# On neighborhoods of the Kuratowski imbedding beyond the first extremum of the diameter functional

by

**Mikhail Katz** (Bloomington, Ind.)

**Abstract.** We determine the homotopy type of $\varepsilon$-neighborhoods of the Kuratowski imbedding of the circle for $\lambda_1 < 2\varepsilon < \lambda_2$, where $\lambda_i$ is the $i$th critical value of the diameter functional on the power set of the circle. We do the same for the complex projective space.

**Introduction.** A compact connected metric space $V$ can be isometrically imbedded in the space $L^\infty = L^\infty(V)$ of bounded functions on $V$ with the sup norm $\|\,\|$. The first explicit definition of this imbedding was given by Kuratowski in volume 25 of the present journal ([15], p. 543), and it is referred to as Kuratowski's imbedding in [10], p. 27. We study the $\varepsilon$-neighborhoods $U_\varepsilon V \subset L^\infty$ in the case of the rank one symmetric spaces, starting with the circle.

For small $\varepsilon > 0$, these neighborhoods have the homotopy type of $V$, provided $V$ is locally contractible. As $\varepsilon$ increases, the homotopy type changes for the first time. It is the new homotopy type (after the first change) that arises in the calculation of a Riemannian invariant called the *Filling Radius* of $V$, if $V$ has a fundamental homology class (cf. [6], [11], [14]). If $V$ is the circle $S^1$ of length 1, the homotopy type of $U_\varepsilon S^1$ first changes when $\varepsilon = \frac{1}{6}$. (Note that for neighborhoods of the equatorial imbedding $S^1 \subset S^2$ in the sphere of curvature $4\pi^2$, the change occurs when $\varepsilon = \frac{1}{4}$.) The new homotopy type is that of $S^3$. To prove this, we decompose $U_\varepsilon S^1$ into the union of two pieces, analogous to the decomposition of $S^3$ along a standard imbedded torus. We then construct a retraction on polyhedral approximations to each of the pieces and their intersection, and appeal to standard theorems from algebraic topology.

This retraction is akin to the flow on a manifold in the absence of critical points of the (typically non-smooth) distance function (cf. [8], [5], [7]). The role of the critical points is played by the extrema of the diameter functional $\delta$ on $2^V$ (cf. § 4). To mediate between $2^V$ and $L^\infty(V)$, we use a predistance construction following an idea of [14], Appendix A (cf. § 3). Denote by $A_1$ the orbit of the first extremum of $\delta$ under the isometry group of $V$. If $V$ is rank one symmetric, one expects the new homotopy type to be that of the "partial join" of $V$ and $A_1$ (cf. § 8). We verify this for the complex projective space $CP^n$.

If $Y \in 2^V$ and $\delta(Y) = d$, the $d$-neighborhood of $Y \subset V$ may be called the *basin* of $Y$. It would be interesting to find suitable conditions on a homogeneous $V$ such that the new homotopy type is that of the subset of the join of $V$ and $A_1$ which consists of intervals joining pairs $x \in V$, $Y \in A_1$ such that $x \in \mathrm{basin}(Y)$.

Theorem 1.1 of Section 1 is the main result for the circle, and its proof occupies Sections 3, 5, and 6. In Section 2, we introduce the unit speed deformation used to exhibit a space of the desired homotopy type inside $U_r$. In Section 4 we define the extrema of the diameter functional, classify them for the circle, and prove a Morse-type lemma (4.8) used in the case of complex projective space, as well. In Section 7 we find the first two extrema of the diameter functional on the power set of the 2-sphere $S^2$, and construct a suitable deformation of sets which does not increase the diameter. In Section 8 we treat the case of $CP^n$ by making use of complex projective trigonometry. We note the related recent articles [1], [2], [9], [16].

## § 1. The homological argument.

Let $S^1$ be the Riemannian circle of length 1, and let $\mathrm{dist}_{S^1}$ be the Riemannian distance in $S^1$. Consider the Kuratowski imbedding $S^1 \subset L^\infty$, $x \mapsto d_x$, where $d_x(y) = \mathrm{dist}_{S^1}(x, y)$ for all $y \in S^1$. This imbedding is isometric, i.e. $\mathrm{dist}_{S^1}(x, y) = \|d_x - d_y\|$ for all $x, y \in S^1$. Note that as a metric space, $S^1$ cannot be isometrically imbedded in Euclidean space. Let $U_r = U_r S^1 \subset L^\infty$ be the $r$-neighborhood of $S^1$. Let $\lambda_k = k/(2k+1)$. This $\lambda_k$ is the Riemannian diameter of the set of vertices of a regular $(2k+1)$-gon inscribed in $S^1$ (cf. Lemma 4.3). The notation of $\lambda_k$ is designed to lay bare the analogy with the case of $S^2$ (cf. Section 7).

THEOREM 1.1. *Let $r > 0$, and assume $\lambda_1/2 < r < \lambda_2/2$. Then $U_r S^1$ has the homotopy type of $S^3$.*

Proof. Let $S^1/\boldsymbol{Z}_3$ be the set of equilateral triangles inscribed in $S^1$. For $p \in S^1/\boldsymbol{Z}_3$, let $|p| \subset S^1$ be the set of vertices of $p$. Let $X$ be the topological join of $S^1$ and $S^1/\boldsymbol{Z}_3$, so that $X \approx S^3$. We will imbed $X$ in $\bar{U}_{\lambda_1/2} \subset U_r$ in Lemma 2.6.

Define an open set $C \subset U_r$ to consist of $f \in U_r$ such that the set $\{x \in S^1 \mid \|f - d_x\| \leqslant r\} \subset S^1$ is contained in some semicircle. Define an open set $A \subset U_r$ to consist of $f \in U_r$ such that the set $\{x \in S^1 \mid \|f - d_x\| < \lambda_2/2\} \subset S^1$ is not contained in any semicircle. Then $U_r = A \cup C$. Let $B = A \cap C$. We will show in Propositions 3.2, 5.1, and 6.1 that the inclusion of $X$ in $U_r$ is a homotopy equivalence on each of the three pieces $X \cap A$, $X \cap B$, $X \cap C$. Consider the Mayer–Vietoris homology exact sequences for $X$ and $U_r$:

$$\begin{array}{ccccccccc} H_n(X\cap B) & \to & H_n(X\cap A)\oplus H_n(X\cap C) & \to & H_n(X) & \to & H_{n-1}(X\cap B) & \to & H_{n-1}(X\cap A)\oplus H_{n-1}(X\cap C) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_n(B) & \to & H_n(A)\oplus H_n(C) & \to & H_n(U_r) & \to & H_{n-1}(B) & \to & H_{n-1}(A)\oplus H_{n-1}(C) \end{array}$$

The 5-lemma now yields an isomorphism in homology between $X$ and $U_r$, and the relative Hurewicz theorem gives isomorphism in homotopy.

## § 2. Reduced functions.

DEFINITION 2.1. A function $f \in L^\infty(S^1)$ is called *reduced* if $f(x) = \|f - d_x\|$ for all $x \in S^1$, or equivalently, $f(x) - f(y) \leqslant \mathrm{dist}_{S^1}(x, y) \leqslant f(x) + f(y)$ for all $x, y \in S^1$.



Thus the values of $f$ at fixed points $x, y \in S^1$ lie in the diagonal half-strip of Fig. 1. The map $L^\infty \to L^\infty$, $f \mapsto \|f - d_x\|$ is a distance non-increasing projection to the separable convex set of reduced functions. This set is invariant under adding positive constants, and is properly contained in the set of 1-Lipschitz functions.

LEMMA 2.2. *Let $f, g \in L^\infty$ be reduced, and let $x \in S^1$. Then*

$$f(x) \leqslant g(x) + \|f - g\|.$$

Proof. This is the triangle inequality in $L^\infty$.

EXAMPLE 2.3. Let $p \in S^1/\boldsymbol{Z}_3$ with $|p| = \{p_1, p_2, p_3\}$, and define the "mountain range" function $MR(p, v) \in L^\infty$ depending on a vector parameter $v = (v_1, v_2, v_3) \in \boldsymbol{R}^3_+$ (the "valleys") by the formula

$$MR(p, v)(x) = \min_{j=1,2,3} \left(\mathrm{dist}_{S^1}(x, p_j) + v_j\right). \tag{2.4}$$

Then $MR(p, v)$ is reduced if and only if $v_i - v_j \leqslant \lambda_1 \leqslant v_i + v_j$ for $1 \leqslant i, j \leqslant 3$, $i \neq j$. Let $v_1 \leqslant v_2 \leqslant v_3$. Assume $v_1 < r$, so that $MR(p, v) \in U_r$. In such case, $MR(p, v) \in C$ if and only if $v_3 > r$, while $MR(p, v) \in A$ if and only if $v_3 < \lambda_2/2$. Note that these are both open conditions.

Remark 2.5. If $MR(p, v) \in B$ then $v_1 < r$ while $r < v_3$. Thus we have a strict inequality $v_1 < v_3$.

LEMMA 2.6. *The imbedding $S^1 \to \bar{U}_{\lambda_1/2}$ can be extended to the join $X$ of $S^1$ and $S^1/\boldsymbol{Z}_3$.*

Proof. Given $f, g \in L^\infty$, we define a unit speed deformation, $US$, of $f$ to $g$ by the formula

$$US(f, g, t)(x) = \begin{cases} \max(f(x) - t, g(x)) & \text{if } f(x) \geqslant g(x), \\ \min(f(x) + t, g(x)) & \text{if not,} \end{cases}$$

for all $x \in S^1$ and $t \geqslant 0$ (see Figure 1).

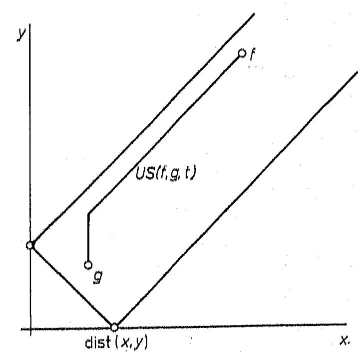


Fig. 1. The unit speed deformation of $f$ to $g$

Define $f_p \in L^\infty(S^1)$ by $f_p(x) = \mathrm{dist}_{S^1}(x, |p|) + \lambda_1/2$. Thus $f_p = MR(p, v)$ when $v_j = \lambda_1/2$ for all $j$. Then for all $x \in S^1$, we have $US(d_x, f_p, 3\lambda_1/2) = f_p$. It follows from [11], p. 506, that the family $\{US(d_x, f_p, t)\}$, where $x \in S^1$, $p \in S^1/\mathbf{Z}_3$, and $t \in [0, 3\lambda_1/2]$, imbeds $X$ in $\bar{U}_{\lambda_1/2} \subset U_r$.

Remark. The part of $X$ corresponding to a fixed $p \in S^1/\mathbf{Z}_3$ is a cone on $S^1$ with vertex $f_p$, i.e. a disk. The choice of $p$ "polarizes" the disk, reducing the rotational symmetry group to $\mathbf{Z}_3$. This phenomenon has no Euclidean or spherical analog: a round circle spans a disk with full rotational symmetry, and it is unclear how breaking the symmetry could fill the circle in a smaller neighborhood.

Lemma 2.7. (i) *$S^1 \subset C$ is a deformation retract of $X \cap C$;*
(ii) *$S^1/\mathbf{Z}_3 \subset A$ is a deformation retract of $X \cap A$;*
(iii) *$(S^1/\mathbf{Z}_3) \times S^1 \subset B$ is a deformation retract of $X \cap B$.*

Proof. (i) We have $f_p(p_j) = \lambda_1/2 < r$, $j = 1, 2, 3$, so that $|p| \subset \{x \in S^1 \mid f_p(x) \leqslant r\}$. Since $|p|$ is not contained in any semicircle, $f_p \notin C$, and $X \cap C \subset X - (S^1/\mathbf{Z}_3)$. But $X - (S^1/\mathbf{Z}_3)$ is homotopic to $S^1 \subset X$.

(ii) Let $f = d_y$, where $y \in S^1$. Then $\{x \in S^1 \mid f(x) < \lambda_2/2\} = B(y, \lambda_2/2) \subset B(y, \frac{1}{4})$, a semicircle. Hence $d_y \notin A$, and $X \cap A \subset X - S^1$. But $X - S^1$ is homotopic to $S^1/\mathbf{Z}_3$.

(iii) Let $x \in S^1$ and $p \in S^1/\mathbf{Z}_3$. Then $B$ cuts out a nonempty connected open set in each interval $\{US(d_x, f_p, t)\}$, $0 \leqslant t \leqslant 3\lambda_1/2$. This set can be retracted to its midpoint.

## § 3. Deformation using predistance.

Definition 3.1. Let $C \subset U_r$ be the set consisting of reduced $f \in U_r$ such that the sublevel set $\{x \in S^1 \mid f(x) \leqslant r\} \subset S^1$ lies in a semicircle.

In this section we construct a deformation retraction of $C \subset U_r$ to $S^1$. It is tempting to retract to the (suitably weighted) center of mass of the sublevel set $f \leqslant r$, which is contained in a semicircle. However, to produce a deformation retraction, more work has to be done.

Proposition 3.2. *Any polyhedron $P \subset C$ can be retracted in $C$ to $S^1$.*

*Discussion.* The idea of the proof can be illustrated as follows. Define a family $\{Y^t\}$, $0 \leqslant t \leqslant 1$, of subsets of $\mathbf{R}$ by setting $Y^t = \{0, 2\}$ if $0 \leqslant t \leqslant \frac{1}{2}$, and $Y^t = \{0\}$ if $\frac{1}{2} < t \leqslant 1$. Thus at time $t = \frac{1}{2}$, $Y$ loses its point $x = 2$. We may think of the family $\{Y^t\}$ as a closed subset of the Cartesian product $[0, 1] \times \mathbf{R}$. Let $f_{Y^t}(x) = \mathrm{dist}(x, Y^t) + 1$. The family $\{f_{Y^t}\} \subset L^\infty$ is discontinuous in $t$, since, for example, at $x = 3$, we have $f_{Y^t}(3) = 1$ for $t = \frac{1}{2}$ but $f_{Y^t}(3) = 3 > 1$ for $t > \frac{1}{2}$. Whenever $t_0 > \frac{1}{2}$, the point $x = 3$ is far from $Y^t$ along the vertical line $t = t_0$ in the $(t, x)$-plane. On the other hand, if $(t, x)$ is allowed first to shift horizontally to $(\frac{1}{2}, x)$, then $\{Y^t\}$ is again nearby. Thus we can turn $\{f_{Y^t}\}$ into a continuous family by allowing some mobility in the $t$ direction. The precise construction using a predistance is described in step 4 below.

Proof of Proposition 3.2. We deform the inclusion map $P \to C$ to a map $m\colon P \to S^1 \subset C$, defined by skeletons. This will be done in a way compatible (up to homotopy) with the deformation of Lemma 2.7(i). By compactness, there is an $r_0 < r$ such that $P \subset U_{r_0} \subset U_r$. Let $\varepsilon = (r - r_0)/6$, and $r_1 = (r + r_0)/2$. Assume $P$ is triangulated into simplices of diameter $\leqslant \varepsilon$. Note that if $f \in C$, the set of minima of $f$ lies in a semicircle.



*Step* 1. For $f \in C$, let $M(f) \in S^1$ be the rightmost point at which $f$ achieves its minimum. Then $f(M(f)) < r_0$. Note that $M$ is not continuous as a function of $f \in P$. Let $P^0 \subset P$ be the 0-skeleton. We define the map $m$ on $P^0$ by setting $m(f) = M(f)$. Now let $f \in P$, and let $f_1, \ldots, f_k$ be the vertices of the simplex containing $f$. Let $m_i = m(f_i)$. By Lemma 2.2, $f(m_i) \leqslant f_i(m_i) + \varepsilon$. Therefore

$$\mathrm{dist}_{S^1}(M(f), m_i) \leqslant f(M(f)) + f(m_i) < 2r_0 + \varepsilon.$$

We have $f \leqslant r$ on the set $\{M(f), m_1, \ldots, m_k\}$. By definition of $C$, this set lies in a semicircle.

*Step* 2. We extend $m$ by linearity to the simplex. This defines $m$ inductively on all of $P$, in such a way that $\mathrm{dist}_{S^1}(M(f), m(f)) \leqslant 2r_0 + \varepsilon$. Note that if $f \notin P^0$, we may have $f(m(f)) > r$.

*Step* 3. We join $M(f)$ linearly with $m(f)$ by the family $M^t(f)$, where $M^0(f) = M(f)$, $M^1(f) = m(f)$.

Let $g \in P$ with $\|f - g\| \leqslant \varepsilon$, and let $Q \subset P$ be the union of the two simplices containing $f$ and $g$, so that $\mathrm{diam}(Q) \leqslant 3\varepsilon$. Hence by Lemma 2.2, we have $f(M(q)) \leqslant r_0 + 3\varepsilon = r_1$ for all $q \in Q$. It follows that the set $m(Q) \subset S^1$ has diameter $\leqslant 2r_1$, and similarly for the convex hull of $m(Q)$ in $S^1$. Thus for all $s, t \in [0, 1]$,

$$(*) \qquad \text{if } \|f - g\| \leqslant \varepsilon \text{ then } \mathrm{dist}_{S^1}(M^s(f), M^t(g)) \leqslant 2r_1.$$

We now use the $M^t(f)$ to define a continuous deformation in $L^\infty$.

*Step* 4. Let $V = (P \times [0, 1]) \cup_m (S^1)$ be the mapping cylinder of $m\colon P \to S^1$. We will construct a distance function dist on $V$ such that $\mathrm{dist}((f, 0), x) = f(x)$ and $\mathrm{dist}|_{S^1} = \mathrm{dist}_{S^1}$, where $x \in S^1 \subset V$, $(f, 0) \in P \times \{0\} \subset V$, and $\mathrm{dist}|_{S^1}$ is the restriction of dist to $S^1$. The desired deformation of $f \in P$ is then $f^t(x) = \mathrm{dist}((f, t), x)$.

**3.3.** *Distances and predistances.* Recall [4] that a *distance function* dist on a space $V$ is a symmetric map $V \times V \to \mathbf{R}^+$ satisfying the triangle inequality. Given a subset $W \subset V \times V$, any symmetric function $d\colon W \to \mathbf{R}^+$ is called a *predistance*. There is a dist canonically associated with $d$, namely

$$\mathrm{dist}(x, y) = \inf(d(x_0, x_1) + d(x_1, x_2) + \ldots + d(x_{n-1}, x_n)),$$

where the infimum is taken over arbitrarily long sequences $x_0, \ldots, x_n$ with $x_0 = x$, $x_n = y$, and $(x_i, x_{i+1}) \in W$.

Continuing with the proof of Proposition 3.2, we choose a large number $N > 0$. We define $d$ on the mapping cylinder as follows. For all $x, y \in S^1$, $f, g \in P$, and $s, t \in [0, 1]$, set

$$(1) \qquad d((f, 0), x) = f(x);$$

$$(2) \qquad d(x, y) = \mathrm{dist}_{S^1}(x, y);$$

$$(3) \qquad d((f, t), x) = \mathrm{dist}_{S^1}(x, M^t(f)) + r_1;$$

$$(4) \qquad d((f, s), (g, t)) = N(\|f - g\|_{L^\infty} + \mathrm{dist}_{S^1}(m(f), m(g)) + |t - s|).$$

Let dist be the corresponding distance.

LEMMA 3.4. *The restriction of* dist *to* $S^1$ *coincides with the original distance* $\mathrm{dist}_{S^1}$.

Proof. We need to verify that for all $x, y \in S^1$ we have

(3.5) $$\mathrm{dist}_{S^1}(x, y) \leqslant d(x, (f, s)) + d((f, s), (g, t)) + d((g, t), y),$$

i.e. one cannot take a shortcut from $x$ to $y$ by going through two points in the mapping cylinder.

Let $\varepsilon$ be as in inequality (*), and choose $N > (\mathrm{diam}\, S^1)/\varepsilon$ in part (4) of the definition of $d$. If $\|f-g\| \geqslant \varepsilon$ then $d((f, s), (g, t)) \geqslant \mathrm{diam}\, S^1$ and inequality (3.5) is satisfied. If $\|f-g\| < \varepsilon$, then we have inequality (*). By the triangle inequality for $\mathrm{dist}_{S^1}$, we have

$$\mathrm{dist}_{S^1}(x, y) \leqslant \mathrm{dist}_{S^1}(x, M^s(f)) + \mathrm{dist}_{S^1}(M^s(f), M^t(g)) + \mathrm{dist}_{S^1}(M^t(g), y).$$

Now part (3) of the definition of $d$ together with the inequality (*) give

$$\begin{aligned}\mathrm{dist}_{S^1}(x, y) &\leqslant d(x, (f, s)) - r_1 + 2r_1 + d(y, (g, t)) - r_1 \\ &\leqslant d(x, (f, s)) + d(y, (g, t)),\end{aligned}$$

proving (3.5) in this case.

LEMMA 3.6. *For all* $f \in P$ *and* $x \in S^1$, $\mathrm{dist}((f, 0), x) = f(x)$.

Proof. To show that dist $= d$ for pairs of points of type (1), we need to verify that for all $(g, t) \in V$, the predistance measured along the sequence of points $(f, 0) \to (g, t) \to x$ is at least $f(x)$. By part (4) of the definition of $d$, we have $d((f, 0), (g, t)) \geqslant \|f-g\| + Nt$. If $N$ is greater than the maximum length of a path from $M(f)$ to $m(f)$, it follows from the definition of $M^t(f)$ in step 3 that $\mathrm{dist}(M^0(g), M^t(g)) \leqslant Nt$. Therefore by the triangle inequality in $L^\infty(S^1)$, we have

$$\begin{aligned} f(x) &\leqslant \|f-g\| + g(M^0(g)) + \mathrm{dist}_{S^1}(M^0(g), M^t(g)) + \mathrm{dist}_{S^1}(M^t(g), x) \\ &\leqslant \|f-g\| + r_1 + Nt + \mathrm{dist}_{S^1}(M^t(g), x) \\ &\leqslant d((f, 0), (g, t)) + d((g, t), x).\end{aligned}$$

Lemmas 3.4 and 3.6 show that $f^0 = f$ and $f^1 = d_{m(f)}$. A similar calculation (cf. also [4], page 37) shows that all functions lie in $C \subset U_r$. This proves Proposition 3.2.

**§ 4. Extrema of the diameter functional.** The power set $2^V$ of a Riemannian manifold $(V, \mathrm{dist})$ may be viewed as a metric space, with respect to the Hausdorff distance $\mathrm{dist}_H$ among (closed) sets $Y \subset V$. The diameter functional $\delta$ on $2^V$ associates to each $Y \subset V$ its diameter $\delta(Y) = \max_{x,y \in Y} \mathrm{dist}(x, y)$.

DEFINITION 4.1. We call $Y$ an *extremum* of $\delta$ if every perturbation of $Y$ decreases $\delta(Y)$ at most quadratically in the size of the perturbation.

Thus we cannot decrease $\delta(Y)$ linearly. To explain what this means, we consider the notion of extremality at a point.

Given $a, b \in V$ with a minimizing geodesic $\gamma$ joining them, denote by $u_{ab} \in T_a V$ the tangent vector to $\gamma$ at $a$. Let $a \in Y$ and let $d = \delta(Y)$. We say that $Y$ is *$\delta$-extremal at* $a$ if the following two equivalent conditions are satisfied:

(i) for every $v \in T_a V$ there is a point $b \in Y$ such that $\mathrm{dist}(a, b) = d$ and $\langle v, u_{ab} \rangle \geqslant 0$;



(ii) a perturbation of $Y$ which displaces $a$ by a distance $\varepsilon > 0$ and keeps all other points fixed, decreases the quantity $\max_{b \in Y}(\mathrm{dist}(a, b))$ at most quadratically in $\varepsilon$.

**4.2.** Now suppose $Y$ has $k$ points: $Y = \{y_1, \ldots, y_k\}$. We view $Y$ as a point of the parameter space $V^k = V \times \ldots \times V$ (the product metric is commensurate with $\mathrm{dist}_H$). Each pair of points $a = y_i$, $b = y_j$ at distance $d$ defines a function $\mathrm{dist}(\pi_i, \pi_j)$ on the parameter space, taking the value $d$ at $Y \in V^k$, where $\pi_j$ is the projection to the $j$th factor. The gradient of this function has the form $(u_{ab}, u_{ba}) \in T_a V \times T_b V \subset T_Y(V^k)$. We use these gradients instead of the vectors $u_{ab}$ in (i), and allow all points to vary instead of just $a$ in (ii), to define $\delta$-extremality of $Y$.

An extremum of $\delta$ typically looks like the Cartesian product of an ordinary smooth extremum, with a piecewise linear function greater than a positive constant multiple of the absolute value function. A critical value of $\delta$ is the value at an extremum. We say that $Y$ is the *first extremum* of $\delta$ if $\delta(Y)$ is the smallest nonzero critical value. It is possible to do a kind of Morse theory in $2^V$ (and in $L^\infty(V)$) with this notion of extremum replacing the ordinary smooth one (cf. Lemma 4.8).

LEMMA 4.3. *Every extremum of* $\delta$ *on the power set of* $S^1$ *is the set of vertices of a regular odd polygon inscribed in* $S^1$.

Proof. If $Y \subset S^1$ is an extremum with $\delta(Y) = \lambda$, every point $a \in Y$ can be included in an isosceles triangle with two points $b, c \in Y$ so that $\mathrm{dist}(a, b) = \mathrm{dist}(a, c) = \lambda$. The lemma follows by induction.

DEFINITION 4.4. Given a finite set $Y \subset S^1$ containing no pair of antipodal points, we define a relation $\sim$ among points of $Y$ as follows: $x \sim y$ if and only if for all $z \in Y$, the set $\{x, y, z\}$ is contained in some semicircle (cf. [13], page 126).

LEMMA 4.5. *The relation* $\sim$ *is an equivalence relation. If* $\delta(Y) < \lambda_2$, *then the number of equivalence classes is either* 1 *or* 3.

Proof. The set $D$ of finite subsets of $S^1$ containing no antipodal points has countably many connected components. Each component of $D$ contains a unique odd polygon (up to congruence). Two points of $Y$ are in the same equivalence class of $\sim$ if and only if they flow to the same vertex of the polygon under the (downward) gradient flow of $\delta$. The only regular polygon with diameter $< \lambda_2$ is the triangle. Hence a set $Y \in D$ with $\delta(Y) < \lambda_2$ is in the connected component of either a point (if $Y$ lies in some semicircle) or an equilateral triangle.

DEFINITION 4.6. The equivalence classes of $Y$ will be called *clusters*.

We may view a cluster as a smeared vertex of a regular odd polygon (Figure 2).

LEMMA 4.7. *Suppose* $Y \subset S^1$ *contains no pair of antipodal points. Let* $y, z \in Y$ *with* $y \sim z$. *Let* $\gamma \subset S^1$ *be the shortest arc joining* $y$ *and* $z$. *Then* $\mathrm{diam}(Y \cup \gamma) = \mathrm{diam}(Y)$.

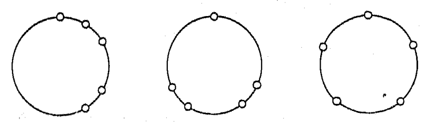

Fig. 2. A 5-point set on $S^1$ may have either 1, 3, or 5 clusters

Proof. Taking the convex hull in $S^1$ of each cluster does not increase the diameter (cf. Lemma 7.2).

Returning to the general situation, let $A_1 \subset \delta^{-1}(\lambda_1) \subset 2^V$ be the set of the first extrema. Assume that the isometry group of $V$ is transitive on $A_1$ (this is true for all two-point homogeneous spaces). Let $k = \text{card}(Y)$ for $Y \in A_1$ be the number of points in the first extremum. Let $\lambda < \lambda_2$ and let $W \subset V^k$ be the connected component of the sublevel set $\delta^{-1}([0, \lambda))$ of $\delta\colon V^k \to \mathbf{R}^+$ which contains $A_1$.

LEMMA 4.8. *$A_1$ is a deformation retract of $W$.*

Proof. Choose the tangent vector to $W$ at $Y \in W$ which forms the minimax angle with the gradients $(u_{ab}, u_{ba})$ (cf. 4.2). This angle is acute unless $Y \in A_1$. The resulting vector field may be regularized to a continuous vector field on $W$ vanishing along $A_1$, but we will not need this. It is clear that the flow generated by this vector field is continuous, proving the lemma.

**§ 5. Deformation of the component of the first extremum.** Let $r > 0$ be as in Theorem 1.1. Let $A \subset U_r$ be the set consisting of reduced $f \in U_r$ such that the sublevel set $\{x \in S^1 \mid f(x) < \lambda_2/2\} \subset S^1$ lies in no semicircle.

The first extremum of the diameter functional on $2^{S^1}$ is the set of vertices of an equilateral triangle inscribed in $S^1$. The set of the first extrema is $A_1 = S^1/\mathbf{Z}_3 \subset 2^{S^1}$. We identify $S^1/\mathbf{Z}_3$ with its image in $A$ by the map $p \mapsto f_p$, where $f_p(x) = \text{dist}_{S^1}(x, |p|) + \lambda_1/2$.

PROPOSITION 5.1. *Any polyhedron $P \subset A$ can be retracted in $A$ to $S^1/\mathbf{Z}_3$.*

Proof. The argument is similar to that for Proposition 3.2. There are two main differences: the range of $m$ will now be a larger space; the retraction in step 4 will now be in two movements. The retraction will be compatible (up to homotopy) with the deformation of Lemma 2.7(ii).

DEFINITION 5.2. Let $T$ be the subset of the third symmetric power of $S^1$ consisting of triples $p = \{p_1, p_2, p_3\}$ such that $p$ does not lie in any closed semicircle.

Let $v = (v_1, v_2, v_3) \in \mathbf{R}^3_+$, and define a function $MR(p, v) \in L^\infty$ by formula (2.4).

DEFINITION 5.3. Let $MR \subset L^\infty(S^1)$ be the collection of reduced functions $MR(p, v)$ such that $p \in T$, $\min_{j=1,2,3} v_j < r$ and $\max_{j=1,2,3} v_j < \lambda_2/2$.

Clearly, $MR \subset A$. Let $h\colon MR \to T$ be the projection to the first factor: $h(MR(p, v)) = p$.

Let $P \subset A \subset U_r$. By compactness, there is an $r_2 < \lambda_2/2$ such that the set $\{x \in S^1 \mid f(x) < r_2\}$ is still not contained in any semicircle, for all $f \in P$. Let $r_3 = (r_2 + \lambda_2/2)/2$. Assume $P$ is triangulated into simplices of diameter $\leqslant \varepsilon = \min(r_1 - r_0, r_3 - r_2)$.

We modify Step 1 as follows. Let $f \in P$. The set $\{x \in S^1 \mid f(x) < r_2\}$ has 3 clusters by Lemma 4.5. Choose the point $p_j$ in each cluster to be the rightmost point at which $f$ achieves its minimum in the cluster, and let $v_j = \max(f(p_j) + \varepsilon, \lambda_1/2)$. This defines the triples $p$ and $v$, and we let $M\colon P \to MR \subset A$ be the discontinuous map $M(f) = MR(p, v)$.

We let $m(f) = M(f)$ on $P^0$, and extend, by linearity within each cluster, to all of $P$. We define the family of functions $M^t(f)$ by joining the underlying 3-point sets of $M(f)$ and $m(f)$, again linearly within each cluster.



We modify step 4 by setting $V = C_m \amalg S^1$, where $C_m = (P \times [0, 1]) \cup_m (MR)$ is the mapping cylinder of $m\colon P \to MR$, and $\amalg$ stands for disjoint union. We define $d$ as follows. Equations (1) and (2) remain the same. In equation (4), $\text{dist}_{S^1}(m(f), m(g))$ must be replaced by the Hausdorff distance between sets $h(m(f))$ and $h(m(g))$ in $S^1$. Equation (3) is replaced by

$$d((f, t), x) = M^t(f)(x). \tag{3'}$$

The inequality (*) is replaced by

$$\text{diam}\big(h(M^s(f)) \cup h(M^t(g))\big) \leqslant 2r_3 \tag{**}$$

if $f$ and $g$ are sufficiently close. This inequality is immediate from Lemma 2.2.

Let dist be the corresponding distance. One checks that $\text{dist}((f, 0), x) = f(x)$ and $\text{dist}((f, 1), x) = m(f)(x)$.

Here a slight refinement of (**) is necessary, namely an upper bound of $v_i + v_j$ for the distance between a point in the $i$th and a point in the $j$th cluster of $h(M^s(f)) \cup h(M^t(g))$, for all $1 \leqslant i, j \leqslant 3$. One also checks that the resulting deformation of $P$ into $MR$ lies entirely in $A \subset U_r$. Proposition 5.1 now follows from Lemma 4.8 with $A_1 = S^1/\mathbf{Z}_3$, where the flow must be modified by including all except the shortest distance among those involved in the definition of the minimax angle. In our situation, this flow consists of only two parts: (i) move the endpoints of the shortest side of the triangle away from each other until the two shortest sides come to have the same length; (ii) move the endpoints of the longest side of the triangle toward each other, keeping the third vertex fixed, until the longest two sides come to have the same length.

By step 1, $v_j \geqslant \lambda_1/2$, and $v_i + v_j \geqslant \lambda_1$, $\forall\, i, j$. Since the shortest side is always $\leqslant \lambda_1$, part (i) does not take us out of the class of reduced functions. A similar remark applies to part (ii).

Once all triangles have been deformed to equilateral ones, we take care of the $v_j$ by deforming all three to $\lambda_1/2$. This completes the retraction of $MR$ to $S^1/\mathbf{Z}_3$.

**§ 6. Deformation of the overlap.** Let $B \subset U_r S^1$ consist of reduced $f \in U_r$ such that the sublevel set $f \leqslant r$ lies in a semicircle, but the set $f < \lambda_2/2$ does not.

PROPOSITION 6.1. *Any polyhedron $P \subset B$ can be retracted in $B$ to $(S^1/\mathbf{Z}_3) \times S^1$.*

Proof. The argument is the same as for the set $A \subset U_r$, except for the last paragraph of Section 5. Let $p \in S^1/\mathbf{Z}_3$, and assume that $MR(p, v) \in B$. Let $a = \min_j v_j$, $b = \max_j v_j$. Then $a < b$ by Remark 2.5. We use the linear map

$$\frac{\lambda_1}{2}\frac{b-t}{b-a} + \frac{\lambda_2}{2}\frac{t-a}{b-a}$$

to deform $a, b$ to $\lambda_1/2, \lambda_2/2$. Let us assume that $a = \lambda_1/2$, $b = \lambda_2/2$.

DEFINITION 6.2. The *skew-hexagon* $H \subset \mathbf{R}^3$ is the set

$$H = \{(v_1, v_2, v_3) \in \mathbf{R}^3 \mid \min_j v_j = a,\ \max_j v_j = b\},$$

consisting of the six edges of the cube $\{(v_1, v_2, v_3) \in \mathbf{R}^3 \mid a \leqslant v_j \leqslant b\}$ whose midpoints lie in the plane $v_1 + v_2 + v_3 = (a + 3b)/2$.

Given $p \in S^1$, we identify the skew-hexagon with $S^1$ as follows. We map $(a, b, b)$ to $p_1$, $(b, a, a)$ to the point opposite $p_1$, and similarly for $p_2$ and $p_3$; and then extend linearly over the 6 edges. This gives a map

$$\{MR(p, v) \mid p \in S^1/\mathbf{Z}_3,\ v \in H\} \to (S^1/\mathbf{Z}_3) \times S^1.$$

We must show that this is homotopic to the map $X \cap B \to (S^1/\mathbf{Z}_3) \times S^1$ of Lemma 2.7 (iii), where the domains of the two maps are identified by retracting $X \cap B$ as in Section 5.

Let $p \in S^1/\mathbf{Z}_3$ be fixed, and let $d_x^t = US(d_x, f_p, t)$. Given $x \in S^1$, let $t_0 > 0$ be the largest $t$ such that $d_x^t \notin B$, or equivalently, the smallest $t$ such that the sublevel set $\{y \in S^1 \mid d_x^t(y) \leqslant b\}$ has three clusters. Let $\hat{v}_j$ be the minimum value of $d_x^{t_0}$ on the cluster near $p_j$, and $v_j = \max(\hat{v}_j, a)$. Our choice of $t_0$ implies that $\max_j v_j = b$. Let $B(p_j, b-a) \subset S^1$ be the open ball of radius $(b-a)$ centered at $p_j$, $j = 1, 2, 3$.

LEMMA 6.3. *The map* $g\colon S^1 \to H$, $x \mapsto (v_1, v_2, v_3)$ *has degree* 1. *It is injective on* $\bigcup_j B(p_j, b-a)$, *and collapses the complement* $S^1 - \bigcup_j B(p_j, b-a)$ *to the three vertices* $(a, a, b)$, $(a, b, a)$, $(b, a, a)$ *of* $H$.

Proof. We identify $S^1$ with the interval $[0, 1]$ so that $p_1 = 0 = 1$, $p_2 = \frac{1}{3}$, $p_3 = \frac{2}{3}$. The graphs of $d_{p_1}$ and $f_p$ meet at points $p_2 - \frac{1}{12}$ and $p_3 + \frac{1}{12}$. We have $d_{p_1} > f_p$ on the interval $(p_2 - \frac{1}{12}, p_3 + \frac{1}{12})$. For $0 \leqslant t < \frac{1}{6}$, we have $d_{p_1}^t > f_p$ on the interval $(p_2 - \frac{1}{12} + \frac{t}{2}, p_3 + \frac{1}{12} - \frac{t}{2})$. For small $x$, $d_x > f_p$ on $(p_2 - \frac{1}{12} + \frac{x}{2}, p_3 + \frac{1}{12} + \frac{x}{2})$. Similarly, $d_x^t > f_p$ on $I_{x,t} = (p_2 - \frac{1}{12} + \frac{t}{2} + \frac{x}{2}, p_3 + \frac{1}{12} - \frac{t}{2} + \frac{x}{2})$.

We have $f_p(p_3 + \frac{1}{30}) = \frac{1}{5}$. As $t$ increases, the interval $I_{x,t}$ shrinks and the common value of $d_x^t$ and $f_p$ at both endpoints of $I_{x,t}$ decreases. The function $d_x^t$ enters $B$ (and $A$) when the value at the right endpoint reaches $\frac{1}{5}$: $f_p(p_3 + \frac{1}{12} - \frac{t}{2} + \frac{x}{2}) = \frac{1}{5}$, or equivalently $\frac{1}{12} - \frac{t}{2} + \frac{x}{2} + \frac{1}{6} = \frac{1}{5}$. Solving this for $t = t_0$, we obtain $t_0 = \frac{1}{10} + x$. When $t = t_0$, the value at the left endpoint of $I_{x,t}$ is $v_2 = f_p(p_2 - \frac{1}{12} + \frac{t_0}{2} + \frac{x}{2}) = \frac{1}{12} - \frac{t_0}{2} - \frac{x}{2} + \frac{1}{6} = \frac{1}{5} - x$. The cluster of the set $\{y \in S^1 \mid d_x^{t_0}(y) \leqslant \frac{1}{5}\}$ near $p_3$ is a single point $\hat{p}_3 = p_3 + \frac{1}{30}$. We have $v_1 = \frac{1}{6}$, $v_2 = \frac{1}{5} - x$, $v_3 = d_x^{t_0}(\hat{p}_3) = f_p(\hat{p}_3) = \frac{1}{5}$. Therefore $g$ maps the arc $[0, \frac{1}{30}] \subset S^1$ to the edge from $(a, b, b)$ to $(a, a, b)$.

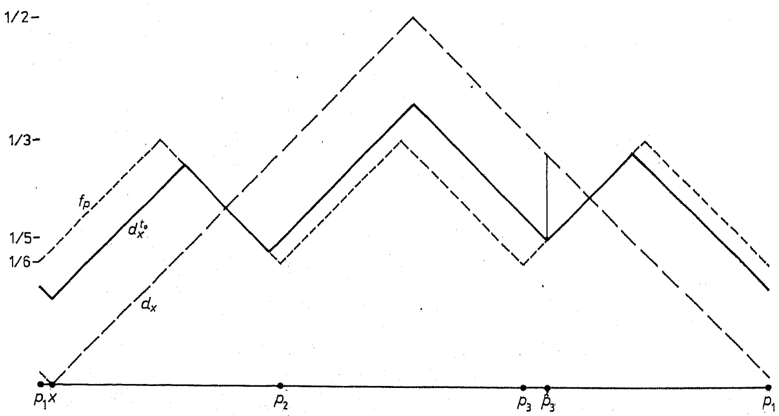


Fig. 3. Graph of $d_x^{t_0}$



To summarize, $g(p_1) = (a, b, b)$, $g(p_2) = (b, a, b)$, $g(p_3) = (b, b, a)$; the ball $B(p_1, b-a)$ is mapped injectively onto the union of the two edges joining $(a, b, b)$ with $(a, a, b)$ and with $(a, b, a)$; and the balls $B(p_2, b-a)$ and $B(p_3, b-a)$ similarly cover the other 4 edges.

Figure 3 shows the graph of $d_x^{t_0}$ for $x = \frac{1}{60}$ by a solid line, and $d_x$ and $f_p$ by broken lines. Also shown is the vertical interval of values of $d_x^t$ at $p_3 + \frac{1}{30}$ for $0 \leqslant t \leqslant t_0$.

**§ 7. The case of the two-sphere.** Let $S^2$ be the sphere of constant curvature $+1$ and Riemannian diameter $\pi$. Let $\alpha_1 = \arccos(-1/3)$ and $\alpha_2 = \arccos(-1/\sqrt{5})$. The number $\alpha_1$ is the spherical diameter of an equilateral 4-tuple, i.e. the set of vertices of a regular tetrahedron inscribed in $S^2$. The number $\alpha_2$ is the spherical diameter of the set of vertices of an inscribed pyramid on a pentagon, such that all pairwise distances among the vertices are equal except for pairs of adjacent vertices of the pentagon. We have $\pi/2 < \alpha_1 < \alpha_2 < 2\pi/3$. Recall that the binary tetrahedral group $E_6$ naturally acts on the sphere $S^3$. The quotient $S^3/E_6$ parametrizes equilateral 4-tuples in $S^2$ (cf. [14], Section 2).

THEOREM 7.1. *Let* $r > 0$, *and assume that* $\alpha_1/2 < r < \alpha_2/2$. *Then* $U_r S^2 \subset L^\infty$ *has the homotopy type of the topological join of* $S^2$ *and* $S^3/E_6$.

Proof. The proof is modeled on that of Theorem 1.1 in Sections 3, 5, and 6. Since taking the geodesic convex hull of sets on $S^2$ increases their diameter, we replace Lemma 4.7 by Lemma 7.2 below. In Lemmas 7.4 and 7.5 we show that $\alpha_1$ and $\alpha_2$ are the first two critical values of the diameter functional. The analog of the skew-hexagon of § 6 is the 2-sphere homeomorphic to the subset of the boundary of the hypercube $0 \leqslant v_j \leqslant 1$, $1 \leqslant j \leqslant 4$, which consists of the 2-faces satisfying $\min_j v_j = 0$, $\max_j v_j = 1$. The flow of Lemma 4.8 must be modified as follows. The collection $G$ of gradients $(u_{ab}, u_{ba})$ must include not only the pairs $a, b$ at maximal distance, but all distances other than the smallest distance. The flow is then generated by the tangent vector to the parameter space which forms the minimax angle with the vectors in $G$. Since the smallest distance among 4 points on $S^2$ is always $\leqslant \alpha_1$ by a packing argument, this flow will never increase distances that are greater than $\alpha_1$.

We iterate this procedure, increasing the multiplicity of the least distance until the 4-tuple becomes equilateral.

LEMMA 7.2. *Let* $Y \subset S^2$ *with* $\operatorname{diam} Y < 2\pi/3$. *Let* $y, z \in Y$ *and assume that for all* $p, q \in Y$, *the set* $\{y, z, p, q\}$ *lies in some hemisphere. Then there is a curve* $\gamma \subset S^2$ *joining* $y$ *and* $z$ *such that* $\operatorname{diam}(Y \cup \gamma) = \operatorname{diam} Y$.

Proof. Let $D \subset S^2$ be the disc built on $yz$ as a diameter. We will show that $y$ and $z$ lie in the same connected component of $D - (\bigcup_{p \in Y} B'_p)$, where $B'_p$ denotes the ball of radius $\pi - \operatorname{diam} Y$ centered at the point $p'$ opposite $p$. If $y$ and $z$ are in different components, then there must be overlapping balls $B'_p$ and $B'_q$, where $p$ and $q$ are separated by the great circle through $y$ and $z$ (Figure 4).

Let $\varphi$ be the arc of the great circle from $p'$ to $q'$ passing through the overlap. Then

$$\operatorname{length}(\varphi) \leqslant \operatorname{rad} B'_p + \operatorname{rad} B'_q = 2(\pi - d) < \pi.$$

Hence $\varphi$ is minimizing. It is clear that $\varphi$ meets the shortest arc $yz$. We reach

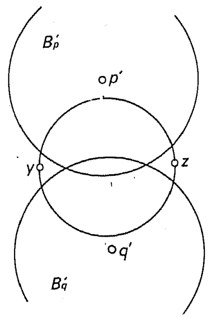


Fig. 4. Points $y$ and $z$ are separated by removing $B_p'$, $B_q'$

a contradiction by concluding that the set $\{y, z, p, q\}$ lies in no hemisphere, which follows from the following lemma.

LEMMA 7.3. *A set $\{a, b, c, d\} \subset S^2$ lies in no hemisphere if and only if the shortest arcs $ab$ and $c'd'$ meet.*

Proof. If $ab$ meets $c'd'$, then the arcs $ab$ and $cd$ contain a pair of antipodal points.

LEMMA 7.4. *The first critical value of the diameter functional on the power set of $S^2$ is $\alpha_1$.*

Proof. This is equivalent to the classical theorem of Jung and Molnár (cf. [3], page 92 and [12], page 452).

LEMMA 7.5. *The second critical value of the diameter functional on the power set of $S^2$ is $\alpha_2$.*

Proof. If $Y \subset S^2$ is an extremum with $\operatorname{diam} Y < 2\pi/3$, then each point of $Y$ is at maximal distance (equal to $\operatorname{diam} Y$) from at least three points of $Y$. If $\operatorname{card}(Y) = 4$, then $Y$ is equilateral and $\operatorname{diam} Y = \alpha_1$. If $\operatorname{card}(Y) = 5$ and the graph of maximal distances has valence 3 at every vertex, then $e = 3v/2 = 15/2$, a contradiction. Hence one of the points must be at maximal distance from all four other points, which therefore lie on a circle. Clearly, they must form an extremum of the diameter functional on the power set of the circle. But the circle has no extrema with an even number of points by Lemma 4.3.

Suppose $Y \subset S^2$ is an extremum of the diameter functional with $\operatorname{diam} Y \leqslant \alpha_2$ and $\operatorname{card} Y = 6$. Then for all $y, z \in Y$, $\operatorname{dist}(y, z') \geqslant \pi - \alpha_2$. Furthermore, $\operatorname{dist}(y, z) \geqslant \pi - \alpha_2$ by [13], Lemma 6.3. Thus $Y \cup Y'$ is a $(\pi - \alpha_2)$-separated 12-point set. It follows by a packing argument that $Y \cup Y'$ is the set of vertices of a regular inscribed icosahedron. The only such extremum $Y$ is the pyramid on a pentagon such that the distance from the top of the pyramid to each of the vertices of the base equals the distance between two non-adjacent vertices of the base.

The possibility $\operatorname{card} Y \geqslant 7$, $\operatorname{diam} Y \leqslant \alpha_2$ is ruled out by the packing argument above.

**§ 8. The case of the complex projective space.** Let $CP^n$ be the complex projective space with curvature $1/4 \leqslant K \leqslant 1$. Then a complex projective line $CP^1 \subset CP^n$ is a 2-sphere of curvature $+1$. Denote by $A_1$ the space of equilateral 4-tuples in complex projective lines in $CP^n$. Denote by $X$ the "partial join" of $A_1$ and $CP^n$, where $a \in A_1$ and $x \in CP^n$ are joined by an interval if and only if $x$ lies in the projective line containing $a$. Let $\lambda_1 < \lambda_2$ be the first two nonzero critical values of the diameter functional on the power set of $CP^n$. We have $\lambda_1 = \alpha_1 = \arccos(-1/3)$ (cf. Section 7) but the value of $\lambda_2$ is unknown.



THEOREM 8.1. *Let $r > 0$, and assume that $\lambda_1/2 < r < \lambda_2/2$. Then $U_r CP^n \subset L^\infty$ has the homotopy type of $X$.*

Proof. We use Lemma 4.8 as before. To state an analog of Lemma 7.2, we need to introduce an equivalence relation on a set $Y \subset CP^n$ with diameter less than $2\pi/3$.

A triangle $abc$ in $CP^n$ with perimeter $< \pi$ can be spanned by a surface, by joining the point $a$ by minimizing geodesics with each point on the minimizing geodesic $bc$.

Let $p_j \in CP^n$, $1 \leqslant j \leqslant 4$, with $\operatorname{dist}(p_i, p_j) < 2\pi/3$. Let $\hat{p}_j \subset CP^n$ be the triangle spanned by the three points other than $p_j$. Let $[p_1 p_2 p_3 p_4] \in H_2(CP^n, Z_2) = Z_2$ be the class of the 2-cycle $\sum_{j=1}^{4} \hat{p}_j$. One verifies that this class is independent of the ordering of the four points.

DEFINITION 8.2. Given $p \in CP^n$ with $\operatorname{dist}(p, CP^1) < \pi$, denote by $p_0 \in CP^1$ the point nearest $p$. Given $m \in CP^1$, let $m' \in CP^1$ denote the point opposite $m$.

LEMMA 8.3. *Let $a, b, c, d \in CP^n$ with pairwise distances $< 2\pi/3$. Let $CP^1$ be the projective line through $a$ and $b$. Then the following three conditions are equivalent:*

(i) $[abcd] = 0$;

(ii) *the set $\{a, b, c, d\} \subset CP^n$ flows to a point under the (downward) diameter flow on the power set of $CP^n$;*

(iii) *the set $\{a, b, c_0, d_0\} \subset CP^1$ is contained in some hemisphere.*

Proof. An extremum with $\leqslant 4$ points of the diameter functional on the power set of $CP^n$ is either a point or an equilateral 4-tuple in $CP^1 \subset CP^n$, whose class in $H_2(CP^n, Z_2)$ is 1. The class is constant under the flow. We must verify that property (iii) is also preserved by the flow. Suppose a transition occurs toward not being contained in any hemisphere. When this happens, three of the four points, say $b$, $c_0$, and $d_0$, must lie on a great circle $\gamma \subset CP^1$ in such a way that no semicircle of $\gamma$ contains all three.

Consider the totally geodesic submanifold $RP^2 \subset CP^n$ (with curvature $+\frac{1}{4}$) containing $\gamma$ and the point $c \in CP^n$. Then the perimeter of the triangle $bc_0d_0$ represents the generator of $\pi_1(RP^2)$. Let $CP^2 \subset CP^n$ be the complex projective plane containing $RP^2$. We replace $d$ by its projection to $CP^2$; this does not increase its distance to $b$ or $c$, and does not change its nearest point $d_0 \in CP^1$. Let $p \in CP^2$ be the point opposite $CP^1$.

Let $\hat{d} \in CP^2$ be the point with the following properties: (i) $\hat{d}_0 = d_0$; (ii) $\hat{d}d_0 = dd_0$; (iii) the distance $c\hat{d}$ is minimal among all points $\hat{d}$ satisfying (i) and (ii).

Then $\hat{d} \in RP^2$ (cf. [12], Lemma 4.12); note that from the theorem of cosines of P. A. Shirokov [17], we have

$$\cos cd = \cos pc \cos pd + \sin pc \sin pd \cos\alpha - 2\sin^2\frac{pc}{2}\sin^2\frac{pd}{2}\sin^2\theta$$

$$\leqslant \cos pc \cos pd + \sin pc \sin pd \cos\theta - 2\sin^2\frac{pc}{2}\sin^2\frac{pd}{2}\sin^2\theta$$

$$= \cos c\hat{d}$$

where $\alpha$ and $\theta$ are the two angles defined in [12], Section 4 at the vertex $p$ of the triangle $cpd$. The angles $\alpha$ and $\theta$ correspond to $\lambda$ and $\varphi$ of [2]. See also [1], [9].

Finally the perimeter of the triangle $bc\hat{d}$ represents the generator of $\pi_1(\boldsymbol{RP}^2)$ and hence has length $\geqslant 2\pi$, contradicting the assumption that all pairwise distances are $< 2\pi/3$.

Let $Y \subset CP^n$ with diam $Y < 2\pi/3$. Let $y, z \in Y$. We write $y \sim z$ if for all $p, q \in Y$, one has $[yzpq] = 0$. Then we have the following analog of Lemmas 4.7 and 7.2.

LEMMA 8.4. *Let $Y \subset CP^n$ with* diam $Y < 2\pi/3$, *let $y, z \in Y$, and assume $y \sim z$. Let $CP^1 \subset CP^n$ be the projective line through $y$ and $z$. Then there is a curve $\gamma \subset CP^1$ joining $y$ and $z$ such that* $\operatorname{diam}(Y \cup \gamma) = \operatorname{diam} Y$.

Proof. Let $D \subset CP^1$ be the disc built on $yz$ as a diameter. Given a point $m \in CP^n$, let $B_m = B(m, d) \cap CP^1$, where $d = \operatorname{diam} Y$. Then $\operatorname{rad}(B_m) \leqslant d$. Let $p \in Y$. Then $y, z \in B_p$. Let $B'_p \subset CP^1$ be the complement of $B_p$. Note that unlike the $S^2$ case, we do not know that $\operatorname{rad} B'_p < \pi/2$. On the other hand,

$$\operatorname{rad}(B'_p) = \pi - \operatorname{rad}(B_p) \geqslant \pi - d > d/2 \geqslant \tfrac{1}{2}\operatorname{dist}(y, z) = \operatorname{rad}(D).$$

Meanwhile, $\operatorname{diam} B_p \geqslant \operatorname{dist}(y, z) = \operatorname{diam} D$, hence $\operatorname{rad}(B_p) \geqslant \operatorname{rad}(D)$. It follows that the circle $\partial D$ is shorter than $\partial B_p$, so $\partial B_p$ cannot be contained in $D$.

We view the great circle through $y$ and $z$ as the equator. Suppose that after removing the balls $B'_p$ from $D$, as $p$ ranges over $Y$, the points $y$ and $z$ turn out to be in different connected components. Then $y$ and $z$ are actually disconnected by removing a pair of overlapping balls $B'_p$ and $B'_q$, with centers $p'_0$ north of the equator and $q'_0$ south of it. Hence the shortest arc $p'_0 q'_0$ meets the equator.

We will show that $p'_0 q'_0$ actually meets the shortest arc $yz$. Then Lemmas 8.3 and 7.3 imply $[yzpq] \neq 0$, contradicting the assumption $y \sim z$.

Let $\varphi$ be the great circle through $p'_0$ and $q'_0$. It is clear that the overlap is contained in $D$, i.e. $B'_p \cap B'_q \subset D$. Of the two points of the intersection $\varphi \cap \partial B'_p$, let $e = \varphi \cap \partial B'_p \cap B'_q \subset D$, and let $a$ be the other point. Since length $\partial B'_p \geqslant$ length $\partial D$, only one of the two points can be in $D$. Thus $a \notin D$. Hence the arc $ae = \varphi \cap B'_p$ must meet $\partial D$. Let $b = ae \cap \partial D \in B'_p$. Since $\partial D \cap B'_p$ is a connected arc of the northern semicircle of $\partial D$, the point $b$ lies north of the equator. We similarly define points $c, g \in \varphi \cap \partial B'_q$, and $f \in \partial D \cap B'_q$ south of the equator (see Figure 5).

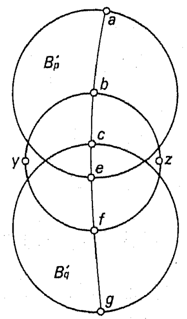


Fig. 5



(Note that the position of $p'_0$ relative to $b$ and $c$ is not known.) We conclude that the "chords" $bf$ and $yz$ of $D$ must meet.

Let $x = bf \cap yz$. The shortest arc joining the centers of overlapping balls must pass through the overlap. Therefore the shortest arc $p'_0 q'_0$ is a subarc of the geodesic segment $ag = abcefg$. We orient $ag$ from $a$ to $g$ by the unit tangent vector $u$. Then $u(x)$ points southward.

Suppose $p'_0 q'_0$ meets the equator in $x'$ instead of $x$. (Note that the union $B'_p \cup B'_q$, and thus the arc $ag$, may well contain both $x$ and $x'$.) Since $p'_0 q'_0$ is a subarc of $ag$, it follows that $u(x')$ points northward. But $p'_0$ comes before $q'_0$ along $ag$. This forces $p'_0$ into the southern hemisphere. The contradiction shows that $p'_0 q'_0$ meets the equator at $x \in yz$. Lemma 8.4 is proved.

## References


[1] H. Aslaksen, *Laws of trigonometry on SU*(3), Trans. Amer. Math. Soc. 317 (1990), 127–142.
[2] U. Brehm, *The shape invariant of triangles and trigonometry in two-point homogeneous spaces*, Geom. Dedicata 33 (1990), 59–76.
[3] Yu. D. Burago and V. A. Zalgaller, *Geometric Inequalities*, Springer-Verlag, 1988.
[4] M. Gromov, *Structures métriques pour les variétés Riemanniennes*, Cedic, Paris 1981.
[5] — *Curvature, diameter and Betti numbers*, Comment. Math. Helv. 56 (1981), 179–195.
[6] — *Filling Riemannian manifolds*, J. Differential Geom. 18 (1983), 1–147.
[7] K. Grove and P. Petersen, *Bounding homotopy types by geometry*, Ann. of Math. 128 (1988), 195–206.
[8] K. Grove and K. Shiohama, *A generalized sphere theorem*, ibid. 106 (1977), 201–211.
[9] W.-Y. Hsiang, *On the laws of trigonometries of two-point homogeneous spaces*, Ann. Global Anal. Geom. 7 (1989), 29–45.
[10] S. T. Hu, *Homotopy Theory*, Academic Press, New York 1959.
[11] M. Katz, *The filling radius of two-point homogeneous spaces*, J. Differential Geom. 18 (1983), 505–511.
[12] — *Jung's theorem in complex projective geometry*, Quart. J. Math. Oxford (2) 36 (1985), 451–466.
[13] — *Diameter-extremal subsets of spheres*, Discrete Comput. Geom. 4 (1989), 117–137.
[14] — *The rational filling radius of complex projective space*, Topology Appl. (1991), to appear.
[15] C. Kuratowski, *Quelques problèmes concernant les espaces métriques non-séparables*, Fund. Math. 25 (1935), 534–545.
[16] E. Leuzinger, *On the trigonometry of symmetric spaces*, Zürich, preprint, 1990.
[17] P. A. Shirokov, *On a certain type of symmetric spaces* (in Russian), Mat. Sb. 41 (1957), 361–372.


DEPARTMENT OF MATHEMATICS
Swain Hall East
Bloomington, Indiana 47405
U.S.A.

Current address:
INSTITUT DES HAUTES ETUDES SCIENTIFIQUES
35, route de Chartres
91440 Bures-sur-Yvette
France

*Received 24 August 1989;*
*in revised form 12 March and 5 June 1990*