Z_p-cohomology manifold with no Z_p-resolution

by

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Abstract. We construct a Z_p-Čech-cohomology manifold with no cohomology resolution over Z_p.

Let X be a cohomology manifold over Z_p (see [B]). A map $f\colon M \to X$ is *acyclic over* Z_p if for every $x \in X$, $\tilde{\check{H}}_i(f^{-1}(x), Z_p) = 0$ for $i > 0$. We say that X *has a cohomology resolution over* Z_p if there exists a closed manifold M and a map $f\colon M \to X$ acyclic over Z_p. In [D] A. Dranishnikov raised the question whether every cohomology Z_p-manifold has a resolution over Z_p. The aim of this note is to give, for every prime p and every odd n, an example of a Z_p-cohomology manifold of dimension n which has no Z_p-resolution. The example X is constructed as an infinite connected sum $X = L_q \# L_q \# L_q \# \ldots$ where L_q is any n-dimensional lens space with $\pi_1(L_q) = Z_q$ such that $q \neq p$ and q is an odd prime. More precisely, we remove a countable family $B_1, B_2, B_3, \ldots$ of bicollared n-cells from an n-sphere S^n such that $\operatorname{diam}(B_i) \to 0$ and such that the cells B_i converge to a fixed point $x \in S^n$, that is, $\operatorname{dist}(B_i, x) \to 0$. Then we take

$$X = \Big(S^n \setminus \bigcup_{i \in N} \mathring{B}_i\Big) \cup \bigcup_{i \in N} L_q^i$$

where every L_q^i is a copy of $L_q \setminus \mathring{D}^n$ and $D^n \subset L_q$ is a bicollared n-cell.

We assume moreover that ∂L_q^i is identified with ∂B_i and that $\operatorname{diam}(L_q^i) \to 0$.

It is easy to see that for any neighbourhoods $U \supset V \ni x$ the natural homomorphism

$$j_{UV}\colon H^i(X, X \setminus V; Z_p) \to H^i(X, X \setminus U; Z_p)$$

is an isomorphism and that both groups are 0 for $i \neq 0, n$ and Z_p for $i = 0, n$ so the local Betti numbers at x are $p^1(x, Z_p) = p^n(x, Z_p) = 1$ and $p^i(x, Z_p) = 0$ for $i \neq 0, n$ (see [B], pp. 7–9) and X is Z_p-orientable so that X is a Z_p-cohomology manifold.

THEOREM. *X has no cohomology resolution over Z_p.*

Proof. Suppose that $f\colon M \to X$ is a cohomology Z_p-resolution over X and so f is Z_p-acyclic and $\dim M = n$. Then by the Vietoris–Begle theorem ([S]), $H^n(M, Z_p) \approx Z_p$ so M is Z_p-orientable. Now, we construct a Z_p-acyclic map

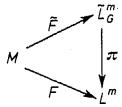
$$p\colon X \to \underbrace{L_q \# L_q \# \ldots \# L_q}_{m \text{ summands}} = L^m$$

which shrinks to a point a neighbourhood of X containing all but a finite number m of the sets L_q^i and which is a homeomorphism elsewhere.

It is easy to deduce from the Vietoris–Begle theorem that the map $F = pf$ is Z_p-acyclic.

Let $G = F_{\#}(\pi_1(M))$. Then G is finitely generated; let m_G be the minimal number of generators of G.

Let $\tilde{L}_G^m$ be the covering of L^m corresponding to the subgroup $G \subset \pi_1(L^m)$. Obviously F is covered by some map $\tilde{F}$:



$$\pi\tilde{F} = F$$

We can choose a number m of summands in L^m as large as we wish and it is enough to show that $\tilde{L}_G^m$ is not compact if m is large enough. Actually, if $\tilde{L}_G^m$ is not compact then $\check{H}^n(\tilde{L}_G^m, Z_p) = 0$ and consequently $(\pi\tilde{F})^* = 0$ so $(\pi\tilde{F})^* = F^*\colon \check{H}(L^m, Z_p) \to \check{H}(M, Z_p)$ is the zero map. This contradicts the Vietoris–Begle theorem ([S]) because both L^m and M are Z_p-orientable.

We will now prove that $\tilde{L}_G^m$ is non-compact for $m > m_G$; to do this we prove that the index of G in $\pi_1(L^m) \approx \underbrace{Z_q * Z_q * \ldots * Z_q}_{m}$ is infinite.

Lemma. *Let $G \subset \pi_1(L^m)$ be a subgroup of index $1 < k < \infty$. If $m \geqslant 3$ then $m_G \geqslant m$.*

Proof. Let $P^m = P_1 \vee \ldots \vee P_m$ with $P_\lambda = L_q$ for $\lambda = 1, \ldots, m$. Let $\pi\colon P_G^m \to P^m$ be the covering map corresponding to $G \subset \pi_1(L^m) = \pi_1(P^m)$. From the Kurosh Theorem [M–S, Thm. VII.5.2] it follows that

$$G \cong F_t * \underbrace{Z_q * Z_q * \ldots * Z_q}_{s},$$

where F_t is a free group of rank t. From [M–S, Prop. VII.5.3] we know that $t = 1 + k(m-1) - \sum_{\lambda=1}^{m} c_\lambda$ where c_λ is the number of components in $\pi^{-1}(P_\lambda)$.

Let a_λ denote the number of simply connected components in $\pi^{-1}(P_\lambda)$. From $q \geqslant 3$ it follows that $a_\lambda \leqslant k/3$. It is also obvious that $s = \sum_{\lambda=1}^{m}(c_\lambda - a_\lambda)$. Hence

$$\begin{aligned} m_G \geqslant t + s &= 1 + k(m-1) - \sum_\lambda c_\lambda + \sum_\lambda (c_\lambda - a_\lambda) \\ &= 1 + k(m-1) - \sum_\lambda a_\lambda \geqslant 1 + k(m-1) - km/3 \\ &= k(2m/3 - 1) + 1 \geqslant 2(2m/3 - 1) + 1 \geqslant m \quad \text{for } m \geqslant 3. \end{aligned}$$

This completes the proof of the lemma and of the theorem.

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References

[B] A. Borel, *Seminar on Transformation Groups*, Ann. of Math. Stud. 46, Princeton, New Jersey, 1960.

[D] A. N. Dranishnikov, *Homological theory of dimension* (in Russian), Uspekhi Mat. Nauk 43 (4) (262), (1988).

[M–S] W. S. Massey and J. Stallings, *Group Theory and Three-dimensional Manifolds*, Yale University Press, New Haven–London 1971.

[S] E. H. Spanier, *Algebraic Topology*, McGraw-Hill, New York–San Francisco 1966.

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