Functors preserving tameness

by

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Abstract. Let $\Lambda = k[Q]/I$ be a basic and connected finite-dimensional algebra over an algebraically closed field k. For each dimension vector $z \in N^{Q_0}$, we denote by $\mathrm{mod}_\Lambda(z)$ the variety of Λ-modules of dimension type z and by $\mathrm{ind}_\Lambda(z)$ the constructible subset of indecomposable modules. We prove that Λ is a tame algebra if and only if for each $z \in N^{Q_0}$, any constructible subset C of $\mathrm{ind}_\Lambda(z)$ is at most one-dimensional provided different modules in C are not isomorphic. We apply this criterion to show that tameness is preserved by Ext functors and under suitable assumptions by Galois covering functors.

Let k be an algebraically closed field. Let Λ be a finite-dimensional, basic and connected k-algebra. Following [11], we write $\Lambda = k[Q]/I$, where Q is a finite oriented graph (= quiver) and I is an admissible ideal of the path algebra $k[Q]$.

By mod_Λ we denote the category of finite-dimensional left Λ-modules. We may identify the Λ-modules with representations of Q satisfying the relations of I. If $M \in \mathrm{mod}\,\Lambda$, we set $\underline{\dim}\, M = (\dim M(x))_{x \in Q_0}$, where Q_0 is the set of vertices of Q. A vector $z \in N^{Q_0}$ is called a *dimension vector*.

An algebra Λ is said to be *tame* if for every dimension $z \in N^{Q_0}$, there is a finite family of Λ-$k[t]$-bimodules M_i which are free as right $k[t]$-modules such that every indecomposable Λ-module of dimension z is isomorphic to $M_i \otimes_{k[t]} S$ for some i and some simple $k[t]$-module S (see [3, 4, 7, 8, ...]).

Let $z \in N^{Q_0}$ be a dimension vector. By $\mathrm{mod}_\Lambda(z)$ we denote the *variety of Λ-modules* of dimension type z (see [4, 9, 10, 15, 16, 19]). The *indecomposable modules* in $\mathrm{mod}_\Lambda(z)$ form a constructible subset denoted by $\mathrm{ind}_\Lambda(z)$. In this note we show that an algebra Λ is tame if and only if for each dimension $z \in N^{Q_0}$, any constructible subset C of $\mathrm{ind}_\Lambda(z)$ is at most one-dimensional provided different modules in C are not isomorphic.

This characterization is well-suited for applications. Applying it, we show that tameness is preserved by Ext functors (Section 2) and under a suitable assumption by the Galois covering functors (Section 3). It is possible to give other (similar) applications.

1. A criterion for tameness.

1.1. Let $z = (z(x))_{x \in Q_0} \in N^{Q_0}$ be a dimension vector. The *variety of Λ-modules* of dimension z is the closed subset $\mathrm{mod}_\Lambda(z)$ of the affine space $\prod_{(x \xrightarrow{\alpha} y) \in Q_1} k^{z(x)\, z(y)}$ (where

Q_1 = set of arrows of Q) formed by the tuples $X = (X(\alpha))_{\alpha \in Q_1}$ such that for any relation $\varrho = \sum_{i=1}^{r} \lambda_i \alpha_{j_i}^{(i)} \ldots \alpha_1^{(i)} \in I(x, y)$ the $z(x) \times z(y)$-matrix $X(\varrho) = \sum_{i=1}^{r} \lambda_i X(\alpha_{j_i}^{(i)}) \ldots X(\alpha_1^{(i)})$ is zero.

The affine *algebraic group* $G(z) = \prod_{x \in Q_0} Gl_{z(x)}(k)$ acts on the variety $\mathrm{mod}_A(z)$ in such a way that two points (= modules) belong to the same orbit if and only if they are isomorphic.

Let $\mathrm{ind}_A(z)$ denote the subset of $\mathrm{mod}_A(z)$ formed by the indecomposable modules. Then $\mathrm{ind}_A(z)$ is a constructible subset of $\mathrm{mod}_A(z)$ [4]. We recall the argument: let W be the constructible subset of $\mathrm{mod}_A(z) \times (\prod_{x \in Q_0} k^{z(x)z(x)})$ formed by those pairs (X, f) such that $f \in \mathrm{End}_A(X)$, $f^2 = f$ and $0_X \neq f \neq 1_X$. Consider the canonical projection $p: \mathrm{mod}_A(z) \times (\prod_{x \in Q_0} k^{z(x)z(x)}) \to \mathrm{mod}_A(z)$. Then $p(W)$ is constructible. Since $\mathrm{ind}_A(z) = \mathrm{mod}_A(z) \backslash p(W)$, this set is also constructible. For the elementary notions of algebraic geometry that we use the reader may see [14, Chapters I, II] and [22, Chapter I].

1.2. Let $A = k\langle t_1, \ldots, t_m \rangle$ be the free associative algebra generated by $t_1, \ldots, t_m$. Let M be a Λ-A-bimodule which is finitely generated free as A-module. Let $z \in N^{Q_0}$ be the dimension vector such that $z(x)$ is the rank of the free right A-module $M(x)$. The functor $M \otimes_A (\text{-}): \mathrm{mod}_A \to \mathrm{mod}_\Lambda$ induces a regular map $f_M: \mathrm{mod}_A(1) \to \mathrm{mod}_\Lambda(z)$ (see [4]). The variety $\mathrm{mod}_A(1)$ may be identified with k^m. Therefore, $\mathrm{Im} f_M$ is a constructible subset of $\mathrm{mod}_\Lambda(z)$ and $\dim \mathrm{Im} f_M \leqslant m$.

1.3. Theorem. *Let $\Lambda = k[Q]/I$ be a finite-dimensional k-algebra. The following are equivalent*:

(a) *Λ is tame.*

(b) *For each dimension $z \in N^{Q_0}$, there is a constructible subset C of $\mathrm{ind}_\Lambda(z)$ with $\dim C \leqslant 1$ and such that $G(z)C = \mathrm{ind}_\Lambda(z)$.*

(c) *For each dimension $z \in N^{Q_0}$, if C is a constructible subset of $\mathrm{ind}_\Lambda(z)$ which intersects each orbit of $G(z)$ in at most one point, then $\dim C \leqslant 1$.*

Proof. (a)$\Rightarrow$(b): Let $z \in N^{Q_0}$. Let $M_1, \ldots, M_s$ be the Λ-$k[t]$-bimodules such that M_i is a free finitely generated $k[t]$-module and any $X \in \mathrm{ind}_\Lambda(z)$ is isomorphic to $M_i \otimes_{k[t]} S$ for some i and some simple $k[t]$-module S. Therefore, the functor $M_i \otimes_{k[t]} (\text{-})$ induces a regular map $f_i: \mathrm{mod}_{k[t]}(1) \to \mathrm{mod}_\Lambda(z)$, $i = 1, \ldots, s$. The set

$$C = \bigcup_{i=1}^{s} (\mathrm{Im} f_i \cap \mathrm{ind}_\Lambda(z))$$

is a constructible subset of $\mathrm{ind}_\Lambda(z)$ with $\dim C \leqslant 1$ and $G(z)C = \mathrm{ind}_\Lambda(z)$.

(b)$\Rightarrow$(c): This follows from Lemma 1.4.

(c)$\Rightarrow$(a): Assume that Λ is not tame. By [7, 8] (see also [3]), the algebra Λ is *wild*. That is, there exists a Λ-$k\langle u, v\rangle$-bimodule M which is free finitely generated as right $k\langle u, v\rangle$-module and such that the functor $M \otimes_{k\langle x, y\rangle} (\text{-}): \mathrm{mod}_{k\langle u, v\rangle} \to \mathrm{mod}_\Lambda$ preserves indecomposability and reflects isomorphisms.

Let $z \in N^{Q_0}$, where $z(x)$ is the rank of the free $k\langle u, v\rangle$-module $M(x)$. We get a regular map $f_M: \mathrm{mod}_{k\langle u, v\rangle}(1) \to \mathrm{mod}_\Lambda(z)$. By definition, $\mathrm{Im} f_M$ is a constructible subset of $\mathrm{ind}_\Lambda(z)$ which intersects each orbit of $G(z)$ in at most one point. Moreover, f_M is injective. Therefore, $\dim \mathrm{Im} f_M = 2$. ■

1.4. Lemma (compare with [7, Lemma]). *Let V be an algebraic variety and G be an algebraic group acting on V. Let U_1, U_2 be two constructible subsets of V satisfying*:

(i) $GU_1 = V$,

(ii) *U_2 intersects each orbit of G in at most one point.*

Then $\dim U_2 \leqslant \dim U_1$.

Proof. We may assume that U_2 is irreducible. Consider the following algebraic maps:

$$\begin{array}{ccc} (g, u) & G \times U_1 \xrightarrow{\varphi} V, & (g, u) \mapsto gu \\ \downarrow & \pi \downarrow & \\ u & U_1 & \end{array}$$

Take an irreducible component Z of $\varphi^{-1}(U_2) \subset G \times U_1$ and $W_1 = \overline{\pi(Z)}$. Then the maps $f = \mathrm{res}\,\varphi: Z \to U_2$ and $p = \mathrm{res}\,\pi: Z \to W_1$ are dominant. By [22, Chap. I (6)], there is an open dense subset Y of Z such that for any $y \in Y$

$$\dim f^{-1}(f(y)) = \dim Z - \dim U_2 \quad \text{and} \quad \dim p^{-1}(p(y)) = \dim Z - \dim W_1.$$

Let $y \in Y$. We show that $p^{-1}(p(y)) \subset f^{-1}(f(y))$. In fact, assume that $y = (g, u) \in Z$ and let $y' = (h, u) \in p^{-1}(p(y))$. Therefore, $gu = f(y)$ and $hu = f(y')$ both lie in U_2. Since $hg^{-1}(f(y)) = f(y')$, we have $f(y) = f(y')$, that is, $y' \in f^{-1}(f(y))$. We get that $\dim p^{-1}(p(y)) \leqslant \dim f^{-1}(f(y))$ and $\dim U_2 = \dim Z - \dim f^{-1}(f(y)) \leqslant \dim W_1 \leqslant \dim U_1$. ■

1.5. The following result on algebraic varieties will be used in the following sections. It is a particular case of [10, (4.2)].

Lemma. *Let V be an algebraic variety over an uncountable algebraically closed field k. Let $(C_n)_{n \in N}$ be a family of constructible subsets of V such that $\bigcup_{n \in N} C_n = V$. Then there is a number $N \in N$ such that $V = \bigcup_{n \leqslant N} C_n$.* ■

2. Ext-functors preserve tameness.

2.1. The purpose of this section is to prove the following:

Proposition. *Let Λ be a finite-dimensional k-algebra over an uncountable algebraically closed field k. For each $n \in N$, let Γ_n be a finite-dimensional k-algebra, let T_n be a finite-dimensional Γ_n-Λ-bimodule and $s_n \in N$. Assume that every indecomposable $X \in \mathrm{mod}_\Lambda$ is isomorphic to $\mathrm{Ext}_{\Lambda_n}^{s_n}(T_n, Y)$ for some $n \in N$ and $Y \in \mathrm{mod}_{\Gamma_n}$. If each Γ_n is a tame algebra, then Λ is tame.*

Not all functors have such a nice behaviour as the following *example* shows.

Let Λ be a wild algebra and let A be the hereditary k-algebra with quiver $\bullet \rightrightarrows \bullet$. Therefore A is tame. We will construct a functor $F: \mathrm{mod}_A \to \mathrm{mod}_\Lambda$ such that every indecomposable $X \in \mathrm{mod}_\Lambda$ is isomorphic to FY for some A-module Y. Let $(X_\lambda)_{\lambda \in k}$ be a set of representatives of the isoclasses of finite-dimensional indecomposable Λ-modules. Let $(S_\lambda)_{\lambda \in k}$ be a set of representatives of the isoclasses of simple regular A-modules. Set $S_\lambda^* = \mathrm{Hom}_k(S_\lambda, k)$. Consider the Λ-A-bimodule

$$M = \bigoplus_{\lambda \in k} (X_\lambda \otimes_k S_\lambda^*).$$

Let $Y \in \mathrm{mod}_A$. Then $Y = Y_p \oplus Y_r \oplus Y_i$, where Y_p (resp. Y_r, Y_i) is a direct sum of preprojective (resp. regular, preinjective) A-modules. We define $FY = M \otimes_A Y_r$ and similarly in morphisms. Since

$$FS_\mu \simeq \bigoplus_{\lambda \in k} [X_\lambda \otimes_k \mathrm{Hom}_A(S_\lambda, S_\mu)] \simeq X_\mu,$$

we see that $F\colon \mathrm{mod}_A \to \mathrm{mod}_A$ is well defined and satisfies the desired property.

If the field k is countable, it is easy to give examples where the proposition fails.

2.2. Let A and Γ be two finite-dimensional k-algebras. Let $A = k[t]$. Let T be a finite-dimensional Γ-A-bimodule and let M be a Γ-A-bimodule which is a finitely generated free right A-module. Fix $s \in N$. Consider the functor

$$\mathrm{Ext}^s_\Gamma(T, M \otimes_A (\text{-}))\colon \mathrm{mod}_A \to \mathrm{mod}_A.$$

Let $\ldots \to P_n \xrightarrow{\partial_n} P_{n-1} \to \ldots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} T \to 0$ be a finite-dimensional projective resolution of Γ-A-bimodules. Set $\partial_n^* = \mathrm{Hom}_\Gamma(\partial_n, M)$ (and $\partial_0^* = 0$). For each $\alpha \in Q_1$, we have induced morphisms $\bar{\alpha} \in \mathrm{End}_\Gamma(T)$ and $\alpha_s \in \mathrm{End}_\Gamma(P_s)$ such that $\alpha_s \partial_s = \partial_s \alpha_{s+1}$ and $\partial_0 \alpha_0 = \bar{\alpha}\partial_0$. Then the induced linear maps $\bar{\alpha}_s \in \mathrm{End}_k(\ker(\partial^*_{s+1} \otimes_A X)/\mathrm{Im}(\partial^*_s \otimes_A X))$ yield a left A-module structure on $\ker(\partial^*_{s+1} \otimes_A X)/\mathrm{Im}(\partial^*_s \otimes_A X)$.

The following is an elementary fact but we will need later some details of the proof.

LEMMA. *$\mathrm{Ext}^s_\Gamma(T, M \otimes_A X) \simeq \ker(\partial^*_{s+1} \otimes_A X)/\mathrm{Im}(\partial^*_s \otimes_A X)$ as left A-modules.*

Proof. Consider the natural transformation

$$\varphi_T\colon \mathrm{Hom}_\Gamma(T, M) \otimes_A X \to \mathrm{Hom}_\Gamma(T, M \otimes_A X),$$

$$f \otimes x \mapsto \varphi_T(f \otimes x)\colon T \to M \otimes_A X, \quad t \mapsto f(t) \otimes x.$$

In fact φ is natural in the three variables. Clearly, φ_T is an isomorphism if T is projective.

If $s \geqslant 1$, we get the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Hom}_\Gamma(P_{s-1}, M) \otimes_A X & \xrightarrow{\partial^*_s \otimes X} & \mathrm{Hom}_\Gamma(P_s, M) \otimes_A X & \xrightarrow{\partial^*_{s+1} \otimes X} & \mathrm{Hom}_\Gamma(P_{s+1}, M) \otimes_A X \\
\sim\downarrow \varphi_{P_{s-1}} & & \sim\downarrow \varphi_{P_s} & & \sim\downarrow \varphi_{P_{s+1}} \\
\mathrm{Hom}_\Gamma(P_{s-1}, M \otimes_A X) & \longrightarrow & \mathrm{Hom}_\Gamma(P_s, M \otimes_A X) & \longrightarrow & \mathrm{Hom}_\Gamma(P_{s+1}, M \otimes_A X).
\end{array}$$

The calculation of the homology in the middle terms gives the result. The case $s = 0$ is similar. ■

2.3. We keep the notation of 2.2.

LEMMA. *Let $z \in N^{Q_0}$. The set*

$$\mathscr{E}(z) = \{X \in \mathrm{mod}_A(z)\colon X \simeq \mathrm{Ext}^s_\Gamma(T, M \otimes_A S) \text{ for some simple } A\text{-module } S\}$$

is constructible in $\mathrm{mod}_A(z)$. If C is a constructible subset of $\mathscr{E}(z)$ which intersects each orbit of $G(z)$ in at most one point, then $\dim C \leqslant 1$.

Proof. Consider the complex of finitely generated free right A-modules

$$U = \mathrm{Hom}_\Gamma(P_{s-1}, M) \xrightarrow{f = \partial^*_s} V = \mathrm{Hom}_\Gamma(P_s, M) \xrightarrow{g = \partial^*_{s+1}} W = \mathrm{Hom}_\Gamma(P_{s+1}, M),$$

together with the morphisms $\alpha_i^* \in \mathrm{End}_A(\mathrm{Hom}_\Gamma(P_i, M))$, $\alpha \in Q_1$. Let $u = \mathrm{rank}_A U$, $v = \mathrm{rank}_A V$ and $w = \mathrm{rank}_A W$.

The set $Z = \{(a, b) \in k^{uv} \times k^{uw}\colon a = f \otimes_A S,\ b = g \otimes_A S$ for some simple A-module $S\}$ is constructible and $\dim Z \leqslant 1$.

Let $0 \leqslant v' \leqslant u' \leqslant v$ be such that $u' - v' = |z| = \sum_{x \in Q_0} z(x)$. Set $v'' = v - v'$. Let $F(u', v')$ be the subset of $Z \times k^{u'v'} \times k^{vv''} \times k^{v'v} \times k^{v'u'} \times k^{u'|z|}$ formed by the tuples $((a, b), \mu, \nu, \gamma, i, h)$ such that the diagrams:

$$\begin{array}{ccccccccc}
0 & \to & k^{u'} & \xrightarrow{\mu} & k^v & \xrightarrow{b} & k^w & & \\
0 & \to & k^{v'} & \xrightarrow{\gamma} & k^v & \xrightarrow{\nu} & k^{v''} & \to & 0 \\
 & & & {\scriptstyle\iota}\searrow & k^{u'} & \nearrow{\scriptstyle\mu} & & &
\end{array}
\qquad
\begin{array}{ccccccccc}
k^u & \xrightarrow{a} & k^v & \xrightarrow{\nu} & k^{v''} & \to & 0 & & \\
0 & \to & k^{v'} & \xrightarrow{i} & k^{u'} & \xrightarrow{h} & k^{|z|} & \to & 0
\end{array}$$

are exact and commutative. Clearly, $F(u', v')$ is constructible. Moreover, given a tuple $((a, b), \mu, \nu, \gamma, i, h) \in F(u', v')$, there are uniquely determined maps $\bar{\alpha} \in \mathrm{End}_k(k^{|z|})$ induced by α_s^* $(\alpha \in Q_1)$. These maps yield a structure of left A-module on $k^{|z|}$; we denote this module by $X((a, b), \mu, \nu, \gamma, i, h)$.

Given $Y \in \mathrm{mod}_A(z)$, we denote by ${}^{\oplus}Y$ the left A-module on $k^{|z|}$ induced by Y. Then the set

$$G(u', v') = \{((a, b), \mu, \nu, \gamma, i, h, Y) \in F(u', v') \times \mathrm{mod}_A(z)\colon X((a, b), \mu, \nu, \gamma, i, h) \simeq {}^{\oplus}Y\}$$

is constructible. Consider the canonical projection $\pi_7\colon F(u', v') \times \mathrm{mod}_A(z) \to \mathrm{mod}_A(z)$. Then $E(u', v') = \pi_7(G(u', v'))$ is constructible and

$$\mathscr{E}(z) = \bigcup_{\substack{0 \leqslant v' \leqslant u' \leqslant v \\ u' - v' = |z|}} E(u', v')$$

is constructible in $\mathrm{mod}_A(z)$. Let C be a constructible subset of $\mathscr{E}(z)$ which intersects each orbit of $G(z)$ in at most one point. It suffices to show that $\dim(C \cap E(u', v')) \leqslant 1$. Thus we may assume that $C \subset E(u', v')$ and that C is irreducible. Consider the canonical projections

$$\begin{array}{ccc}
F(u', v') \times \mathrm{mod}_A(z) & \xrightarrow{\pi_1} & Z \\
{\scriptstyle\pi_7}\downarrow & & \\
\mathrm{mod}_A(z) & &
\end{array}$$

Take an irreducible component D of $\pi_7^{-1}(C)$ and set $Z' = \overline{\pi_1(D)}$. Then the restriction maps $p_7\colon D \to C$ and $p_1\colon D \to Z'$ are dominant. Let $y \in D$. We show that $p_1^{-1}(p_1(y)) \subset p_7^{-1}(p_7(y))$. Let $y = ((\alpha, \beta), \mu, \nu, \gamma, i, h, Y)$ and $((\alpha, \beta), \mu', \nu', \gamma', i', h', Y') \in p_1^{-1}(p_1(y))$. Then, by definition, $Y, Y' \in C$ and ${}^{\oplus}Y \simeq {}^{\oplus}Y'$. Therefore, $Y = Y'$. That is, $y' \in p_7^{-1}(p_7(y))$. As in the proof of Lemma 1.4 this implies that

$$\dim C \leqslant \dim Z \leqslant 1. \ ■$$

2.4. Proof of Proposition 2.1. Since Γ_n is tame, for each $d \in N$, there exists a family of Γ_n-$k[t]$-bimodules $M_1^{(n,d)}, \ldots, M_{s(n,d)}^{(n,d)}$ such that every indecomposable Γ_n-module Y with $\dim_k Y = d$ is isomorphic to $M_j^{(n,d)} \otimes_{k[t]} S$ for some j and some simple $k[t]$-module S. By 2.3, for each $z \in N^{Q_0}$, $n, d \in N$ and $1 \leqslant j \leqslant s(n, d)$, the set

$$\mathscr{E}(z, n, d, j) = \{X \in \mathrm{mod}_A(z)\colon X \simeq \mathrm{Ext}_{\Gamma_n}^{s_n}(T_n, M_j^{(n,d)} \otimes_{\Gamma_n} S)$$
$$\text{for some simple } k[t]\text{-module } S\}$$

is constructible and every constructible subset $C \subset \mathscr{E}(z, n, d, j)$ which intersects each orbit of $G(z)$ in at most one point satisfies $\dim C \leqslant 1$. By hypothesis, $\mathrm{ind}_A(z) = \bigcup_{n,d,j} (\mathscr{E}(z, n, d, j) \cap \mathrm{ind}_A(z))$. By 1.5, there are finite sets of indices $n_i \in N$, $d_i \in N$, $1 \leqslant j_i \leqslant s(n_i, d_i)$ such that $\mathrm{ind}_A(z) = \bigcup_i (\mathscr{E}(z, n_i, d_i, j_i) \cap \mathrm{ind}_A(z))$. Therefore, if $C \subset \mathrm{ind}_A(z)$ is a constructible subset which intersects each orbit of $G(z)$ in at most one point, then $\dim C \leqslant 1$. Our criterion 1.3 implies that A is tame. ■

2.5. Let A be a finite-dimensional k-algebra. A module $T \in \mathrm{mod}_A$ is called a *tilting module* if the following conditions are satisfied (see [1, 13, 18]):

(a) $\mathrm{Ext}_A^i(M, \text{-}) = 0$ for $i \gg 0$.

(b) $\mathrm{Ext}_A^i(M, M) = 0$ for $i > 0$.

(c) There exists an exact sequence

$$0 \to {}_A A \to T_0 \to T_1 \to \ldots \to T_r \to 0$$

where $T_i \in \mathrm{add}(T)$.

Let $T \in \mathrm{mod}\, A$ be a tilting module and $B = \mathrm{End}_A(T)$. Let $\mathscr{X}_i$ be the full subcategory of mod_A formed by the modules with concentrated ith T-extension, that is,

$$\mathrm{obj}\, \mathscr{X}_i = \{X \in \mathrm{mod}_A\colon \mathrm{Ext}_A^j(T, X) = 0 \text{ if } j \neq i\}.$$

Let $\mathscr{Y}_i$ be the full subcategory of mod_B formed by the modules with concentrated ith T-torsion, that is,

$$\mathrm{obj}\, \mathscr{Y}_i = \{Y \in \mathrm{mod}_B\colon \mathrm{Tor}_j^B(Y, T) = 0 \text{ if } j \neq i\}.$$

Theorem [1, 18]. *In the above situation, the functors*

$$\mathrm{Ext}_A^i(T, \text{-})\colon \mathscr{X}_i \to \mathscr{Y}_i \quad \textit{and} \quad \mathrm{Tor}_i^B(\text{-}, T)\colon \mathscr{Y}_i \to \mathscr{X}_i$$

are equivalences of categories, inverse to each other. ■

Corollary. *In the above situation, assume that k is uncountable and every indecomposable B-module belongs to $\mathscr{Y}_i$ for some $i \in N$. If A is tame, then B is tame.* ■

The hypotheses of the Corollary are satisfied if A is hereditary.

3. Functors defined by the action of groups.

3.1. Let $A = k[Q]/I$ be a finite-dimensional algebra.

Proposition. *Assume that k is an uncountable algebraically closed field. For each $n \in N$, let Γ_n be a finite-dimensional k-algebra and $F_n\colon \mathrm{mod}_{\Gamma_n} \to \mathrm{mod}_A$ be a right exact functor. Assume that each indecomposable A-module is a direct summand of $F_n(Y)$ for some $n \in N$ and some Γ_n-module Y. If each Γ_n is a tame algebra, then A is tame.*

Proof. As in the proof of 2.1 in 2.4, it is enough to show the following:

Let M be a Γ_n-$k[t]$-bimodule which is finitely generated free as right $k[t]$-module. Then the set

$$\mathscr{F}(z) = \{Y \in \mathrm{mod}_A(z)\colon Y \text{ is a direct summand of } F_n(M \otimes_{k[t]} S)$$
$$\text{for some simple } k[t]\text{-module } S\}$$

is constructible and if C is a constructible subset of $\mathscr{F}(z)$ which intersects each orbit of $G(z)$ in at most one point, then $\dim C \leqslant 1$.

Since F_n is right exact, there is a A-Γ_n-bimodule N such that $F_n = N \otimes_{\Gamma_n} (\text{-})$. Then $F_n(M \otimes_{k[t]} S) \xrightarrow{\sim} (N \otimes_{\Gamma_n} M) \otimes_{k[t]} S$. By [4], there is a finite family of A-$k[t]$-bimodules $L_1, \ldots, L_s$ which are finitely generated free as right $k[t]$-modules and such that for each simple $k[t]$-module S, we have $(N \otimes_{\Gamma_n} M) \otimes_{k[t]} S \xrightarrow{\sim} L_i \otimes_{k[t]} S$ for some i. Therefore, it is enough to show our claim for each of the sets

$$\mathscr{F}^i(z) = \{Y \in \mathrm{mod}_A(z)\colon Y \text{ is a direct summand of } L_i \otimes_{k[t]} S$$
$$\text{for some simple } k[t]\text{-module } S\}.$$

Let $e_i \in N^{Q_0}$ be such that $e_i(x)$ is the rank of the free $k[t]$-module $L_i(x)$. Let $f_i\colon \mathrm{mod}_{k[t]}(1) \to \mathrm{mod}_A(e_i)$ be the regular map induced by $L_i \otimes_{k[t]} (\text{-})$. Then $Z = \mathrm{Im} f_i$ is a constructible subset of $\mathrm{mod}_A(e_i)$ with $\dim Z \leqslant 1$.

Let K_1 be the subset of

$$Z \times \big(\prod_{x \in Q_0} k^{e_i(x)}\big) \times \mathrm{mod}_A(z) \times \big(\prod_{x \in Q_0} \mathrm{Hom}_k(k^{z(x)}, k^{e_i(x)})\big)$$

formed by the tuples (X, f, Y, j) satisfying:

$$f \in \mathrm{End}_A(X), \quad f^2 = f \quad \text{and} \quad 0 \to Y \xrightarrow{j} X \xrightarrow{f} X \text{ is exact.}$$

Clearly, K_1 is constructible. Therefore, the subset K of $Z \times \mathrm{mod}_A(z)$ formed by the pairs (X, Y) such that Y is a direct summand of X, is constructible. Consider the canonical projections

$$\begin{array}{ccc} K & \xrightarrow{\pi_1} & Z \\ {\scriptstyle \pi_2}\downarrow & & \\ \mathscr{F}^i(z) & & \end{array}$$

Then $\mathscr{F}^i(z) = \mathrm{Im}\, \pi_2$ is constructible. Let C be a constructible subset of $\mathscr{F}^i(z)$ intersecting each orbit of $G(z)$ in at most one point. By the Krull–Schmidt Theorem the induced regular map $\mathrm{res}\, \pi_1\colon \pi_2^{-1}(C) \to Z$ is finite. This implies that $\dim C \leqslant \dim \pi_2^{-1}(C) \leqslant \dim Z = 1$. ■

3.2. For the terminology used in the next result see [2, 5, 12, 17].

Proposition. *Let $A = k[Q]/I$ be an algebra over an uncountable algebraically closed field k of characteristic $p \geqslant 0$. Let $F\colon (\tilde{Q}, \tilde{I}) \to (Q, I)$ be a Galois covering of bounded quivers given by the action of a p-residually finite group Π which acts freely on $\tilde{Q}$. Suppose that $\tilde{A} = k[\tilde{Q}]/\tilde{I}$ is locally support-finite. Then $\tilde{A}$ is tame if and only if A is tame.*

Proof. Let $F_\lambda: \mathrm{mod}_{\tilde{\Lambda}} \to \mathrm{mod}_\Lambda$ be the push-down functor. Consider a sequence $(\Gamma_n)_{n\in N}$ of finite full subcategories of $\tilde{\Lambda}$ such that $\bigcup_n \Gamma_n = \tilde{\Lambda}$ and if $\tilde{\Lambda}(x, \Gamma_n) \neq 0$ or $\tilde{\Lambda}(\Gamma_n, x) \neq 0$, then $x \in \Gamma_{n+1}$. The restriction functor $\varepsilon^n_\cdot: \mathrm{mod}_{\tilde{\Lambda}} \to \mathrm{mod}_{\Gamma_n}$ has a left adjoint $\varepsilon^n_\lambda: \mathrm{mod}\,\Gamma_n \to \mathrm{mod}\,\tilde{\Lambda}$ such that $\varepsilon^n_\cdot \varepsilon^n_\lambda = \mathrm{id}_{\mathrm{mod}_{\Gamma_n}}$ (see [6]). Therefore the functor

$$F_n = F_\lambda \varepsilon^n_\lambda: \mathrm{mod}_{\Gamma_n} \to \mathrm{mod}_\Lambda$$

is right exact.

Let Y be an indecomposable Λ-module. Since $\tilde{\Lambda}$ is locally support-finite, by [6, 2.5] the pull-up $F_\cdot Y$ decomposes as $F_\cdot Y \simeq \bigoplus_{i\in I} X_i$, where X_i an indecomposable finite-dimensional $\tilde{\Lambda}$-module. Thus $F_\lambda F_\cdot Y \simeq \bigoplus_{i\in I} F_\lambda X_i$. Since Π is p-residually finite and acts freely on $\tilde{\Lambda}$, Y is a direct summand of $F_\lambda F_\cdot Y$ (see [5]). Therefore Y is a direct summand of $F_\lambda X_i$ for some $i \in I$.

There is a number $n \in N$ such that $\mathrm{supp}\, X_i \subset \Gamma_n$. Then by [5, Lemma 2], $X_i \simeq \varepsilon^{n+1}_\lambda \varepsilon^{n+1}_\cdot (X_i)$. Thus Y is a direct summand of $F_{n+1}(\varepsilon^{n+1}_\cdot (X_i))$.

If $\tilde{\Lambda}$ is tame, by [4], each Γ_n is tame. Therefore, 3.1 implies that Λ is tame. For the converse, assume that Λ is tame. By [4], it is enough to show that each Γ_n is tame. Consider the right exact functor

$$H_n = \varepsilon^n_\cdot F_\cdot: \mathrm{mod}_\Lambda \to \mathrm{mod}_{\Gamma_n}.$$

Let Y be an indecomposable Γ_n-module. Thus $X = F_\lambda \varepsilon^n_\lambda Y \in \mathrm{mod}_\Lambda$. We get

$$H_n X = \varepsilon^n_\cdot \big(\bigoplus_{g\in\Pi} {}^g(\varepsilon^n_\lambda Y)\big) \cong \bigoplus_{g\in S} \varepsilon^n_\cdot {}^g(\varepsilon^n_\lambda Y) = Y \oplus \big(\bigoplus_{g\in S-\{1\}} \varepsilon^n_\cdot {}^g(\varepsilon^n_\lambda Y)\big),$$

where S is the finite set of $g \in \Pi$ such that $\mathrm{supp}\, {}^g(\varepsilon^n_\lambda Y) \cap \Gamma_n \neq \emptyset$. Again, 3.1 implies that Γ_n is tame. ∎

Remark. The Proposition removes the hypothesis "Π acts freely on the isoclasses of indecomposable finitely generated $\tilde{\Lambda}$-modules" of the main theorem of [5].

3.3. For the terminology used in the next result see [20].

Proposition. *Let G be a finite group of automorphisms of Λ such that $\mathrm{char}\, k \nmid o(G)$ and consider the associated skew group algebra $\Lambda[G]$. If Λ is tame, then $\Lambda[G]$ is tame.*

Proof. Consider the functor $F = \Lambda[G] \otimes_\Lambda (\text{-}): \mathrm{mod}_\Lambda \to \mathrm{mod}_{\Lambda[G]}$ which is exact and such that every module $X \in \mathrm{mod}_{\Lambda[G]}$ is a direct summand of $F(\mathrm{res}\, X)$, where $\mathrm{res}: \mathrm{mod}_{\Lambda[G]} \to \mathrm{mod}_\Lambda$ is the forgetful functor. Similar arguments to those given before show that $\Lambda[G]$ is tame whenever Λ is so. ∎

Corollary. *Let G be an abelian finite group of automorphisms of Λ such that $\mathrm{char}\, k \nmid o(G)$. Then Λ is tame if and only if $\Lambda[G]$ is tame.*

Proof. By [20], the group of characters $X(G)$ of G acts on $\Lambda[G]$ and $\mathrm{mod}_{\Lambda[G][X(G)]} \simeq \mathrm{mod}_\Lambda$. The proposition gives the result. ∎

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