is a set of targets for the huge cardinal $\varkappa$ with $|A| > 1$. We consider the problem of making $\varkappa \to (\alpha)$ simultaneously resurrectable for all $\alpha \in A$. The central problem in generalizing the method of the previous section is that the associated generic sets that do the resurrecting must be made to cohere.

We first present a theorem which gives sufficient criteria to allow us to make many targets simultaneously resurrectable. Then, we give as corollaries two examples illustrating how these criteria may be satisfied.

THEOREM 3. *Suppose that A is a set of targets for the huge cardinal $\varkappa$. Assume the following hold:*

1. $\bar{A}$ *is a limit ordinal.*

2. *A contains none of its limit points.*

3. *If* $\{\alpha_\delta\colon \delta < \bar{A}\}$ *is an increasing enumeration of the elements of A, then, for some* $f\colon \varkappa \to V_\varkappa$, $L(f, \varkappa, \alpha_\delta, \alpha_{\delta+1})$ *holds for every* $\delta < \bar{A}$.

Then $\varkappa \to (\alpha)$ *for every* $\alpha \in A$ *can be made simultaneously resurrectable after* $\varkappa$*-directed closed forcing of smaller cardinality. That is, there exists a* $\varkappa$*-chain condition partial order P such that if G is P-generic over V, and* $P' \in V[G]$ *is such that* $V[G] \models P'$ *is a* $\varkappa$*-directed closed partial order, then the following holds:*

For any G' *which is* P'*-generic over* $V[G]$, *there exists a* $P'' \in V[G*G']$ *such that* $V[G*G'] \models P''$ *is a* $|P'|^+$*-directed closed partial order, and there exists a set* G'' *which is* P''*-generic over* $V[G*G']$ *and is such that for any* $\delta < \bar{A}$ *with* $V[G] \models |P'| < \alpha_\delta$, *we have* $V[G*G'*G''] \models \varkappa \to (\alpha_\delta)$.

Proof. Let f be as in the statement of the theorem. Define P from f precisely as in the previous section, and let G be P-generic over V. Assume $P' \in V[G]$ is such that $V[G] \models P'$ is a $\varkappa$-directed closed partial order, and suppose G' is P'-generic over $V[G]$.

Let $B = \{\beta \in A\colon |P'| < \beta\}$. We will construct a $P'' \in V[G*G']$ such that $V[G*G'] \models P''$ is $|P'|^+$-directed closed, and a G'' which is P''-generic over $V[G*G']$ and is such that $V[G*G'*G''] \models \varkappa \to (\beta)$ for every $\beta \in B$.

Let $\{\beta_\delta\colon \delta < \bar{B}\}$ be an increasing enumeration of the elements of B. Then, for each $\delta < \bar{B}$, $L(f, \varkappa, \beta_\delta, \beta_{\delta+1})$ holds. We will construct a sequence of partial orders $\{R(\delta)\colon \delta \leqslant \bar{B}\}$ in V from which we will eventually define P''. Each $R(\delta)$ will be an iterated forcing construction. We will not directly define each $R(\delta)$ as an iteration, but shall define it as the image, under an elementary embedding, of a previously defined iterated forcing construction. At each stage of the construction, we shall have $R(\delta) \in V_{\beta_\delta}$, and $[R(\delta)]_\varkappa = P$.

Since in $V[G]$, $|P'| < \beta_0$, we may assume that $P' \in (V_{\beta_0})_{V[G]}$. Let π be a P-name for P'. As in the previous section, we can choose π with $\pi \in V_{\beta_0}$, $|\pi| > \varkappa$, and $|\pi|$ a singular strong limit cardinal. Then $2^{|\pi|} = |\pi|^+$. Let $R(0) = P*\pi$. Then $R(0) \in V_{\beta_0}$ and $[R(0)]_\varkappa = P$.

Assume now that $R(\delta) \in V_{\beta_\delta}$ has been defined, where $[R(\delta)]_\varkappa = P$. Then $R(\delta)\backslash[R(\delta)]_\varkappa$ is a partial order in $V[G]$. Let π_δ be a P-name for this partial order, where, as above, we can choose π_δ with $\pi_\delta \in V_{\beta_\delta}$, $|\pi_\delta| > \varkappa$, and $|\pi_\delta|$ a singular strong limit cardinal. Then $2^{|\pi_\delta|} = |\pi_\delta|^+$. We note that the π of the previous paragraph is a legitimate choice for π_0. Let us make this choice, and set $\pi_0 = \pi$.

By our assumption that $L(f, \varkappa, \beta_\delta, \beta_{\delta+1})$ holds, let U_δ be a normal ultrafilter witnessing $\varkappa \to (\beta_\delta; |\pi_\delta|, \beta'_\delta)$ for some $\beta'_\delta < \beta_{\delta+1}$, and satisfying $[i_{U_\delta}(f)](\varkappa) = \pi_\delta$. For notational simplicity, we shall write i_δ and M_δ instead of i_{U_δ} and M_{U_δ} respectively.

Define $R(\delta+1) = i_\delta(P*\pi_\delta)$. Then, since i_δ fixes all sets of rank less than $\varkappa$, $[R(\delta+1)]_\varkappa = [i_\delta(P*\pi_\delta)]_\varkappa = P$. Also, $R(\delta+1) = i_\delta(P*\pi_\delta) \in i_\delta(V_{\beta_\delta}) = V_{i_\delta(\beta_\delta)} \subseteq V_{\beta_{\delta+1}}$, where the last relationship follows by using straightforward cardinality arguments in the ultrapower $\prod V/U_\delta$ to show that $i_\delta(\beta_\delta) < \beta_{\delta+1}$. Hence, $R(\delta+1) \in V_{\beta_{\delta+1}}$. Also, by our choice of U_δ, by the elementarity of i_δ, and by the way P was defined from f in V, it follows that the iteration $R(\delta+1)$ properly extends the iteration $R(\delta)$.

Next, suppose $\langle R(\gamma)\colon \gamma < \delta\rangle$ has been defined, where $\delta \leqslant \bar{B}$ is a limit ordinal. Let $R(\delta)$ be the inverse limit of this sequence. By our assumption that A (and hence B) contains none of its limit points, it follows that $R(\delta) \in V_{\beta_\delta}$. Then $R(\delta)$ is an iterated forcing construction that properly extends each $R(\gamma)$, for $\gamma < \delta$. If $\delta = \bar{B}$, set $R = R(\delta)$.

It is clear from our construction that each $R(\delta)$ begins with $P*\pi_0$. Define $P(\delta)$, for every $\delta < \bar{B}$, by $P(\delta) = R(\delta)\backslash(P*\pi_0)$. Also, define $P'' = R\backslash(P*\pi_0)$. Then P'' is an iterated forcing construction in $V[G*G']$. Also, P'' is $|P'|^+$-directed closed. This follows from the way the σ_γ's were used in the definition of P, and the way the $R(\delta)$'s were defined from P using elementary embeddings.

Next, we wish to define a set G'' which is P''-generic over $V[G*G']$ and is such that for any $\delta < \bar{B}$, $V[G*G'*G''] \models \varkappa \to (\beta_\delta)$. We must define G'' so as to respect the masterconditions associated with each embedding i_δ. In other words, it must be the case that, for any $\delta < \bar{B}$, and any p, $p \in G*G'*G''_\delta$ if and only if $i_\delta(p) \in G*G'*G''_{\delta+1}$.

Intuitively, what we wish to do is to define inductively a sequence $\{G''_\delta\colon \delta < \bar{B}\}$ such that each such G''_δ is $P(\delta)$-generic over $V[G*G']$ and is such that the appropriate mastercondition is respected at each stage. The problem with this direct approach is that it is not true, in general, that taking limits of generic sets results in a generic set. Hence, this method will work only if B is finite. If B is infinite, we must be somewhat more subtle, and iteratively obtain names for these masterconditions. Ultimately, we shall obtain one mastercondition for P''.

Before beginning the construction, let us try to clear up a possible source of confusion ahead. If $\gamma < \delta < \bar{B}$ and $p \in P(\delta)$, then we have $p^\gamma \in P(\delta)\backslash P(\gamma)$. Hence p^γ is a forcing condition associated with the iteration from stage γ up to, but not including, stage δ. We emphasize that this does not, in general, correspond with any forcing occurring between the actual ordinals γ and δ, but relates, instead, to β_γ and β_δ.

We will inductively define an $r \in P''$. Hence, r will be a function with domain $\bar{B}$. For each δ with $0 < \delta < \bar{B}$ we shall have that $r{\restriction}\delta \in P(\delta)$, and $r(\delta)$ is a $P(\delta)$-name for an element of $P(\delta+1)\backslash P(\delta)$. In particular, we shall define r such that for any such δ, $r{\restriction}\delta \Vdash r(\delta)$ has the required mastercondition property. More precisely (but perhaps less clearly), we shall have that $r{\restriction}\delta \Vdash r(\delta)$ extends every element of $\{q^\delta \in (P(\delta+1)\backslash P(\delta))^\wedge\colon q = \hat{i}_\delta(p)$ for some $p \in \hat{G}*\hat{G}'*\Gamma_\delta\}$, where Γ_δ is the canonical name for a set which is $P(\delta)$-generic over $V[G*G']$.

We begin by letting $r(0)$ be some element of $P(1)$ which extends every element of $\{q \in P(1)\colon q = i_0(p)$ for some $p \in G*G'\}$. That such an $r(0)$ exists follows by precisely the same method which we used in the last section. Note that we have $r{\restriction}1 \in P(1)$.

Now, suppose that we have defined $r{\restriction}\delta \in P(\delta)$ for some $\delta < \bar{B}$. We wish to define $r(\delta)$. Let H be any set which is $P(\delta)$-generic over $V[G*G']$ with $r{\restriction}\delta \in H$. Our methods of the last section enable us to obtain an $s \in P(\delta+1)\backslash P(\delta)$ such that $V[G*G'*H] \models s$ extends every element of $\{q^\delta \in P(\delta+1)\backslash P(\delta)\colon q = i_\delta(p)$ for some $p \in G*G'*H\}$. Then, since such an s exists for any such H, we can define $r(\delta)$ to be a canonical name for such an s. Then $r{\restriction}\delta \Vdash r(\delta)$ extends every element of $\{q^\delta \in (P(\delta+1)\backslash P(\delta))^\wedge\colon q = \hat{i}_\delta(p)$ for some $p \in \hat{G}*\hat{G}'*\Gamma_\delta\}$.

In this manner, we obtain $r \in P''$. We note that it was not necessary to distinguish between the successor and limit cases in our construction. The fact that we always take inverse limits in defining P'' guarantees that if, for a limit ordinal $\delta \in \bar{B}$, $r \upharpoonright \gamma \in P(\gamma)$ for every $\gamma < \delta$, then $r \upharpoonright \delta \in P(\delta)$.

Let G'' be any set which is P''-generic over $V[G*G']$ with $r \in G''$. Pick any $\delta < \bar{B}$. We must show that $V[G*G'*G''] \models \kappa \to (\beta_\delta)$.

We first claim that $V[G*G'*G''_{\delta+1}] \models \kappa \to (\beta_\delta)$. Since $r \upharpoonright \delta \in G''_\delta$, $V[G*G'*G''_\delta] \models [r(\delta)]_{G''_\delta}$ extends every element of $\{q^\delta \in P(\delta+1) \backslash P(\delta)\colon q = i_\delta(p)$ for some $p \in G*G'*G''_\delta\}$. Then, since i_δ fixes all sets of rank less than or equal to κ, and since $r \upharpoonright \delta+1 \in G''_{\delta+1}$, it follows that, given any p, $p \in G*G'*G''_\delta$ if and only if $i_\delta(p) \in G*G'*G''_{\delta+1}$.

Recall that, by our construction, the partial order in V over which $G*G'*G''_{\delta+1}$ is generic, is $P*\pi_{\delta+1}$. Also, $P*\pi_{\delta+1}$ is an initial part of $i_\delta(P*\pi_{\delta+1})$. Hence, it makes sense to consider the partial order $i_\delta(P*\pi_{\delta+1}) \backslash (P*\pi_{\delta+1})$ in $M_\delta[G*G'*G''_{\delta+1}]$.

We need to obtain an appropriate mastercondition. That is, we need to find an element of this partial order which extends every element of $\{q^{\delta+1} \in i_\delta(P*\pi_{\delta+1}) \backslash (P*\pi_{\delta+1})\colon q = i_\delta(p)$ for some $p \in G*G'*G''_{\delta+1}\}$. That such a mastercondition exists follows by precisely the same methods as we used in the last section.

Next, we note that, again precisely as in the last section, $M_\delta[G*G'*G''_{\delta+1}]$ is closed enough with respect to $V[G*G'*G''_{\delta+1}]$ and, in $M_\delta[G*G'*G''_{\delta+1}]$, the partial order $i_\delta(P*\pi_{\delta+1}) \backslash (P*\pi_{\delta+1})$ is closed enough, so that we can, within $V[G*G'*G''_{\delta+1}]$, construct a set $H(\delta+1)$ which is $i_\delta(P*\pi_{\delta+1}) \backslash (P*\pi_{\delta+1})$-generic over $M_\delta[G*G'*G''_{\delta+1}]$, and contains our mastercondition.

It is then the case that, for any p, $p \in G*G'*G''_{\delta+1}$ if and only if

$$i_\delta(p) \in G*G'*G''_{\delta+1}*H(\delta+1).$$

Silver's method now allows us to extend $i_\delta\colon V \to M_\delta$ to

$$i^*_\delta\colon V[G*G'*G''_{\delta+1}] \to M_\delta[G*G'*G''_{\delta+1}*H(\delta+1)].$$

Next, we claim $i^*_\delta[\beta_\delta] \in M_\delta[G*G'*G''_{\delta+1}]$. Since P has the κ-chain condition, and M_δ is closed under β'_δ-sequences with respect to V, the closure lemma implies that $M_\delta[G]$ is closed under β'_δ-sequences with respect to $V[G]$. Note that $G'*G''_{\delta+1}$ is $R(\delta+1)$-generic over $V[G]$ and also over $M_\delta[G]$. The following calculation shows that $R(\delta+1)$ has the $(\beta'_\delta)^+$-chain condition: $|R(\delta+1)| = |i_\delta(P*\pi_\delta) \backslash P| = |i_\delta(P*\pi_\delta)| = |i_\delta(\pi_\delta)| = i_\delta|\pi_\delta| = \beta'$. Hence, applying the closure lemma again, we conclude that $M_\delta[G*G'*G''_{\delta+1}]$ is closed under β'_δ-sequences with respect to $V[G*G'*G''_{\delta+1}]$. Then certainly, since $\beta_\delta < \beta'_\delta$, $i^*_\delta[\beta_\delta] \in M_\delta[G*G'*G''_{\delta+1}]$.

It follows that $i^*_\delta[\beta_\delta] \in M_\delta[G*G'*G''_{\delta+1}*H(\delta+1)]$. We define $W_\delta = \{x \subseteq P_{=\kappa}(\beta_\delta)\colon i^*_\delta[\beta_\delta] \in i^*_\delta(x)\}$. Then, since $M_\delta[G*G'*G''_{\delta+1}*H(\delta+1)]$ and i^*_δ have been defined in $V[G*G'*G''_{\delta+1}]$, it follows that $W_\delta \in V[G*G'*G''_{\delta+1}]$. It is straightforward to verify that, in $V[G*G'*G''_{\delta+1}]$, W witnesses that $\kappa \to (\beta_\delta)$.

We must show that $V[G*G'*G''] \models \kappa \to (\beta_\delta)$. We can view $V[G*G'*G'']$ as having been obtained from $V[G*G'*G''_{\delta+1}]$ by forcing over $V[G*G'*G''_{\delta+1}]$ with $P'' \backslash P(\delta+1)$. In $V[G*G'*G''_{\delta+1}]$, this partial order is β^+_δ-directed closed. Hence, no new subsets of $P_{=\kappa}(\beta_\delta)$ are added by this forcing. Consequently, W_δ witnesses that $\kappa \to (\beta_\delta)$ in $V[G*G'*G'']$. ■

We note that for each $\delta < \bar{B}$, each $M_\delta[G*G'*G''_{\delta+1}]$ is closed under β'_δ-sequences with respect to $V[G*G'*G''_{\delta+1}]$. Hence, we could have obtained a normal ultrafilter W_δ witnessing $\kappa \to (\beta_\delta;\ |\pi_\delta|,\ \beta'_\delta)$ instead of just $\kappa \to (\beta_\delta)$, in $V[G*G'*G''_{\delta+1}]$. Also, in $V[G*G'*G''_{\delta+1}]$, $P'' \backslash P(\delta+1)$ is a $(\beta'_\delta)^+$-directed closed partial order, and so W_δ witnesses $\kappa \to (\beta_\delta;\ \kappa'_\delta,\ \beta'_\delta)$ in $V[G*G'*G'']$. This is strictly stronger than what we needed to show.

Before giving corollaries to the theorem, we comment briefly on the different types of limits that we used in our forcing constructions. In defining P, we took direct limits at inaccessible cardinals, and inverse limits at limit ordinals which are not inaccessible cardinals. This is exactly the kind of limits that were needed to preserve the necessary closure and chain conditions. In taking limits of the $P(\delta)$'s, we took inverse limits everywhere. This was necessary, as we have already pointed out, in order to obtain our mastercondition $r \in P''$. The fact that this means that we cannot use the usual techniques to conclude that $P(\delta)$ for limit $\delta < \bar{B}$ has the $(\sup_{\gamma<\delta} \beta_\gamma)$-chain condition is not a problem. The central fact here is that, by our assumption that A, and hence B, contains none of its limit points, $\sup_{\gamma<\delta} \beta_\gamma < \beta_\delta$. Thus P_δ, for limit $\delta < \bar{B}$, does not involve forcing up to some element of B. Hence, chain conditions for $P(\delta)$ are not needed.

We next give, as corollaries to the theorem, two examples to show how the premises of the theorem might be satisfied.

COROLLARY 1. *Suppose C is some collection of targets for the huge cardinal κ such that*:

1. *$\bar{C}$ is a regular cardinal bigger than 2^κ.*
2. *If $\{\gamma_\delta\colon \delta < \bar{C}\}$ is an enumeration of the elements of C in increasing order, then, for each $\delta < \bar{C}$, there exists a λ_δ with $\gamma_\delta < \lambda_\delta \leqslant \gamma_{\delta+1}$ and $\kappa \to (\gamma_\delta, \lambda_\delta)$.*

Then there is an $A \subseteq C$ which is cofinal in C, and is such that $\kappa \to (\gamma)$, for all $\gamma \in A$, can be made simultaneously resurrectable after κ-directed closed forcing of smaller cardinality. In other words, the conclusion of the theorem holds for this A.

Proof. By our second assumption, the lemma of the previous section implies that for each $\delta < \bar{C}$, there exists a function $f_\delta\colon \kappa \to V_\kappa$ such that $L(f_\delta, \kappa, \gamma_\delta, \lambda_\delta)$ holds. Then certainly $L(f_\delta, \kappa, \gamma_\delta, \gamma_{\delta+1})$ holds. Since there are 2^κ many functions from κ to V_κ, the regularity of $\bar{C}$ implies that for some function $f\colon \kappa \to V_\kappa$, and some $D \subseteq C$ with $\bar{D} = \bar{C}$, we have $f = f_\delta$ for every δ such that $\gamma_\delta \in D$. Now let $A \subseteq D$ be such that $\bar{A} = \bar{D}$ and A contains none of its limit points. This A satisfies the premises of the theorem, and hence its conclusion. ■

COROLLARY 2. *Suppose $\kappa \to (\lambda_0, \lambda_1)$. Then, for some set A which is cofinal below λ_0, $\kappa \to (\alpha)$ for every $\alpha \in A$, and every such $\kappa \to (\alpha)$ can be made simultaneously resurrectable after κ-directed closed forcing of smaller cardinality. In other words, the conclusion of the theorem holds for this A.*

Proof. Assume $\kappa \to (\lambda_0, \lambda_1)$. By the lemma of the previous section, for some $f\colon \kappa \to V_\kappa$, $L(f, \kappa, \lambda_0, \lambda_1)$ holds. Let $i\colon V \to M$ witness that $\kappa \to (\lambda_0, \lambda_1)$. By elementarity, $M \models \lambda_0 \to (\lambda_1, i(\lambda_1))$. By closure considerations, $\lambda_0 \to (\lambda_1)$ holds in V. Let U witness $\lambda_0 \to (\lambda_1)$. By closure considerations again, $f \in M_U$ and $M_U \models L(f, \kappa, \lambda_0, \lambda_1)$.

But then $\{\beta < \lambda_0: L(f, \varkappa, \beta, \lambda_0)\}$ is cofinal below λ_0. By closure considerations yet again, $\{\beta < \lambda_0: M_U \models L(f, \varkappa, \beta, \lambda_0)\}$ is cofinal below λ_0. Then $\{\beta < \lambda_0: \{\gamma < \lambda_0: L(f, \varkappa, \beta, \gamma)\}$ is cofinal below $\lambda_0\}$ is cofinal below λ_0. It is then easy to find a set A such that A is cofinal below λ_0, A contains none of its limit points, and, if $\{\alpha_\delta: \delta < \lambda_0\}$ is an increasing enumeration of the elements of A, then, for any $\delta < \lambda_0$, $L(f, \varkappa, \alpha_\delta, \alpha_{\delta+1})$ holds. This A satisfies the premises of the theorem, and hence its conclusion. ∎

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Received 12 April 1989;
in revised form 23 November 1989

On the l-equivalence of metric spaces

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Abstract. In this paper we present topological properties of metric spaces which are preserved by l-equivalence. Furthermore, we present an isomorphic classification of the function spaces $C_p(X)$ where X is any countable metric space with scattered height less than or equal to ω.

0. Introduction. *By a space we mean a Tikhonov space.* For a space X we define $C(X)$ ($C^*(X)$) to be the set of all continuous (bounded continuous) real valued functions on X. We can topologize these function spaces in several natural ways. Whenever we endow $C(X)$ ($C^*(X)$) with the compact-open topology we denote it by $C_0(X)$ ($C_0^*(X)$), and if we endow $C(X)$ ($C^*(X)$) with the topology of pointwise convergence we denote it by $C_p(X)$ ($C_p^*(X)$).

In [10] van Mill proved that for a countable metric space X which is not locally compact we have $C_p^*(X) \approx \sigma_\omega$, where $\sigma_\omega = (l_f^2)^\infty$ and $l_f^2 = \{x \in l^2 | \ x_i = 0$ for all but finitely many $i\}$ (l^2 denotes separable Hilbert space). Furthermore in [5] it was proved that under the same conditions $C_p(X) \approx \sigma_\omega$. It is easily seen that for an infinite countable discrete space X, $C_p(X) \approx \mathbf{R}^\infty$. The gap between "discrete" and "not locally compact" was filled in by Dobrowolski, Gul'ko and Mogilski in [7]. They proved that for every countable metric nondiscrete space X, $C_p^*(X) \approx C_p(X) \approx \sigma_\omega$. After these results it is interesting to study linear homeomorphism between the function spaces $C_p(X)$ ($C_p^*(X)$), for countable metric spaces X.

In [12] Pelant gives an example of two countable metric spaces X and Y, which are both not locally compact, such that $C_p^*(X)$ and $C_p^*(Y)$ are not linearly homeomorphic. In [5], Baars, de Groot, van Mill and Pelant gave an example of two countable metric spaces X and Y, which are both not locally compact, such that $C_p(X)$ and $C_p(Y)$ are not linearly homeomorphic. In [3] Baars and de Groot presented an isomorphic classification of the function spaces $C_p(X)$, for zero-dimensional locally compact separable metric spaces X. These classification results depend strongly on results in [1] of Arkhangel'skiĭ and on the isomorphic classification of the function spaces $C_0(X)$, where

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1985 *Mathematics Subject Classification*: 54C35, 57N17.
Key words and phrases: function space, l-equivalence, countable metric space.