A Cantor–Bendixson theorem for the space $\omega_1^{\omega_1}$

by

Jouko Väänänen (Helsinki)

Abstract. Consider the Baire space $\mathcal{N} = \omega^\omega$, i.e. the product of countably infinitely many copies of the discrete space ω of natural numbers. The Cantor–Bendixson theorem says that any closed subset of ω^ω can be uniquely expressed as the disjoint union of a perfect (i.e. closed and dense-in-itself) set and a countable set. We study the generalized Baire space $\mathcal{N}_1 = \omega_1^{\omega_1}$ obtained from the cartesian product of ω_1 copies of ω_1 by letting basic neighborhoods of any $f\colon \omega_1 \to \omega_1$ be sets of the form $N(f, \alpha) = \{g\colon \omega_1 \to \omega_1 \mid g(\beta) = f(\beta) \text{ for } \beta < \alpha\}$ where $\alpha < \omega_1$. This is an ω_1-metrizable space in the sense of Sikorski [8]. We study the cardinality of closed subsets of this space. We show among other things that a natural generalization of the Cantor–Bendixson theorem to the space $\mathcal{N}_1$ is consistent relative to the consistency of a measurable cardinal.

1. Introduction. The purpose of this paper is to use game-theoretic ideas to generalize Cantor–Bendixson type constructions from the space $\mathcal{N}$ to the analogous space $\mathcal{N}_1$. Our motivation comes from logic where the latter space can serve as a domain for the study of classes of models of cardinality ω_1 (see e.g. [7]).

Throughout the paper we work in the space $\mathcal{N}_1$ defined in the above abstract. However, the results can be generalized to ω_1-metric spaces (see [8]) which satisfy a natural completeness condition (considered in [8]) and which have a dense set of cardinality $\leqslant 2^\omega$.

For the appropriate generalization of the Cantor–Bendixson theorem from the space $\mathcal{N}$ to the space $\mathcal{N}_1$ we introduce the concepts of *α-perfectness* and *α-scatteredness*. An ω_1-perfect set is a canonical example of a subset of $\mathcal{N}_1$ of cardinality 2^{ω_1}. In Section 2 we introduce a hypothesis $\mathrm{I}(\omega)$ which implies that any closed subset of $\mathcal{N}_1$ of cardinality at least ω_2 contains an ω_1-perfect subset. In Section 3 we use forcing to construct an $(\omega+1)$-scattered subset of $\mathcal{N}_1$ of cardinality ω_2. In Section 4 we show that trees can play in $\mathcal{N}_1$ a role similar to the role of Cantor–Bendixson ranks in $\mathcal{N}$.

The cardinality of a set A is denoted by $|A|$. The restriction of $f \in \mathcal{N}_1$ to an ordinal α is denoted by $f|\alpha$. By a *tree* we mean a partial order in which every element has a well-ordered set of predecessors. A tree may have many roots, i.e. minimal elements. The *rank* of an element of a tree is the order-type of the set of its predecessors. The set of elements of a tree T of rank α is denoted by T^α. If x is an element of a tree, then $x|\alpha$ denotes the set of predecessors of x of rank $< \alpha$. The set of elements of rank $< \alpha$ is denoted by $T|\alpha$. The *height* of a tree T is the least ordinal greater than the ranks of elements of T. The supremum of the ranks of elements of T is denoted by $\gamma(T)$. For the definition of a *normal* tree we refer to [4]. If T is a tree, σT is the tree of all initial segments of branches of T ordered by end-extension. If T_1 and T_2 are trees and there is an order-preserving mapping $T_1 \to T_2$, we write $T_1 \leqslant T_2$. A subset of ω_1 is *closed* if it contains supremums of ascending chains of its elements, *stationary* if it meets every closed and unbounded set, and *bistationary* if it is stationary and its complement is stationary. Continuum Hypothesis $2^{\omega} = \omega_1$ is denoted by CH and the Generalized Continuum Hypothesis $\forall\alpha(2^{\omega_\alpha} = \omega_{\alpha+1})$ by GCH.

The author is indebted to Hugh Woodin for suggesting the proof of Theorem 3 and discussing its details and to Taneli Huuskonen for reading the manuscript and making useful comments.

2. Perfectness generalized. Perfectness of a closed subset E of $\mathcal{N}$ can be characterized by the following simple game $G(E, x_0)$. There are two players I and II. Player I plays natural numbers in ascending order and player II plays points of E (see Figure 1). Player I starts the game with some n_1. Whenever I has played n_i, and ($i = 1$ or) II has played $x_j, j < i$, II plays a point x_i from E in such a way that

$$x_i \neq x_j \quad \text{but} \quad x_i | n_{j+1} = x_j | n_{j+1}$$

I	II
n_1	
	x_1
n_2	
	x_2
$\vdots$	$\vdots$

Fig. 1. $G(E, x_0)$

for $j < i$. Player II *wins* the game if he can make all his ω moves, otherwise player I wins. It is easy to see that E is perfect if and only if player II has a winning strategy in $G(E, x_0)$ for all $x_0 \in E$.

I	II
α_1	
	x_1
α_2	
	x_2
$\vdots$	$\vdots$
α_ξ	
	x_ξ
$\vdots$	$\vdots$
$(\xi < \delta)$	$(\xi < \delta)$

Fig. 2. $G(E, x_0, \delta)$

We shall generalize the notion of perfectness by simply letting the above game go on for δ moves for some countable ordinal δ. So let E be a subset of $\mathcal{N}_1$ and δ an infinite ordinal $\leqslant \omega_1$. The game $G(E, x_0, \delta)$ is defined as $G(E, x_0)$ above except that now I plays countable ordinals α_ξ for $0 < \xi < \delta$ in ascending order (see Figure 2). At limits I has to play the supremum of his previous moves. Again II wins if he can make all his moves, otherwise I wins.

DEFINITION 1. A subset E of $\mathcal{N}_1$ is *δ-perfect* if it is closed and player II has a winning strategy in $G(E, x_0, \delta)$ for all $x_0 \in E$.

Note that E is dense-in-itself in the ordinary sense if and only if it is ω-perfect. While $G(E, x_0)$ is determined by the Gale–Steward theorem, there is no reason why $G(E, x_0, \alpha)$ should be determined for infinite α. For an example, let $A \subset \omega_1$ be bistationary and

(1) $$E_A = \{f \in \mathcal{N}_1 \mid f(\alpha) = 0 \text{ except for a closed set of } \alpha \in A\}.$$

Now $G(E_A, x_0, \omega+1)$ is non-determined for all $x_0 \in E_A$. Indeed, if II had a winning strategy, A would contain a closed unbounded set, and if I had a winning strategy, A would avoid a closed unbounded set. Note that E_A is a (closed and) perfect set of cardinality 2^{ω}.

It follows immediately from the definition that if E is both $(\alpha+1)$-perfect and β-perfect, then it is $(\alpha+\beta)$-perfect. Let us call an ordinal δ *indecomposable* if $\alpha < \delta$ implies $\alpha+\delta = \delta$. Thus δ-perfectness implies $(\delta+1)$-perfectness for decomposable ordinals δ. On the other hand, if δ is indecomposable, then the set

(2) $$S_\delta = \{f \in \mathcal{N}_1 \mid \text{the order type of } \{\alpha \mid f(\alpha) \neq 0\} \text{ is } < \delta\}$$

is δ-perfect but not $(\delta+1)$-perfect.

Perfect subsets of $\mathcal{N}$ have cardinality 2^{ω}. This is not true of $\mathcal{N}_1$ as the following example shows. Let

(3) $$E_{\text{fin}} = \{f \in \mathcal{N}_1 \mid f(\alpha) = 0 \text{ except for a finite number of } \alpha < \omega_1\}.$$

The set E_{fin} is clearly (closed and) perfect but has cardinality ω_1 only.

PROPOSITION 1. *Let E be a non-empty δ-perfect subset of $\mathcal{N}_1$. If $\delta > \omega$, then $|E| \geqslant 2^{\omega}$. If $\delta = \omega_1$, then $|E| = 2^{\omega_1}$.*

Proof. Let $x_0 \in E$. Let τ be a winning strategy of II in $G(E, x_0, \delta)$. By repeatedly using the $\omega+1$ first moves provided by τ we can build a full binary tree of height $\omega+1$ inside E. This proves the first claim. Suppose then τ is a winning strategy of II in $G(E, x_0, \omega_1)$. Using again τ repeatedly we can build a system x_s, $s \in 2^{<\omega_1}$, of points of E and a system α_s, $s \in 2^{<\omega_1}$, of ordinals in such a way that if s is an initial segment of s', then $x_s(\beta) = x_{s'}(\beta)$ for $\beta < \alpha_s$ and $x_s(\alpha_s) \neq x_{s'}(\alpha_s)$. Now we choose for any $f \in 2^{\omega_1}$ an element $x_f \in \mathcal{N}_1$ so that $x_f(\beta) = x_s(\beta)$ whenever $\beta < \alpha_s$ and s is $f|\alpha_s$. Since E is closed, all the points $x_f, f \in 2^{\omega_1}$, are in E and are distinct. This proves the second claim. ∎

Closed subsets of $\mathcal{N}_1$ are closely related to trees of cardinality and height ω_1. Let E be a closed subset of $\mathcal{N}_1$. Let $[E]$ be the tree of functions $f|\alpha$, where $f \in E$ and $\alpha < \omega_1$, ordered by $\subseteq$. Then E is the set of uncountable branches of $[E]$. On the other hand, if T is a subtree of $\omega_1^{<\omega_1}$, then the set $[T]$ of uncountable branches of T is a closed subset of $\mathcal{N}_1$. Note that if the levels of the tree T are countable, then $[T]$ cannot contain any $(\omega+1)$-perfect subsets. This follows from the first part of the proof of Proposition 1. If T happens to be a Kurepa tree, then $[T]$ is an example of a closed set of cardinality $\geqslant \omega_2$ which contains no δ-perfect subsets for $\delta > \omega$. With this remark we have proved the following observation which indicates an important limitation to any attempt to generalize the Cantor–Bendixson theorem to the space $\mathcal{N}_1$:

PROPOSITION 2. *If there is a Kurepa tree, then $\mathcal{N}_1$ has a closed subset of cardinality $\geqslant \omega_2$ with no $(\omega+1)$-perfect subsets.*

Another immediate consequence of the relation between closed subsets of $\mathcal{N}_1$ and trees is the equivalence of the following two conditions:

1. $\mathcal{N}_1$ has a closed subset of cardinality λ.
2. There is a tree of cardinality and height ω_1 with exactly λ uncountable branches.

Thus the statement

(CB1) There is no closed subset of $\mathcal{N}_1$ of cardinality λ where $\omega_1 < \lambda < 2^{\omega_1}$

is equiconsistent with the existence of an inaccessible cardinal (see [5], p. 84).

A simple diagonal argument shows that if CH is assumed, then there is a subset of $\mathcal{N}_1$ which has cardinality 2^{ω_1} and which has no non-empty ω_1-perfect subsets. On the other hand, the proof of Theorem 4.8 of [5] shows that if an inaccessible cardinal is collapsed (in a model of GCH) to ω_2, then in the resulting model every closed subset of $\mathcal{N}_1$ of cardinality $\geqslant \omega_2$ contains a non-empty ω_1-perfect subset. Thus the statement

(CB2) Every closed subset of $\mathcal{N}_1$ of cardinality $\geqslant \omega_2$ contains a non-empty ω_1-perfect subset

is equiconsistent with the existence of an inaccessible cardinal.

Let us call a closed subset E of $\mathcal{N}_1$ *σ-closed* if the tree $[E]$ contains a subset D which is dense (i.e. every $t \in T$ has an extension in D) and closed under unions of ascending ω-sequences.

LEMMA 1. *Any perfect σ-closed subset of $\mathcal{N}_1$ is ω_1-perfect.*

Proof. Let E be σ-closed and perfect. Suppose $x_0 \in E$ is given and I starts the game with α_1. Then II finds an extension d_1 of $x_0|\alpha_1$ from D. Now there is some $x_1 \in E$ other than x_0 which extends d_1. This is the first move of II. The strategy of II is to play all his moves as extensions of an ascending sequence of elements of D. Since D is closed under limits of countable ascending sequences, II will be able to play ω_1 moves. ∎

Taneli Huuskonen pointed out that the set of $f: \omega_1 \to 3$ with $f(\alpha) = 2$ for finitely many α only, is ω_1-perfect but not σ-closed.

The *δ-kernel* $\mathrm{Ker}(E, \delta)$ of a subset E of $\mathcal{N}_1$ is defined as the set of x_0 for which II has a winning strategy in $G(E, x_0, \delta)$. It is easy to see that $\mathrm{Ker}(E, \delta)$ is a closed δ-perfect subset of the closure of E, which contains every δ-perfect subset of E. For $\alpha < \beta$, $\mathrm{Ker}(E, \beta) \subseteq \mathrm{Ker}(E, \alpha)$. The perfect kernel of E in the usual sense is, of course, $\mathrm{Ker}(E, \omega)$.

For our next result we need the following set-theoretical hypothesis:

I(ω) There is a normal ω_2-complete ideal $\mathcal{I}$ on ω_2 such that $\mathcal{I}^+ = \{a \subseteq \omega_2 \mid a \notin \mathcal{I}\}$ has a dense subset in which every descending ω-sequence has a lower bound.

R. Laver (unpublished, see [1]) has proved that I(ω) becomes true if a measurable cardinal is Levy-collapsed to ω_2. On the other hand, the ideal given by I(ω) is precipitous so that the consistency of I(ω) implies the consistency of a measurable cardinal. I(ω) implies CH.

THEOREM 1. *Assume I(ω). If E is a subset of $\mathcal{N}_1$ of cardinality $\geqslant \omega_2$, then E meets* $\mathrm{Ker}(E, \omega_1)$.

Proof. Let E be a subset of $\mathcal{N}_1$ of cardinality $\geqslant \omega_2$. Let $S \subseteq E$ have cardinality ω_2. Let $\mathcal{I}$ be an ω_2-complete normal ideal on S such that $\mathcal{I}^+$ has a dense subset K which is closed under descending ω-sequences. Let us call $x \in X$, $X \subseteq S$, an *$\mathcal{I}$-point of X* if every neighborhood N of x satisfies $N \cap X \in \mathcal{I}^+$. We prove at first that every $X \in \mathcal{I}^+$ has an $\mathcal{I}$-point. Suppose not. Then every element x of X has a basic neighborhood N_x with $N_x \cap X \in \mathcal{I}$. Since we have CH, there are but ω_1 basic neighborhoods and we get $X \in \mathcal{I}$, which is a contradiction.

Let now x_0 be an $\mathcal{I}$-point of S. We show that x_0 is in $\mathrm{Ker}(E, \omega_1)$, that is, player II has a winning strategy in $G(E, x_0, \omega_1)$. Suppose I plays α_1. Then, since x_0 is an $\mathcal{I}$-point of S, there is a set $X_1 \in K$ with $X_1 \subseteq N(x_0, \alpha_1)$. The strategy of II is to choose some $\mathcal{I}$-point x_1 of X_1. When I plays α_2, II chooses some $X_2 \in K$ with $X_2 \subseteq X_1 \cap N(x_1, \alpha_2)$. According to the strategy of II he chooses some $\mathcal{I}$-point x_2 of X_2. Since K has infimums for descending ω-sequences, player II can follow this strategy for the whole length of ω_1 moves. ∎

Note that I(ω) implies, by the above theorem, condition (CB2). We shall see later (Theorem 4) that I(ω) implies (CB2) in a particularly strong form.

3. Scatteredness generalized. A space E is *scattered* if each non-empty subspace contains an isolated point. It is easy to see that a subset of $\mathcal{N}$ is scattered iff player I has a winning strategy in $G(E, x_0)$ for all $x_0 \in E$. Thus a closed subset of $\mathcal{N}$ splits into its perfect kernel and a countable scattered part.

DEFINITION 2. A subset E of $\mathcal{N}_1$ is *δ-scattered* if player I has a winning strategy in $G(E, x_0, \delta)$ for all $x_0 \in E$. The *δ-scattered part* of E, $\mathrm{Sc}(E, \delta)$, is the set of $x_0 \in E$ for which I has a winning strategy in $G(E, x_0, \delta)$.

Note that $\mathrm{Sc}(E, \delta)$ is itself δ-scattered and if E is closed, then $E - \mathrm{Sc}(E, \delta)$ is closed. If δ is indecomposable, then the set S_δ defined by equation (2) is a δ-perfect set which is $(\delta+1)$-scattered.

In $\mathcal{N}$ every scattered set is countable. In $\mathcal{N}_1$ we have the following scattered (even discrete) sets of cardinality 2^{ω}:

(4) $$E_\alpha = \{f \in \mathcal{N}_1 \mid f(\beta) = 0 \text{ for } \beta \geqslant \alpha\}.$$

The set E_{fin} defined by equation (3) is an $(\omega+1)$-scattered closed set of cardinality ω_1. The δ-scatteredness of sets of cardinality $\geqslant \omega_1$ leads in general outside the range of ZFC and we have only independence results. The following proposition gives the immediately provable cases.

PROPOSITION 3. *Let E be a subset of* $\mathcal{N}_1$. *If E is countable, then E is* $(\omega+1)$*-scattered. If E is closed and* $|E| \leqslant \omega_1$, *then E is* ω_1*-scattered.*

Proof. Suppose $E = \{f_n \mid n < \omega\}$. The strategy of I is to choose α_{n+1} in such a way that either $x_n = f_n$ or $f_n(\beta) \neq x_n(\beta)$ for some $\beta < \alpha_{n+1}$. Then II cannot play x_ω without breaking the rules. Suppose then $E = \{f_\alpha \mid \alpha < \omega_1\}$ is closed. Now I chooses α_ξ so that either $x_\xi = f_\xi$ or $f_\xi(\beta) \neq x_\xi(\beta)$ for some $\beta < \alpha_\xi$. Let $f \in \mathcal{N}_1$ so that $f(\beta) = x_\xi(\beta)$ for $\beta < \alpha_\xi$. Since E is closed, $f \in E$, a contradiction. ∎

The set E_A from equation (1) is an example of a closed set which has cardinality 2^ω but which is not δ-scattered for any $\delta < \omega_1$. This can be seen as follows: Suppose I has a winning strategy τ in $G(E_A, x_0, \delta)$. There is a forcing notion which adds a closed and unbounded subset C to A but adds no new countable subsets to the universe. In the forcing extension, II has a strategy, based on C, which beats τ. But τ is still a winning strategy of I in the extension. This contradiction establishes the claim. Actually we have shown $\mathrm{Sc}(E_A, \delta) = \emptyset$.

THEOREM 2. *The statement: There is a closed subset E of* $\mathcal{N}_1$ *of size* ω_2 *such that*

$$\mathrm{Ker}(E, \omega+1) = \mathrm{Sc}(E, \omega+1) = \emptyset$$

is consistent with GCH *assuming the consistency of* ZFC.

Proof. We use countable trees as conditions to force a Kurepa tree. This is a standard forcing notion (see e.g. Theorem 55 in [4]). Let $\mathcal{P}_1$ be the set of conditions (T, f) where

(P1) T is a countable normal tree of successor height,

(P2) f is a countable partial function from ω_2 onto $T^{\gamma(T)}$.

We order the conditions as follows: $(T, f) \leqslant (T', f')$ iff

(P3) $T' = T \mid \gamma(T')$,

(P4) $\mathrm{dom}(f') \subseteq \mathrm{dom}(f)$,

(P5) $f'(\alpha) \leqslant f(\alpha)$ if $\alpha \in \mathrm{dom}(f')$.

If G is a $\mathcal{P}_1$-generic set of conditions over V, then

$$T = \{T' \mid (T', f) \in G \text{ for some } f\}$$

is a Kurepa tree. Let E be the closed subset $[T]$ of $\mathcal{N}_1$. We know already that $\mathrm{Ker}(E, \omega+1) = \emptyset$ and $|E| = \omega_2$. To prove $\mathrm{Sc}(E, \omega+1) = \emptyset$, suppose I has a winning strategy τ in $G(E, x_0, \omega+1)$. Let $\mathbf{E}$, $\mathbf{x}_0$ and $\boldsymbol{\tau}$ be names for E, x_0 and τ such that for some $p_0 \in G$

(5) $$p_0 \Vdash (\boldsymbol{\tau} \text{ is a winning strategy of } I \text{ in } G(\mathbf{E}, \mathbf{x}_0, \omega+1)).$$

Let $p_0' \leqslant p_0$ such that for some α_1

$$p_0' \Vdash \boldsymbol{\tau}(\mathbf{x}_0) = \alpha_1.$$

We may assume that $p_0' = (T_0, f_0)$ such that $\gamma(T_0) > \alpha_1$. Let $p_1 \leqslant p_0'$ such that for some x_1

$$p_1 \Vdash \mathbf{x}_1 \mid \alpha_1 = \mathbf{x}_1 \mid \alpha_1 \,\&\, \mathbf{x}_1 \neq \mathbf{x}_0 \,\&\, \mathbf{x}_1 \in \mathbf{E}.$$

We construct inductively $p_n = (T_n, f_n)$, x_n and α_n in such a way that $\gamma(T_n) > \alpha_n$,

$$p_n \Vdash \boldsymbol{\tau}(\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}) = \alpha_n$$

and

$$p_n \Vdash \mathbf{x}_n \mid \alpha_n = \mathbf{x}_{n-1} \mid \alpha_n \,\&\, \mathbf{x}_n \neq \mathbf{x}_{n-1} \,\&\, \mathbf{x}_n \in \mathbf{E}.$$

Now we can construct p_ω and x_ω such that $p_\omega \leqslant p_n$ for all $n < \omega$ and

$$p_\omega \Vdash \mathbf{x}_\omega \mid \alpha_n = \mathbf{x}_n \mid \alpha_n.$$

This contradicts equation (5). ∎

THEOREM 3. *The statement*

There is a closed $(\omega+1)$*-scattered subset of* $\mathcal{N}_1$ *of cardinality* ω_2

is consistent with GCH *assuming the consistency of* ZFC.

Proof. The proof is an elaboration of the proof of Proposition 2. The set of forcing conditions we use was suggested by Hugh Woodin. Recall the definition of $\mathcal{P}_1$ in the proof of Proposition 2. Suppose $(T, f) \in \mathcal{P}_1$. Let $G(T)$ be the following game (which is similar to $G(E, x_0, \omega+1)$): Player I plays an ascending sequence $(\alpha_n)_{n<\omega}$ of ordinals $< \gamma(T)$ and player II plays elements of $\mathrm{dom}(f)$. Player II starts the game with some a_0. Then I plays α_0. Next II has to play a_1 such that $f(a_1) \neq f(a_0)$ and $f(a_1) \mid \alpha_0 = f(a_0) \mid \alpha_0$. Then I plays again some $\alpha_1 > \alpha_0$ and II answers with a_2 such that $f(a_2) \neq f(a_1)$ and $f(a_2) \mid \alpha_1 = f(a_1) \mid \alpha_1$, and so on. Suppose $(a_n, \alpha_n)_{n<m}$ $(m \leqslant \omega)$ has been played. An element a of ω_2 *extends* $(a_n, \alpha_n)_{n<m}$ if $a \neq a_n$ and $f(a) \mid \alpha_n = f(a_n) \mid \alpha_n$ for all $n < \omega$. Player II wins if, after $(a_n, \alpha_n)_{n<\omega}$ has been played, $(a_n, \alpha_n)_{n<\omega}$ has an extension. Otherwise I has won.

Suppose $p = (T, f) \in \mathcal{P}_1$ and τ is a strategy of I in $G(T)$. A sequence $(a_n)_{n<m}$ $(m \leqslant \omega)$ is a (p, τ)*-sequence* if each a_n is in ω_2 and $(a_n)_{n<m}$ is a sequence of moves of II in $G(T)$ when I plays τ. If $a \in \omega_2$, a sequence $(a_n)_{n<m}$ $(m \leqslant \omega)$ is a (p, τ, a)*-sequence* if it is a (p, τ)*-sequence*, and $f(a)$ is an extension of $(a_n, \tau(a_0, \ldots, a_n))_{n<m}$.

A *rank-function* for (p, τ) is a function ϱ such that if $a \in \mathrm{dom}(f)$ and $(a_n)_{n\leqslant m}$ is a (p, τ, a)-sequence, then $\varrho(a, a_0, \ldots, a_n)$ is defined and is $< \omega_1$ and for any (p, τ, a)-sequence $(a_0, \ldots, a_n, a_{n+1})$:

$$\varrho(a, a_0, \ldots, a_n) > \varrho(a, a_0, \ldots, a_n, a_{n+1}).$$

Note that any τ for which (p, τ) has a rank function ϱ is necessarily a winning strategy. Indeed, if II could play $(a_n)_{n<\omega}$ against τ and the sequence had a τ-extension $f(a)$, then

$$\varrho(a, a_0) > \ldots > \varrho(a, a_0, \ldots, a_n) > \ldots,$$

which is impossible.

CLAIM 1. *If* $p = (T, f) \in \mathcal{P}_1$, τ *is a winning strategy of I in* $G(T)$ *and* $a \in \mathrm{dom}(f)$, *then* (p, τ) *has a rank-function.*

Proof. Let R_0^a be the set of (p, τ, a)-sequences $(a_0, \ldots, a_n)$ such that there is no (p, τ, a)-sequence $(a_0, \ldots, a_n, a_{n+1})$. Let $R_{\alpha+1}^a$ be the set of (p, τ, a)-sequences $(a_0, \ldots, a_n)$ such that every (p, τ, a)-sequence $(a_0, \ldots, a_n, a_{n+1})$ is in R_α^a. Finally, let R_ν^a $(\nu = \bigcup \nu)$ be the union of R^a_α, $\alpha < \nu$. Suppose now $(a_0, \ldots, a_n)$ is a (p, τ, a)-sequence. If $(a_0, \ldots, a_n) \notin R^a_{\omega_1}$, then a cardinality argument gives an a_{n+1} such that $(a_0, \ldots, a_n, a_{n+1}) \notin R^a_{\omega_1}$. Iterating this yields a win for II against τ. This contradiction shows that $(a_0, \ldots, a_n) \in R^a_{\omega_1}$. Let

$$\varrho(a, a_0, \ldots, a_n) = \min\{\alpha < \omega_1 \mid (a_0, \ldots, a_n) \in R^a_\alpha\}.$$

This ϱ is a rank-function for (T, f). The claim is proved. ■

Let $\mathscr{P}_2$ be the set of conditions

$$(T, f, \tau, \varrho),$$

where T and f satisfy the above conditions (P1) and (P2) and moreover:

(P6) $\gamma(T)$ is a limit ordinal,

(P7) τ is a winning strategy of I in $G(T)$,

(P8) ϱ is a rank-function for $((T, f), \tau)$.

We order the conditions as follows:

$$(T, f, \tau, \varrho) \leqslant (T', f', \tau', \varrho')$$

iff (P3)–(P5) hold and

(P9) $\tau(a_0, \ldots, a_n) = \tau'(a_0, \ldots, a_n)$ if $a_0, \ldots, a_n \in \operatorname{dom}(f')$,

(P10) $\varrho(a, a_0, \ldots, a_n) = \varrho'(a, a_0, \ldots, a_n)$ if $a, a_0, \ldots, a_n \in \operatorname{dom}(\varrho')$.

CLAIM 2. *$\mathscr{P}_2$ is countably closed.*

Proof. Suppose

$$p_0 \geqslant p_1 \geqslant \ldots \geqslant p_n \geqslant \ldots \quad (n < \omega)$$

is a sequence of elements of $\mathscr{P}_2$ and

$$p_n = (T_n, f_n, \tau_n, \varrho_n).$$

Let $T = \bigcup_{n<\omega} T_n$ and $D = \bigcup_{n<\omega} \operatorname{dom}(f_n)$. We let T_ω be T extended by a top-level which contains a unique extension t_a for each branch $\{f_n(a) \mid n < \omega\}$, $a \in D$. We let $f_\omega(a) = t_a$ for $a \in D$. Then $p_\omega = (T_\omega, f_\omega)$ is in $\mathscr{P}_1$. Player I has a natural strategy τ_ω in $G(T_\omega)$ determined by $\bigcup_{n<\omega} \tau_n$. To define a rank-function for (p_ω, τ_ω), let $(a_n)_{n<m}$ be a $(p_\omega, \tau_\omega, a)$-sequence. For all $n < m$ there is a $k < \omega$ such that for $l \geqslant k$, $\varrho_l(a, a_0, \ldots, a_n)$ is defined and constant. We let $\varrho_\omega(a, a_0, \ldots, a_n)$ be this constant value. Clearly, ϱ_ω is a rank-function for (p_ω, τ_ω). Hence t_ω is a winning strategy and $(T_\omega, f_\omega, \tau_\omega, \varrho_\omega) \in \mathscr{P}_2$. The claim is proved. ■

Let G be a $\mathscr{P}_2$-generic set over V. Let

$$T = \bigcup\{T_0 \mid (T_0, f, \tau, \varrho) \in G \text{ for some } f, \tau, \varrho\},$$

$$b_a = \{f(a) \mid (T, f, \tau, \varrho) \in G \text{ for some } a \in \operatorname{dom}(f) \text{ and some } T, \tau, \varrho\}.$$

CLAIM 3. *T is a Kurepa tree in which each b_a, $a \in \omega_2$, is an uncountable branch.*

It suffices to show that the set

$$\{(T, f, \tau, \varrho) \in \mathscr{P}_2 \mid \gamma(T) \geqslant \alpha, \beta \in \operatorname{dom}(f)\}$$

is dense in $\mathscr{P}_2$ for all $\alpha < \omega_1$ and $\beta < \omega_2$. Since $\mathscr{P}_2$ is countably closed it suffices to prove that any condition (T, f, τ, ϱ) in $\mathscr{P}_2$ has a proper extension $(T', f', \tau', \varrho')$ with $\beta \in \operatorname{dom}(f')$. So let $(T, f, \tau, \varrho) \in \mathscr{P}_2$. Let T' be obtained from T by adding above any top-level node t a copy B_t of the binary tree $2^{<\omega}$ with an extension to all branches of B_t which are eventually zero. Let $s(t)$ denote the extension of the constant zero branch of B_t. We define $f'(a) = s(f(a))$ for $a \in \operatorname{dom}(f)$ and let f' map arbitrary new elements of ω_2 to the other top-level elements of T'. At this point we can make sure $\beta \in \operatorname{dom}(f')$. We have defined $p' = (T', f') \in \mathscr{P}_1$.

Next we define a strategy τ' of I in $G(T')$ as follows. For any sequence $(a_0, \ldots, a_{n-1})$ of moves of II in $G(T')$ from $\operatorname{dom}(f)$ we let

$$\tau'(a_0, \ldots, a_{n-1}) = \tau(a_0, \ldots, a_{n-1}).$$

Suppose then $a_n \in \operatorname{dom}(f') - \operatorname{dom}(f)$. Let $t \in T$ so that $f'(a_n)$ extends a branch in B_t. We let $\tau'(a_0, \ldots, a_{n-1}, a_n)$ be the smallest ordinal γ such that

$$f'(a_n)|\gamma \neq s(t)|\gamma.$$

If $s'(a_n) = s(t)$, we let $\tau'(a_0, \ldots, a_{n-1}, a_n) = \gamma(T)$. For any further sequence

$$(a_0, \ldots, a_{n-1}, a_n, a_{n+1}, \ldots, a_m)$$

we let τ' be determined by some diagonalization strategy based on the countability of T'.

To define ϱ' let $a \in \operatorname{dom}(f')$. If $a \in \operatorname{dom}(f)$ and

$$(a_0, \ldots, a_{n-1}) \in (\operatorname{dom}(f))^n$$

is a (p', τ', a)-sequence, we let

$$\varrho'(a, a_0, \ldots, a_{n-1}) = \varrho(a, a_0, \ldots, a_{n-1}).$$

If $a_n \in \operatorname{dom}(f') - \operatorname{dom}(f)$ so that $(a, a_0, \ldots, a_{n-1}, a_n)$ is still a (p', τ', a)-sequence, then $\varrho'(a, a_0, \ldots, a_{n-1}) > 0$ and we can let

$$\varrho'(a, a_0, \ldots, a_{n-1}, a_n) = 0.$$

Note that our definition of τ' ensures that no $(a, a_0, \ldots, a_{n-1}, a_n, a_{n+1})$ is a (p', τ', a)-sequence. Let then $a \in \operatorname{dom}(f') - \operatorname{dom}(f)$. Now we have full freedom in defining $\varrho'(a, a_0, \ldots, a_{n-1})$ and we can use Claim 1 to do so. This ends the definition of $(T', f', \tau', \varrho')$. The claim is proved. ■

CLAIM 4. *Every uncountable branch of T is a b_α for some $\alpha < \omega_2$.*

Proof. Suppose b is a branch of T which satisfies $b \neq b_\alpha$ for all $\alpha < \omega_2$. Let $\boldsymbol{b}$ and $\boldsymbol{b}_\alpha$ be names for b and b_α. Let $p_n = (T_n, f_n, \tau_n, \varrho_n)$ be a sequence of conditions such that $p_0 \geqslant \ldots \geqslant p_n \geqslant \ldots$ and

$$p_{n+1} \Vdash (\boldsymbol{b}|\beta_\alpha \neq \boldsymbol{b}_\alpha|\beta_\alpha)$$

for all $\alpha \in \operatorname{dom}(f_n)$. Let $p = \inf\{p_n \mid n < \omega\}$, $p = (T, f, \tau, \varrho)$. Let $\gamma = \gamma(T)$. The branch b has an element on level γ. So for some $n < \omega$ and $\alpha \in \operatorname{dom}(f_n)$

$$p \Vdash (b \text{ eventually agrees with } b_\alpha),$$

a contradiction. The claim is proved. ■

CLAIM 5. *$\mathscr{P}_2$ satisfies the $\aleph_2$-chain condition.*

Proof. Suppose W is a subset of $\mathscr{P}_2$ of cardinality $\aleph_2$. By the Δ-lemma (see Lemma 24.4 in [4]) there is a countable $A \subseteq \omega_2$ and $W_1 \subseteq W$ of size $\aleph_2$ such that if

$$(T, f, \tau, \varrho), (T', f', \tau', \varrho') \in W_1$$

are distinct then

$$T = T', \quad \operatorname{dom}(f) \cap \operatorname{dom}(f') = A, \quad f|A = f'|A,$$

$$\{(f(a_0), \ldots, f(a_n), \alpha) \mid \tau(a_0, \ldots, a_n) = \alpha\} = \{(f'(a_0), \ldots, f'(a_n), \alpha) \mid \tau'(a_0, \ldots, a_n) = \alpha\},$$

$$\{(f(a), f(a_0), \ldots, f(a_n), \alpha) \mid \varrho(a, a_0, \ldots, a_n) = \alpha\}$$
$$= \{(f'(a), f'(a_0), \ldots, f'(a_n), \alpha) \mid \varrho'(a, a_0, \ldots, a_n) = \alpha\}.$$

Now it is easy to see that any two elements of W_1 are compatible. The claim is proved. ■

Claims 2 and 5 imply that $\mathscr{P}_2$ preserves cardinals.

We may assume without loss of generality that T is a subtree of $\omega_1^{<\omega_1}$. Let E be the set $[T]$ of uncountable branches of T. In view of Claim 4 we may take E to be the set $\{b_a \mid a < \omega_2\}$. We define a strategy τ of I in $G(E, b_{a_0}, \omega+1)$ as follows:

$$\tau(b_{a_0}, \ldots, b_{a_n}) = \alpha \quad \text{iff} \quad \tau'(a_0, \ldots, a_n) = \alpha \quad \text{for some } (T', f', \tau', \varrho') \in G.$$

Let also for $a < \omega_2$

$$\varrho(a, b_{a_0}, \ldots, b_{a_n}) = \alpha \quad \text{iff} \quad \varrho'(a, a_0, \ldots, a_n) = \alpha \quad \text{for some } (T', f', \tau', \varrho') \in G.$$

We see that τ is a winning strategy of I as follows. Suppose II is able to play ω moves $b_{a_0}, \ldots, b_{a_n}, \ldots$ against τ in such a way that he can continue his play for one more move b_{a_ω}. Now

$$\varrho(a_\omega, b_{a_0}) > \ldots > \varrho(a_\omega, b_{a_0}, \ldots, b_{a_n}) > \ldots,$$

a contradiction. Thus τ is a winning strategy of I in $G(E, b_0, \omega+1)$, whatever $b_0 \in E$. ■

The following is our Cantor–Bendixson theorem for the space $\mathscr{N}_1$. It shows that assuming $\mathrm{I}(\omega)$, any closed subset of $\mathscr{N}_1$ can be made ω_1-perfect by removing up to ω_1 points.

THEOREM 4. *If* $\mathrm{I}(\omega)$, *then*

(CB3) *If E is a closed subset of $\mathscr{N}_1$, then*

$$E = \operatorname{Ker}(E, \omega_1) \cup \operatorname{Sc}(E, \omega_1), \quad \textit{where } |\operatorname{Sc}(E, \omega_1)| \leqslant \omega_1.$$

Proof. In view of Proposition 3 we may assume $|E| \geqslant \omega_2$. Suppose at first $|\operatorname{Sc}(E, \omega_1)| \geqslant \omega_2$. By Theorem 1 $\operatorname{Sc}(E, \omega_1)$ meets its ω_1-kernel. But then it also meets the ω_1-kernel of E, a contradiction. So we have established $|\operatorname{Sc}(E, \omega_1)| \leqslant \omega_1$. Thus the set

$$A = E - \operatorname{Sc}(E, \omega_1)$$

has size $\geqslant \omega_2$. Let $f \in A$. We shall prove that $f \in \operatorname{Ker}(E, \omega_1)$ by producing a winning strategy of II in $G(E, f, \omega_1)$. So suppose I plays α_1 and let

$$B = \overline{N(f, \alpha_1)} \cap A.$$

If $|B| \geqslant \omega_2$, we can use Theorem 1 to get $x_1 \in \operatorname{Ker}(B, \omega_1)$. Then II has a winning strategy in $G(B, x_1, \omega_1)$ which gives a winning strategy in $G(E, x_1, \omega_1)$ and we are done. So suppose $|B| \leqslant \omega_1$. Since B is closed we can use Proposition 3 to get a winning strategy τ of I in $G(A, f, \omega_1)$ starting with the move α_1. We get a contradiction by showing that τ gives rise to a winning strategy of I in $G(E, f, \omega_1)$. The strategy of I is to play τ as long as II plays his moves in A. As soon as II plays some $x_\xi \in \operatorname{Sc}(E, \omega_1)$, I starts to use his winning strategy in $G(E, x_\xi, \omega_1)$. This ends the description of the winning strategy of I. ■

We do not know the exact consistency strength of (CB3). It lies somewhere between the consistency of an inaccessible cardinal and the consistency of a measurable cardinal.

4. Trees and Cantor–Bendixson ranks. The perfect kernel of a closed subset E of a topological space can be expressed as the intersection of the following descending chain of closed sets:

$$\text{(6)} \qquad \begin{cases} E_0 = E, \\ E_{\alpha+1} = \text{limit points of } E_\alpha, \\ E_\nu = \bigcap_{\alpha<\nu} E_\alpha \quad (\nu = \bigcup \nu). \end{cases}$$

This gives rise to a notion of *rank*: If x is not in the perfect kernel of E, there is a unique α such that $x \in E_\alpha - E_{\alpha+1}$. This is the *Cantor–Bendixson rank* of x. In this section we use trees to represent $\operatorname{Ker}(E, \omega_1)$ as an intersection of levels, generalizing the hierarchy (6).

Let E be a subset of $\mathscr{N}_1$, T a tree and $x_0 \in \mathscr{N}_1$. We use

$$G(E, x_0, T)$$

to denote the game which is like $G(E, x_0, \omega_1)$ except that I plays countable ordinals α_ξ and elements t_ξ of the tree T. The elements t_ξ have to be chosen in ascending order from some chain in T (Figure 3). Winning is defined as for $G(E, x_0, \omega_1)$. Note that if

I	II
α_1, t_1	
	x_1
α_2, t_2	
	x_2
$\vdots$	$\vdots$
α_ξ, t_ξ	
	x_ξ
$\vdots$	$\vdots$
$(\xi < \delta)$	$(\xi < \delta)$

Fig. 3. $G(E, x_0, T)$

T consists of just one branch of length δ, then $G(E, x_0, T)$ is the same game as $G(E, x_0, \delta)$ (except that in $G(E, x_0, T)$ I does not have to play consecutive elements of T, but this is irrelevant). Using the game $G(E, x_0, T)$ rather than $G(E, x_0, \delta)$ we can define T-perfectness, T-scatteredness, $\mathrm{Ker}(E, T)$ and $\mathrm{Sc}(E, T)$ for any tree T. This is mainly interesting when T has branches of all lengths $\prec \delta$, δ a limit ordinal, but no branches of length δ.

The set E_A of (1) is $T(A)$-perfect and $(\sigma T(A))$-scattered, where $T(A)$ is the tree of closed sequences of elements of A.

Let B_α be the tree of all non-empty descending chains of elements of the ordinal α. Then (using the notation of (6))

$$\mathrm{Ker}(E, B_\alpha) = E_\alpha.$$

Using the fact that any tree T with no infinite branches satisfies $T \leqslant B_\alpha$ for some α, it is easy to see that

$$\mathrm{Ker}(E, \omega) = \bigcap \{\mathrm{Ker}(E, T) \mid T \text{ has no infinite branches}\}.$$

The following theorem as well as its proof are adaptations of similar results in [2], [3] and [6].

THEOREM 5. *Let E be a closed subset of $\mathcal{N}_1$. Then*

$$\mathrm{Ker}(E, \omega_1) = \bigcap \{\mathrm{Ker}(E, T) \mid T \textit{ has no uncountable branches}\},$$

$$\mathrm{Sc}(E, \omega_1) = \bigcup \{\mathrm{Sc}(E, T) \mid T \textit{ has no uncountable branches}\}.$$

Proof. If T has no uncountable branches, then

$$\mathrm{Ker}(E, \omega_1) \subseteq \mathrm{Ker}(E, T)$$

for trivial reasons. Suppose then $x_0 \notin \mathrm{Ker}(E, \omega_1)$. So II does not have a winning strategy in $G(E, x_0, \omega_1)$. Let T_1 denote the tree of pairs (τ, α) where τ is a winning strategy of II in $G(E, x_0, \alpha)$ and $\alpha < \omega_1$. We order these pairs by $(\tau, \alpha) \leqslant (\tau', \alpha')$ iff $\alpha \leqslant \alpha'$ and τ' coincides with τ on the first α moves. Let $T = \sigma T_1$. Note that T has no uncountable branches. If II has a winning strategy τ in $G(E, x_0, T)$, then a repeated appeal to τ yields an uncountable branch in T_1. Therefore

$$x_0 \notin \mathrm{Ker}(E, x_0, T)$$

and the first part of the claim is proved.

For the second claim, assume I has a winning strategy τ in $G(E, x_0, \omega_1)$. Let T_1 be the tree of sequences $(x_0, x_1, \ldots, x_\alpha)$, where $x_1, \ldots, x_\alpha$ are consecutive moves of II in $G(E, x_0, \omega_1)$ when I plays τ (and II has not lost yet). Let $T = \sigma T_1$. Note that T has no uncountable branches. Now I has a winning strategy in $G(E, x_0, T)$: he just copies sequences of moves of II into T. Thus $x_0 \in \mathrm{Sc}(E, T)$. ■

It is possible to prove representations like in Theorem 5 for arbitrary $\mathrm{Ker}(E, T)$ and $\mathrm{Sc}(E, T)$, but we omit the details (see [6] for similar constructions).

References

[1] F. Galvin, T. Jech and M. Magidor, *An ideal game*, J. Symbolic Logic 43 (1978), 284–292.
[2] T. Hyttinen, *Model theory for infinite quantifier languages*, Fund. Math. 134 (1990), 123–140.
[3] T. Hyttinen and J. Väänänen, *On Scott- and Karp-trees of uncountable models*, J. Symbolic Logic 55 (1990), 897–908.
[4] T. Jech, *Set Theory*, Academic Press, 1978.
[5] I. Juhász, *Cardinal functions* II, in: *Handbook of Set-Theoretic Topology* (eds. K. Kunen and J. E. Vaughan), North-Holland, 1984, 63–109.
[6] J. Oikkonen and J. Väänänen, *Game-theoretic inductive definability*, to appear.
[7] S. Shelah, H. Tuuri and J. Väänänen, *On the number of automorphisms of uncountable models*, to appear.
[8] R. Sikorski, *Remarks on some topological spaces of high power*, Fund. Math. 37 (1949), 125–136.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF HELSINKI
SF-00100 Helsinki
Finland

Received 5 November 1989