Functors preserving tameness

by

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Abstract. Let $A = k\langle Q \rangle/I$ be a basis and connected finite-dimensional algebra over an algebraically closed field $k$. For each dimension vector $z \in \mathbb{N}^{Q_0}$, we denote by $\text{mod}_A(z)$ the variety of $A$-modules of dimension type $z$ and by $\text{ind}_A(z)$ the constructible subset of indecomposable modules. We prove that $A$ is tame if and only if for each $z \in \mathbb{N}^{Q_0}$, any constructible subset $C$ of $\text{ind}_A(z)$ is at most one-dimensional provided different modules in $C$ are not isomorphic. We apply this criterion to show that tameness is preserved by Ext functors and under suitable assumptions by Galois covering functors.

Let $k$ be an algebraically closed field. Let $A$ be a finite-dimensional, basic and connected $k$-algebra. Following [11], we write $A = k\langle Q \rangle/I$, where $Q$ is a finite oriented graph (= quiver) and $I$ is an admissible ideal of the path algebra $k\langle Q \rangle$.

By $\text{mod}_A$ we denote the category of finite-dimensional left $A$-modules. We may identify the $A$-modules with representations of $Q$ satisfying the relations of $I$. If $M \in \text{mod}_A$, we set $\dim M = (\dim M(x))_{x \in Q_0}$, where $Q_0$ is the set of vertices of $Q$.

A vector $z \in \mathbb{N}^{Q_0}$ is called a dimension vector.

An algebra $A$ is said to be tame if for every dimension $z \in \mathbb{N}^{Q_0}$, there is a finite family of $A$-$k\langle z \rangle$-bimodules $M_i$ which are free as right $k\langle z \rangle$-modules such that every indecomposable $A$-module of dimension $z$ is isomorphic to $M_i \otimes_{A\langle z \rangle} S$ for some $i$ and some simple $k\langle z \rangle$-module $S$ (see [3, 4, 7, 8, ...]).

Let $z \in \mathbb{N}^{Q_0}$ be a dimension vector. By $\text{mod}_A(z)$ we denote the variety of $A$-modules of dimension type $z$ (see [4, 9, 10, 15, 16, 19]). The indecomposable modules in $\text{mod}_A(z)$ form a constructible subset denoted by $\text{ind}_A(z)$. In this note we show that an algebra $A$ is tame if and only if for each dimension $z \in \mathbb{N}^{Q_0}$, any constructible subset $C$ of $\text{ind}_A(z)$ is at most one-dimensional provided different modules in $C$ are not isomorphic.

This characterization is well-suited for applications. Applying it, we show that tameness is preserved by Ext functors (Section 2) and under a suitable assumption by the Galois covering functors (Section 3). It is possible to give other (similar) applications.

1. A criterion for tameness.

1.1. Let $z = (x(z))_{z \in Q_0} \in \mathbb{N}^{Q_0}$ be a dimension vector. The variety of $A$-modules of dimension $z$ is the closed subset $\text{mod}_A(z)$ of the affine space $\prod_{z \in Q_0} k^{x(z)}$ (where...
1.4. LEMMA (compare with [7, Lemma]). Let $V$ be an algebraic variety and $G$ be an algebraic group acting on $V$. Let $U_1$, $U_2$ be two constructible subsets of $V$ satisfying:
(i) $GU_1 = V$;
(ii) $U_1$ intersects each orbit of $G$ in at most one point.
Then $\dim U_1 \leq \dim U_1$.

Proof. We may assume that $U_1$ is irreducible. Consider the following algebraic maps:

$$\begin{align*}
    (g, u) \in G \times U_1 &\mapsto V, \\
    (g, u) &\mapsto gu
\end{align*}$$

Take an irreducible component $Z$ of $g^{-1}(U_2) \subset G \times U_1$ and $W = \pi(Z)$. Then the maps $f = \pi: Z \to U_2$ and $p = \pi: Z \to W_1$ are dominant. By [22, Chap. I.6], there is an open dense subset $Y$ of $Z$ such that for any $y \in Y$

$$\dim f^{-1}(f(y)) = \dim Z - \dim U_1$$

and $\dim p^{-1}(p(y)) = \dim Z - \dim W_1$.

Let $y \in Y$. We show that $p^{-1}(p(y)) = f^{-1}(f(y))$. In fact, assume that $y = (g, u) \in Z$ and let $y' = (h, u) \in p^{-1}(p(y))$. Therefore, $gu = f(y)$ and $hu = f(y)$ both lie in $U_2$. Since $h^{-1}g^{-1}f(y) = f(y)$, we have $f(y) = f(y')$, that is, $y' \in f^{-1}(f(y))$. We get that $\dim p^{-1}(p(y)) = \dim f^{-1}(f(y))$ and $\dim U_1 = \dim Z - \dim f^{-1}(f(y))$.

1.5. The following result on algebraic varieties will be used in the next sections. It is a particular case of [10, (4.2)].

LEMMA. Let $V$ be an algebraic variety over an uncountable algebraically closed field $k$. Let $(C_{n \in \omega})_n$ be a family of constructible subsets of $V$ such that $V = \bigcup_{n \in \omega} C_n$. Then there is a number $n \in \omega$ such that $V = \bigcup_{n \in \omega} C_n$.

2. Ext-functors preserve tameness.

2.1. The purpose of this section is to prove the following:

PROPOSITION. Let $A$ be a finite-dimensional $k$-algebra over an uncountable algebraically closed field $k$. For each $n \in \omega$, let $\Gamma_n$ be a finite-dimensional $A$-algebra. Assume that every indecomposable $X \in \mod_A$ is isomorphic to $\text{Ext}^1_{\Gamma_0}(V, T)$ for some $n \in \omega$ and $Y \in \mod_{\Gamma_n}$. If each $\Gamma_n$ is tame, then $A$ is tame.

Not all functors have such a nice behaviour as the following example shows.

Let $A$ be a wild algebra and let $A$ be the hereditary $k$-algebra with quiver $1 \to 2$. Therefore $A$ is tame. We will construct a functor $F: \mod_A \to \mod_A$ such that every indecomposable $X \in \mod_A$ is isomorphic to $FY$ for some $A$-module $Y$. Let $\{X_n\}_n$ be a set of representatives of the isoclasses of finite-dimensional indecomposable $A$-modules. Let $\{S_{n,k}\}$ be a set of representatives of the isoclasses of simple regular $A$-modules. Set $\mathcal{S} = \bigoplus_{n \in \omega} S_{n,k}$. Consider the $A$-$A$-bimodule

$$M = \bigoplus_{n \in \omega} (X_n \otimes \mathcal{S}).$$

Let $Y \in \mod_A$. Then $Y = Y_\ell \otimes Y_\omega \oplus Y_\omega$, where $Y_\ell$ (resp. $Y_\omega$, $Y_\omega$) is a direct sum of
preprojective (resp. regular, preinjective) $A$-modules. We define $FY = M \otimes_A Y$, and similarly in morphisms. Since
\[ FS_a \cong \bigoplus_{k \in k} X_{k} \otimes_A \text{Hom}_A(S_k, S_{j}) \cong X_{a}, \]
we see that $F: \text{mod}_A \to \text{mod}_A$ is well defined and satisfies the desired property.

If the field $k$ is countable, it is easy to give examples where the proposition fails.

2.2. Let $A$ and $I$ be two finite-dimensional $k$-algebras. Let $A = k[1]$. Let $T$ be a finite-dimensional $I$-$A$-bimodule and let $M$ be a $I$-$A$-bimodule which is a finitely generated free right $A$-module. Fix $x \in N$. Consider the functor
\[ \text{Ext}^1(T, M \otimes_A x) : \text{mod}_A \to \text{mod}_A. \]
Let $\ldots \to P_n(x) \to \ldots \to P_1(x) \to P_0(x) \to T \to 0$ be a finite-dimensional projective resolution of $I$-$A$-bimodules. Set $\partial_n^I = \text{Hom}_I(\partial_{n+1}^I, M)$ (and $\partial_0^I = 0$). For each $a \in Q_1$, we have induced morphisms $\delta_a \in \text{End}_I(T)$ and $x \in \text{End}_I(P_1)$ such that $x \delta_a = \partial_a x$ and $\partial_a x = \delta_a x$. Then the induced linear maps $\delta_a \in \text{End}(\text{ker}(\partial_n^I \otimes_A x) / \text{im}(\partial_n^I \otimes_A x))$ yield a left $A$-module structure on $\ker(\partial_n^I \otimes_A x) / \text{im}(\partial_n^I \otimes_A x)$. The following is an elementary fact but we will need later some details of the proof.

**Lemma.** $\text{Ext}^1(T, M \otimes_A x) \cong \ker(\partial_n^I \otimes_A x) / \text{im}(\partial_n^I \otimes_A x)$ as left $A$-modules.

**Proof.** Consider the natural transformation
\[ \varphi_T : \text{Hom}_I(T, M \otimes_A x) \to \text{Hom}_I(T, M \otimes_A x), \]
\[ f \otimes x \mapsto \varphi_T(f \otimes x) : T \to M \otimes_A x, \quad t \mapsto f(t) \otimes x. \]
In fact $\varphi_T$ is natural in the three variables. Clearly, $\varphi_T$ is an isomorphism if $T$ is projective.

If $s > 1$, we get the following commutative diagram:
\[ \begin{array}{ccc}
\text{Hom}_I(P_{s-1}, M) \otimes_A X & \xrightarrow{\varphi_{P_{s-1}}} & \text{Hom}_I(P_s, M) \otimes_A X \\
\downarrow \varphi_{P_{s-1}} & & \downarrow \varphi_{P_s} \\
\text{Hom}_I(P_{s-1}, M) \otimes_A X & \xrightarrow{\varphi_{P_{s}}} & \text{Hom}_I(P_{s+1}, M) \otimes_A X
\end{array} \]
\[ \xrightarrow{\varphi_{P_s}} \text{Hom}_I(P_{s+1}, M) \otimes_A X. \]
The calculation of the homology in the middle terms gives the result. The case $s = 0$ is similar.

2.3. We keep the notation of 2.2. Let $z \in \text{N}^2$. The set
\[ \mathfrak{E}(z) = \{ X \in \text{mod}_A(z) : X \cong \text{Ext}^1(T, M \otimes_A x) \text{ for some simple } A \text{-module } S \} \]
is constructible in $\text{mod}_A(z)$. If $C$ is a constructible subset of $\mathfrak{E}(z)$ which intersects each orbit of $G(z)$ in at most one point, then $\dim C \leq 1$.

**Proof.** Consider the complex of finitely generated free right $A$-modules
\[ U = \text{Hom}_I(P_{s-1}, M) \otimes_R Y, \quad V = \text{Hom}_I(P_s, M) \otimes_R Y, \quad W = \text{Hom}_I(P_{s+1}, M), \]

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Together with the morphisms $\alpha \in \text{End}_A(\text{Hom}_I(P_s, M), \alpha \in Q_1$. Let $u = \text{rank}_A U$, $v = \text{rank}_A V$ and $w = \text{rank}_A W$. The set $Z = \{(a, b) \in \mathfrak{E}(z) : a = \text{Ext}^1(T, M \otimes_A x) \text{ for some simple } A \text{-module } S \}$ is constructible and $\dim Z \leq 1$.

Let $0 \leq u' \leq u < v$ be such that $u' - v = |a| = \sum_{\alpha \in Q_1} |a| \alpha$. Set $v' = u - v$. Let $F(u', v')$ be the subset of $Z \times \mathfrak{E}(z) \times \mathfrak{E}(z) \times \mathfrak{E}(z) \times \mathfrak{E}(z) \times \mathfrak{E}(z)$ formed by the tuples $(a, b, \mu, \nu, \gamma, i, h)$ such that the diagrams:
\[ \begin{array}{ccc}
0 & \to & k' \\
0 & \to & k' \\
0 & \to & k' \\
0 & \to & k' \\
0 & \to & k'
\end{array} \]
are exact and commutative. Clearly, $F(u', v')$ is constructible. Moreover, given a tuple $(a, b, \mu, \nu, \gamma, i, h) \in F(u', v')$, there are uniquely determined maps $\delta \in \text{End}_I(k^i)$ induced by $\delta \in \text{End}_I(k^i)$ which yield a structure of left $A$-module on $k^i$; we denote this module by $X(a, b, \mu, \nu, \gamma, i, h)$. These maps yield a structure of left $A$-module on $k^i$ induced by $Y$. Then the set
\[ G(u', v') = \{(a, b, \mu, \nu, \gamma, i, h) \in F(u', v') : (a, b, \mu, \nu, \gamma, i, h) \in \mathfrak{E}(z) \} \]
is constructible. Consider the canonical projection $\pi_i : F(u', v') \to \text{mod}_A(z)$ and $\pi_i = \pi_i : G(u', v') \to \text{mod}_A(z)$ is constructible and
\[ \delta(z) = \bigcup_{u', v', i} G(u', v') \]
is constructible in $\text{mod}_A(z)$. Let $C$ be a constructible subset of $\mathfrak{E}(z)$ which intersects each orbit of $G(z)$ in at most one point. It suffices to show that $\dim C \leq 1$. Thus we may assume that $C \subset E(u', v')$ and that $C$ is irreducible. Consider the canonical projections
\[ F(u', v') \times \text{mod}_A(z) \xrightarrow{\pi_i} Z \]
is constructible in $\text{mod}_A(z)$. Let $D$ be an irreducible component of $\pi_i^{-1}(C)$ and set $Z' = \pi_i^{-1}(D)$. Then the restriction maps $p_1 : D \to C$ and $p_1 : D \to Z'$ are dominant. Let $y \in D$. We show that $p_1^{-1}(y) \subset p_1^{-1}(y) \subset p_1^{-1}(y)$. Then $y = \{(a, b, \mu, \nu, \gamma, i, h) \in \mathfrak{E}(z) : \text{otherwise} \}. \text{Then, by definition, } Y \in C \text{ and } \mathfrak{E}(z) \subset \mathfrak{E}(z). \text{Therefore, } Y = Y. \text{ That is, }\}

\[ \text{dim } C \leq \text{dim } Z \leq 1. \]
2.4. Proof of Proposition 2.1. Since $F_\gamma$ is tame, for each $d \in \mathbb{N}$, there exists a family of $\Gamma_d \cdot \mathbb{K}[\{\bar{g}\}]$-bimodules $M^{(j)}_{m,n}$ such that every indecomposable $\Gamma_d$-module $Y$ with $\dim Y = d$ is isomorphic to $M^{(j)}_{m,n} \otimes \mathbb{K} \mathbf{S}$. For some $j$ and some simple $k[\{\bar{g}\}]$-module $S$. By 2.3, for each $x \in \mathbb{N}^d$, $n \in \mathbb{N}$ and $1 \leq j \leq s(n,d)$, the set

$$\theta_x(n, d, j) = \{x \in \text{mod}_d(x); X \cong (\mathbb{K} \mathbf{S})^{\oplus \oplus} \otimes S\}$$

for some simple $k[\{\bar{g}\}]$-module $S$ is constructible and every constructible subset $C \subseteq \{x \in \mathbb{N}^d \times \mathbb{N} \mid x \text{ intersects each orbit of } G(x) \text{ in at most one point satisfies dim } C \leq 1$. By hypothesis, $\text{ind}_k(x) = \bigcup_j \theta_x(n, d, j) \cap \text{ind}_k(x)$. By 1.5, there are finite sets of indices $i \in \mathbb{N}$, $d_i \in \mathbb{N}$, $1 \leq j_i \leq s(n, d_i)$ such that $\text{ind}_k(x) = \bigcup_j \theta_x(n_i, d_i, j_i) \cap \text{ind}_k(x)$. Therefore, if $C \subseteq \text{ind}_k(x)$ is a constructible subset which intersects each orbit of $G(x)$ in at most one point, then dim $C \leq 1$. Our criterion 1.3 implies that $A$ is tame.

2.5. Let $A$ be a finite-dimensional $k$-algebra. A module $T \in \text{mod}_d$ is called a tilting module if the following conditions are satisfied (see [1, 13, 18]):

(a) $\text{Ext}_T^i(M, -) = 0$ for $i > 0$.
(b) $\text{Ext}_T^i(M, M) = 0$ for $i > 0$.
(c) There exists a non-zero sequence

$$0 \rightarrow A \rightarrow T_0 \rightarrow T_1 \rightarrow \cdots \rightarrow T_n \rightarrow 0$$

where $T_0 \in \text{add}(T)$.

Let $T \in \text{mod}_d$ be a tilting module and $B = \text{End}_d(T)$. Let $\mathcal{F}_T$ be the full subcategory of $\text{mod}_d$ formed by the modules with concentrated $T$-extension, that is,

$$\text{obj} \mathcal{F}_T = \{X \in \text{mod}_d; \text{Ext}_T^i(T, X) = 0 \text{ if } i \neq 1\}$$

Let $\mathcal{F}_d$ be the full subcategory of $\text{mod}_d$ formed by the modules with concentrated $T$-torsion, that is,

$$\text{obj} \mathcal{F}_d = \{X \in \text{mod}_d; \text{Tor}_T^i(Y, T) = 0 \text{ if } i \neq 1\}$$

**Theorem** [1, 18]. In the above situation, the functors

$$\text{Ext}_T^i(T, -): \mathcal{F}_T \rightarrow \mathcal{F}_d$$

are equivalences of categories, inverse to each other. ■

**Corollary.** In the above situation, assume that $k$ is an uncountable and every indecomposable $B$-module belongs to $\mathcal{F}_d$ for some $i \in N$. If $A$ is tame, then $B$ is tame.

The hypotheses of the Corollary are satisfied if $A$ is hereditary.

3. Functors defined by the action of groups.

3.1. Let $A = k[Q]/I$ be a finite-dimensional algebra.

**Proposition.** Assume that $k$ is an uncountable algebraically closed field. For each $n \in \mathbb{N}$, let $\Gamma_n$ be a finite-dimensional $k$-algebra and $F_n: \text{mod}_d \rightarrow \text{mod}_k$ be a right exact functor. Assume that each indecomposable $\Gamma_n$-module is a direct summand of $F_n(Y)$ for some $n \in \mathbb{N}$ and some $\Gamma_n$-module $Y$. If each $\Gamma_n$ is a tame algebra, then $A$ is tame.

**Proof.** As in the proof of 2.1 in 2.4, it is enough to show the following:

Let $M$ be a $\Gamma_d \cdot \mathbb{K}[\{\bar{g}\}]$-bimodule which is finitely generated free as right $k[\{\bar{g}\}]$-module. Then the set

$$\mathcal{F}(z) = \{Y \in \text{mod}_d(z); Y \text{ is a direct summand of } F_n(M \otimes \mathbb{K} \mathbf{S}) \text{ for some simple } k[\{\bar{g}\}]\text{-module } S\}$$

is constructible and if $C$ is a constructible subset of $\mathcal{F}(z)$ which intersects each orbit of $G(z)$ in at most one point, then dim $C \leq 1$.

Since $F_n$ is right exact, there is a $A \cdot \mathbb{K}[\{\bar{g}\}]$-module $N$ such that $F_n = N \otimes \mathbb{K}$. Then $F_n(M \otimes \mathbb{K} \mathbf{S}) \cong (N \otimes \mathbb{K} \mathbf{S}) \otimes \mathbb{K} \mathbf{S}$. By 2.3, there is a finite family of $A \cdot \mathbb{K}[\{\bar{g}\}]$-modules $L_1, \ldots, L_s$ which are finitely generated free as right $k[\{\bar{g}\}]$-modules and such that for each simple $k[\{\bar{g}\}]$-module $S$, we have $(N \otimes \mathbb{K} \mathbf{S}) \otimes \mathbb{K} \mathbf{S} \cong L_s \otimes \mathbb{K} \mathbf{S}$ for some $i$. Therefore, it is enough to show our claim for each of the sets

$$\mathcal{F}(z) = \{Y \in \text{mod}_d(z); Y \text{ is a direct summand of } F_n(L_i \otimes \mathbb{K} \mathbf{S}) \text{ for some simple } k[\{\bar{g}\}]\text{-module } S\}$$

Let $\sigma_i \in \mathbb{N}^{\mathbb{N}}$ be such that $\sigma_i(x)$ is the rank of the free $k[\{\bar{g}\}]$-module $L_i(x)$. Let $f_i: \text{mod}_d(l_i) \rightarrow \text{mod}_d(k_i)$ be the regular map induced by $L_i \otimes \mathbb{K} \mathbf{S}$. Then $Z = 1$, $l_i$ is a constructible subset of $\text{mod}_d(k_i)$ with dim $Z \leq 1$.

Let $K_1$ be the subset of

$$Z \times \left( \prod_{d \in \mathbb{N}} l^{(d)} \right) \times \prod_{d \in \mathbb{N}} l^{(d)}$$

formed by the tuples $(X, f, Y, j)$ satisfying:

$$f \in \text{End}_d(X), f^2 = f \text{ and } 0 \rightarrow Y \rightarrow X \rightarrow X \text{ is exact}.$$ 

Clearly, $K_1$ is constructible. Therefore, the subset $K$ of $Z \times \text{mod}_d(k)$ formed by the pairs $(X, Y)$ such that $Y$ is a direct summand of $X$ is constructible. Consider the canonical projections

$$K \rightarrow Z \leftarrow \mathcal{F}(z)$$

Then $\mathcal{F}(z) = 1$, $\mathcal{F}(z)$ is constructible. Let $C$ be a constructible subset of $\mathcal{F}(z)$ intersecting each orbit of $G(z)$ in at most one point. By the Krull–Schmidt Theorem the induced regular map res. $\mathcal{F}(z) \rightarrow \mathcal{F}(z)$ is finite. This implies that dim $Z = 1$.

3.2. For the terminology used in the next result see [2, 5, 12, 17].

**Proposition.** Let $A = k[Q]/I$ be an algebra over an uncountable algebraically closed field $k$ of characteristic $0$. Let $F: (\mathbb{Q}, \Gamma) \rightarrow (Q, I)$ be a Galois covering of bounded quivers given by the action of a $p$-residually finite group $\Gamma$ which acts freely on $Q$. Suppose that $\bar{A} = k[Q]/I$ is locally support-finite. Then $\bar{A}$ is tame if and only if $A$ is tame.
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Proof. Let $F_2: \text{mod}_A \to \text{mod}_B$ be the push-down functor. Consider a sequence $(\Gamma_{a\in A})$ of finite full subcategories of $\mathcal{A}$ such that $\bigcup_{a\in A} \Gamma_a = \mathcal{A}$ and if $\mathcal{A}(x, \Gamma_a) \neq 0$ or $\mathcal{A}(\mathcal{A}, \Gamma_a \neq 0$, then $x \in \Gamma_{a+1}$. The restriction functor $\mathcal{A}(\cdot,a) \to \mathcal{A}$ has a left adjoint $\mathcal{A}(\cdot,a) \to \mathcal{A}$ such that $\mathcal{A}(\cdot,a) = \text{id}_{\text{mod}_A}$ (see [5]). Therefore the functor

$$F_a = F_2 \cdot \mathcal{A}(\cdot,a) : \text{mod}_B \to \text{mod}_A$$

is right exact.

Let $Y$ be an indecomposable $A$-module. Since $\mathcal{A}$ is locally support-finite, by [6, 2.5] the pull-up $F_Y$ decomposes as $F_Y = \bigoplus_{a \in A} F_X$, where $X_a$ is an indecomposable finite-dimensional $A$-module. Thus $F_X = \bigoplus_{a \in A} F_X X_a$. Since $\mathcal{A}$ is support-finite and acts freely on $\mathcal{A}$, $Y$ is a direct summand of $F_X Y$ (see [5]). Therefore $Y$ is a direct summand of $F_Y X_a$ for some $a \in A$.

There is a number $N \in \mathbb{N}$ such that supp $X_a \subseteq \Gamma_a$. Then by [5, Lemma 2], $X_a \cong e^{\Gamma_a}_{\Gamma_a} e^{\Gamma_a}_{\Gamma_a}(X_a)$. Thus $Y$ is a direct summand of $F_X e^{\Gamma_a}_{\Gamma_a} e^{\Gamma_a}_{\Gamma_a}(X_a)$.

If $\mathcal{A}$ is tame, by [4], each $\Gamma_a$ is tame. Therefore, 3.1 implies that $A$ is tame. For the converse, assume that $\mathcal{A}$ is tame. By [4], it is enough to show that each $\Gamma_a$ is tame. Consider the right exact functor

$$H_a = \mathcal{A}(\cdot,a) : \text{mod}_B \to \text{mod}_A.$$ 

Let $Y$ be an indecomposable $\Gamma_a$-module. Thus $X = F_a \cdot \mathcal{A}(\cdot,a) \cdot Y \cong \bigoplus_{g \in S} e^{\Gamma_a}_{\Gamma_a}(Y) \cong \bigoplus_{g \in S} e^{\Gamma_a}_{\Gamma_a}(Y) = Y \oplus \bigoplus_{g \in S} e^{\Gamma_a}_{\Gamma_a}(Y)$,

where $S$ is the finite set of $g \in SS$ such that $\mathcal{A}(g)(Y) \cap \Gamma_a \neq 0$. Again, 3.1 implies that $\Gamma_a$ is tame. ■

Remark. The Proposition removes the hypothesis "$F$ acts freely on the isoclasses of indecomposable finitely generated $A$-modules" of the main theorem of [5].

3.3. For the terminology used in the next result see [20].

PROPOSITION. Let $G$ be a finite group of automorphisms of $\mathcal{A}$ such that $\text{char} K \nmid (G)$ and consider the associated skew group algebra $A[G]$. If $A[G]$ is tame, then $A[G]$ is tame.

Proof. Consider the functor $F = A[G] \otimes (-) : \text{mod}_A \to \text{mod}_A(\otimes)$ which is exact and such that every module $X \in \text{mod}_A(\otimes)$ is a direct summand of $F(\text{res} X)$, where res : $\text{mod}_A \to \text{mod}_A$ is the forgetful functor. Similar arguments to those given before show that $A[G]$ is tame whenever $A$ is so. ■

COROLLARY. Let $G$ be an abelian finite group of automorphisms of $\mathcal{A}$ such that $\text{char} K \nmid (G)$. Then $A$ is tame if and only if $A[G]$ is tame.

Proof. By [20], the group of characters $X(G)$ of $G$ acts on $A[G]$ and $\text{mod}_A(\otimes) \cong \text{mod}_A$. The proposition gives the result. ■

References


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