Peak reductions and waist reflection functors

by

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In honour of Professor Hiroyuki Tachikawa
on the occasion of his 60-th birthday

Abstract. We develop methods for reducing arbitrary bimodule matrix problems to the study of representations of partially ordered sets and socle projective modules over right peak rings.

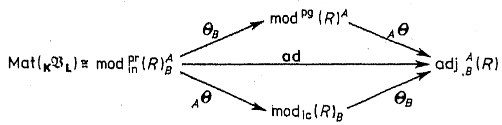
1. Introduction. An important role in the representation theory of finite-dimensional algebras is played by bimodule matrix problems in the sense of Roiter [14] and Drozd [3], i.e. the classification of indecomposables in the category $\mathrm{Mat}({}_{\mathbf{K}}\mathfrak{B}_{\mathbf{L}})$ of matrices over a **K**-**L**-bimodule $\mathfrak{B}$: $\mathbf{K}^{\mathrm{op}} \times \mathbf{L} \to \mathscr{A}b$. If either **K** or **L** is the category of finite-dimensional vector spaces over a division ring then the study of $\mathrm{Mat}({}_{\mathbf{K}}\mathfrak{B}_{\mathbf{L}})$ is equivalent to the study of the subspace category of a vector space category [12, 20] and to the study of the category $\mathrm{mod}_{\mathrm{sp}}(R)$ of finitely generated right socle projective R-modules over a right peak ring R [17]. In this case there is a well-developed theory and there are criteria for determining the representation type [6, 7, 9, 17].

The main aim of this paper is to study bimodule matrix problems reducible to the study of $\mathrm{mod}_{\mathrm{sp}}(R)$, where R is a right peak ring. Our reductions are given in terms of modules over **B**-*traced* rings [20], i.e. semiperfect rings (in general without identity) of the form

(1.1)
$$R = \begin{pmatrix} A & {}_A N_B \\ 0 & B \end{pmatrix},$$

where $A = \bigoplus_{i \in I_A} e_i A$, $B = \bigoplus_{j_* \in I_B} e_{j_*} B$ are semiperfect rings with complete sets of orthogonal primitive idempotents e_i, $i \in I_A$, e_{j_*}, $j_* \in I_B$, ${}_A N_B$ is an A-B-bimodule and $e_i N_B$ is finitely cogenerated for every $i \in I_A$ in the sense that $\mathrm{soc}(e_i N_B)$ is a finitely generated essential submodule of $e_i N_B$. It is proved in [20] that if B is narrow then there are a traced ring R and a commutative diagram

(1.2)



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of full subcategories of $\mathrm{Mod}(R)$ connected by full dense functors having small kernels and preserving the representation type (see also [11]).

Let us recall from [16, 19, 20] that a right R-module

$$X_R = (X'_A, X''_B, \varphi)$$

is *injectively finitely cogenerated* (i.e. X_R is in $\mathrm{mod}_{ic}(R)_B$) if X'_A is in the category $\mathrm{mod}(A)$ of finitely generated right A-modules, X''_B is in the category $\mathrm{inj}(B)$ consisting of injectives in the category $\mathrm{fcg}(B)$ of finitely cogenerated right B-modules and the map

$$\bar{\varphi}\colon X'_A \to \mathrm{Hom}_B({}_AN_B, X''_B)$$

adjoint to φ is injective. The module X_R is *adjusted* (i.e. X_R is in $\mathrm{adj}^A_B(R)$) if X'_A is in $\mathrm{mod}(A)$, X''_B is in $\mathrm{fcg}(B)$, φ is surjective and $\bar{\varphi}$ is injective. The module X is in $\mathrm{mod}^{\mathrm{pr}}_{\mathrm{in}}(R)^A_B$ (resp. in $\mathrm{mod}^{\mathrm{pr}}(R)^A$) if $X' \in \mathrm{pr}(A)$ and $X'' \in \mathrm{inj}(B)$ (resp. $X' \in \mathrm{pr}(A)$ and φ is surjective). The remaining definitions can be found in [20].

In Section 2 we consider the case when B is an Auslander sp-algebra of an sp-representation-finite right multipeak ring and e_iN_B is injective for $i \in I_A$. We associate to R a right multipeak ring Ω^+_R in such a way that $\mathrm{mod}_{ic}(R)_B \cong \mathrm{mod}_{sp}(\Omega^+_R)$. Applications of this reduction are given in Section 3.

In Section 4 we associate to any traced ring R having a waist (Definition 4.1) a right peak ring δR and a waist reflection functor

$$\delta\colon \mathrm{adj}^A_B(R) \to \mathrm{mod}_{sp}(\delta R)$$

which is an equivalence of categories (Theorem 4.12). Let us explain the main idea of it by the following simple example.

EXAMPLE 1.3. Let I and J be finite posets with order $\prec$ [1] and denote by $I \lhd J$ the poset disjoint union of I and J with additional relations $i \prec j$ for $i \in I$ and $j \in J$. Let $R = F(I \lhd J)$ be the path algebra of $I \lhd J$ over a division ring F. Then R is traced of the form (1.1) with $A = FI$, $B = FJ$ and ${}_AN_B$ defined by ${}_iN_j = F$ for all $i \in I$ and $j \in J$. It is easy to see that $\mathrm{Mat}({}_AN_B)$ can be interpreted as the category of matrix representations of the pair I, J [6, 16] and $\mathrm{Mat}({}_AN_B) \cong \mathrm{mod}^{\mathrm{pr}}_{\mathrm{in}}(R)^A_B$. The categories $\mathrm{Mat}({}_AN_B)$ and $\mathrm{adj}^A_B(R)$ have the same representation type and the functor **ad** vanishes only on finitely many objects of the form $(P, 0, 0)$, $(0, Q, 0)$ (see [16; Prop. B7.2, Lemma B7.16] and [20; 5.5, 5.14, 5.17] for details).

Suppose that I has a unique maximal element u and J has a unique minimal element v. Then $I \lhd J$ has the form of Fig. 1 and an adjusted R-module X_R can be identified with a system $X_R = (X_t, {}_s\varphi_t)$ of finite-dimensional F-vector spaces X_t, $t \in I \lhd J$, connected by F-linear maps ${}_s\varphi_t\colon X_t \to X_s$ for all $t \prec s$ in $I \lhd J$ such that ${}_r\varphi_s\,{}_s\varphi_t = {}_r\varphi_t$ for $t \prec s \prec r$, ${}_v\varphi_i$ is injective and ${}_j\varphi_u$ is surjective for all $i \in I$ and $j \in J$.

Let $\delta(I, J)$ be the poset disjoint union of $I - \{u\}$ and $J - \{v\}$. Then the reflection functor

$$\delta\colon \mathrm{adj}^A_B(R) \to \delta(I, J)\text{-sp}$$

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[1] We suppose that $i \prec j$ and $j \prec i$ implies $i = j$.

defined by $\delta(X_R) = (M, M_t)$, where $M = X_v$, $M_i = \mathrm{Im}\,{}_v\varphi_i$ for $i \in I - \{u\}$ and $M_j = \mathrm{Ker}\,{}_j\varphi_v$ for $j \in J - \{v\}$ is an equivalence of categories. In the case I has more than one maximal element the formula above defines a functor $\delta'\colon \mathrm{adj}^A_B(R) \to (I \,\dot\cup\, J - \{v\})\text{-sp}$ which is full, faithful and $\mathrm{Im}\,\delta'$ consists of spaces (M, M_t) such that $M = \sum_{i \in I} M_i$.

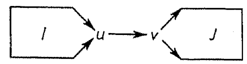


Fig. 1

Applications of the waist reflection functor δ are discussed in Section 5. We show in 5.1 that the study of representations of a pair of posets with zero relations is reduced to the study of I-spaces. In 5.3 a new explanation of the differentiation algorithm for posets [10, 16] and for right peak rings [17, 19] is given in terms of the waist reflection functor. Moreover, we describe in 5.2 a peak triangular reduction for right multipeak path algebras Λ. This allows us to reduce (under some assumptions) the study of $\mathrm{mod}_{sp}(\Lambda)$ to the study of I-spaces (Proposition 5.2d). The reduction can be considered as a counterpart of the differentiation procedure for right multipeak path algebras Λ. It allows us to determine the representation type [1] of $\mathrm{mod}_{sp}(\Lambda)$ for a large class of right multipeak algebras Λ which appear mainly in the study of nonschurian vector space categories $\mathbf{K}_F$ by applying the covering technique as follows. Given such a vector space F-category $\mathbf{K}_F$ we can construct a universal Galois covering

$$f\colon (\tilde{Q}, \tilde{\Omega}) \to (Q, \Omega)$$

of the bound quiver $F(Q, \Omega)$ of the right peak F-algebra $R = R_{\mathbf{K}}$ of $\mathbf{K}_F$ [17] and the induced push-down functor

$$f_\lambda\colon \mathrm{mod}_{sp} F(\tilde{Q}, \tilde{\Omega}) \to \mathrm{mod}_{sp}(R)$$

preserves the representation type [18, 22, 23, 25]. The bound quiver algebra $F(\tilde{Q}, \tilde{\Omega})$ is a multipeak one [18] and in order to determine the representation type of $\mathbf{K}_F$ it is sufficient to determine the type of $\mathrm{mod}_{sp} F(\tilde{Q}, \tilde{\Omega})$. Our reduction in 5.2 allows us to do this for a wide class of vector space categories studied in [21–25]. Let us illustrate the procedure above by the following example.

EXAMPLE 1.3'. Consider a bipartite stratified poset $I^*_\varrho = (I^*, \varrho)$, in the sense of [22, 23], where I^* is the poset of Fig. 2 and ϱ is a relation on pairs (i, j), $i \prec j$, defined by

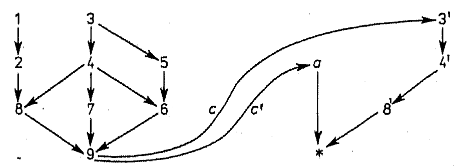


Fig. 2

$(i, i)\varrho(i', i')$, $i = 3, 4, 8$ $(3, 4)\varrho(3', 4')$, $(4, 8)\varrho(4', 8')$ and $(3, 8)\varrho(3', 8')$. We are going to determine the representation type of $\mathrm{mod}_{sp}(R)$, where R is the right peak incidence matrix F-algebra [22, 23]

$$FI_{\varrho}^{*} = \{\lambda = (\lambda_{pq}) \in FI^{*};\ \lambda_{pq} = \lambda_{st} \ \text{if}\,(p, q)\varrho(s, t)\}$$

of I_{ϱ}^{*}. Here FI^{*} is the incidence algebra of the poset I^{*} [7, 16, 22, 23] and F is an algebraically closed field. It follows from [22–25] that $R \cong F(Q, \Omega)$, where Q is the quiver of Fig. 3 and Ω is the ideal in FQ generated by the marked commutativity and zero relations, and by zero relations of the form $c\beta c'$, $c\gamma c$ with paths β, γ (see [23; p. 7]).

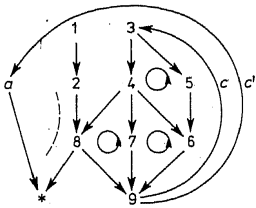


Fig. 3

Moreover, the bound quiver $(\tilde{Q}, \tilde{\Omega})$ with $\tilde{Q}$ as in Fig. 4 and $\tilde{\Omega} = \Omega$ is the universal cover of (Q, Ω) with Galois group $G = \mathbf{Z}$, the induced push down functor f_{λ} preserves the representation type and by [24] the support of any indecomposable module in $\mathrm{mod}_{sp} F(\tilde{Q}, \tilde{\Omega})$ is a full peak subquiver of the two-peak poset of Fig. 5 with the marked zero relation. By applying our waist reflection functor and the peak reduction in 5.2 we show in Example 5.2 d that $\mathrm{mod}_{sp}(FI_{\varrho}^{*+})$ is of tame type by reducing the problem to I'-spaces and by applying the criterion of Nazarova [9]. It follows that $\mathrm{mod}_{sp}(R)$ is of tame type.

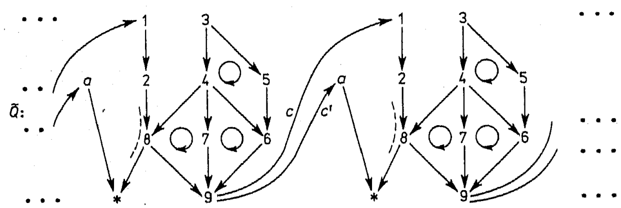


Fig. 4

The waist reflection functors and peak triangular reductions are frequently used in [21–26].

Throughout this paper we suppose that R is a basic semiperfect ring of the form (1.1) and we fix complete sets of primitive orthogonal idempotents e_i, $i \in I_A$, and

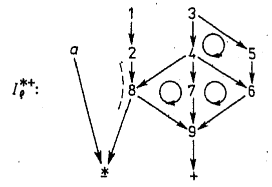


Fig. 5

e_{j_*}, $j_* \in I_B$, of A and B. It follows that R has the following $I_R \times I_R$-matrix form:

$$(1.4) \qquad R = \left[\begin{array}{cc|cc} \cdots A_i \cdots {}_iA_j \cdots & & \cdots {}_iN_{s_*} \cdots {}_iN_{t_*} \cdots & \\ \cdots {}_jA_i \cdots A_j \cdots & & \cdots {}_jN_{s_*} \cdots {}_jN_{t_*} \cdots & \\ \hline & & \cdots B_{s_*} \cdots {}_{s_*}B_{t_*} \cdots & \\ 0 & & {}_{t_*}B_{s_*} \cdots B_{t_*} \cdots & \end{array}\right],$$

where $I_R = I_A \dot{\cup} I_B$, $A_i = e_iAe_i$ and $B_{t_*} = e_{t_*}Be_{t_*}$ are local rings and ${}_i\bar{A}_i = J(A_i)$, ${}_iA_j = e_iAe_j$ for $i \neq j$ in I_A, ${}_{t_*}\bar{B}_{t_*} = J(B_{t_*})$, ${}_{t_*}B_{s_*} = e_{t_*}Be_{s_*}$ for $s_* \neq t_*$ in I_B and ${}_iN_{t_*} = e_iNe_{t_*}$ for $i \in I_A$, $t_* \in I_B$. We shall also write ${}_iN_B = e_iN_B$, ${}_AN_{t_*} = {}_ANe_{t_*}$.

The matrices in (1.4) are assumed to have only finitely many entries different from zero and the multiplication in R is induced by bimodule maps

$$c_{ijk}\colon {}_iA_j \otimes {}_jA_k \to {}_iA_k, \qquad c_{s_*t_*r_*}\colon {}_{s_*}B_{t_*} \otimes {}_{t_*}B_{r_*} \to {}_{s_*}B_{r_*},$$

$$c_{ijt_*}\colon {}_iA_j \otimes {}_jN_{t_*} \to {}_iN_{t_*}, \qquad c_{it_*s_*}\colon {}_iN_{t_*} \otimes {}_{t_*}B_{s_*} \to {}_iN_{s_*}$$

satisfying obvious associativity conditions. Here we use the terminology and notations introduced in [20, 17]. In particular, we call R *faithfully right B-traced* if R is B-traced and ${}_AN$ is faithful. R is left A-traced if R^{op} is right A-traced. We denote by $(I_R, \boldsymbol{d})$ the value scheme of R (see [20; p. 10]).

R is a *right peak* (resp. *multipeak*) ring with peak idempotent(s) e_{t_*}, $t_* \in I_B$, if B is a division ring (resp. a product of division rings) and R is faithfully right B-traced [20, 17, 18]. In this case $\mathrm{mod}_{lc}(R)_B$ is the category $\mathrm{mod}_{sp}(R)$ of finitely generated *socle projective* right R-modules. If R is a left multipeak ring (i.e. R^{op} is right multipeak) then $\mathrm{mod}^{pg}(R)^A$ is the category $\mathrm{mod}_{ti}(R)$ of finitely generated *top injective* right R-modules which are finitely cogenerated [20, 18].

By an R-module we mean a right R-module X such that $XR = X$. We denote by $J(R)$ the Jacobson radical of R and we put $J(X) = XJ(R)$. We denote by $\mathrm{pr}(R)$ the category of finitely generated projective right R-modules. By a *PI-ring* we mean a ring satisfying a polynomial identity. We recall that an artinian basic ring R is a PI-ring if and only if any division factor ring of $R/J(R)$ is finite-dimensional over its center. It follows from [15] that if T is an artinian PI-ring with identity then there is a *Morita duality*

(1.5) $$D\colon \operatorname{mod}(T^{\mathrm{op}}) \to \operatorname{mod}(\tilde{T})^{\mathrm{op}},$$

where $\tilde{T} = \operatorname{End}(\mathfrak{E}_T)^{\mathrm{op}}$ and $\mathfrak{E}_T$ is a minimal injective cogenerator in $\operatorname{mod}(T^{\mathrm{op}})$. It derives the *Nakayama equivalence*

(1.6) $$\mathfrak{n}\colon \operatorname{pr}(T) \to \operatorname{inj}(\tilde{T})$$

defined by the formula $\mathfrak{n}(P) = D\operatorname{Hom}_T(P, T)$ [20; Sec. 1].

2. Simply sp-reducible traced rings. Following [7; Section 6] we associate to any sp-representation-finite artinian right multipeak ring S its *Auslander* sp-*ring*

(2.1) $$A_{\mathrm{sp}}(S) = \operatorname{End}(Y_0 \oplus \ldots \oplus Y_m)$$

where $Y_0, \ldots, Y_m$ is a complete set of pairwise nonisomorphic indecomposable modules in $\operatorname{mod}_{\mathrm{sp}}(S)$. We know from [7] that if, in addition, S is a right peak ring then $A_{\mathrm{sp}}(S)$ is an artinian, both left and right peak ring and $\operatorname{gl.dim} A_{\mathrm{sp}}(S) \leqslant 2$. A ring B will be called an *Auslander* sp-*ring* if $B \cong A_{\mathrm{sp}}(S)$ for some sp-representation-finite right multipeak artinian ring S.

DEFINITION 2.2. A faithfully right B-traced ring R (1.1) is *simply* sp-*reducible* if $e_i N_B$ is in $\operatorname{inj}(B)$ for all $i \in I_A$ and B is an artinian PI-ring which is Morita dual to $A_{\mathrm{sp}}(S)^{\mathrm{op}}$ for some sp-representation-finite right multipeak ring S in the sense that there is a duality

(2.3) $$D\colon \operatorname{mod}(A_{\mathrm{sp}}(S)^{\mathrm{op}}) \to (\operatorname{mod}(B))^{\mathrm{op}}.$$

In this case we have a composed equivalence

(2.4) $$\operatorname{mod}_{\mathrm{sp}}(S) \xrightarrow{\omega} \operatorname{pr}(A_{\mathrm{sp}}(S)) \xrightarrow{\mathfrak{n}} \operatorname{inj}(B),$$

where $\mathfrak{n}$ is the Nakayama equivalence (1.6) and ω is the Yoneda equivalence given by the formula $\omega(X) = \operatorname{Hom}_S(Y_0 \oplus \ldots \oplus Y_m, X)$.

It is easy to see that if S is a right peak ring with a peak P_* then B is a right peak ring and

(2.5) $$\mathfrak{n}\omega(P_*) \cong E_B(P'_*), \qquad \mathfrak{n}\omega(E_S(P_*)) \cong P'_*,$$

where $E_B(P'_*)$ is the B-injective envelope of the right peak P'_* of B.

If R is a simply sp-reducible faithfully B-traced ring then under the notation above we associate to R the ring

(2.6) $$\Omega_R = \begin{pmatrix} A & {}_AM_S \\ 0 & S \end{pmatrix}$$

where ${}_AM_S = \bigoplus_{i \in I_A} (\mathfrak{n}\omega)^{-1}(e_i N_B)$ and the left A-action on M_S is induced by the left multiplication of elements $e_j a e_i$ on $e_i N_B$. Since S is a right multipeak ring, it has a faithfully right F-traced form

$$S = \begin{pmatrix} T & {}_TN''_F \\ 0 & F \end{pmatrix},$$

where F is a product of division rings. If ${}_AM_S = {}_AM'_T \oplus {}_AM''_F$ is the corresponding decomposition of M then

(2.7) $$\Omega_R = \begin{bmatrix} A & {}_AM'_T & {}_AM''_F \\ & T & {}_TN''_F \\ & & F \end{bmatrix} = \begin{pmatrix} A' & {}_{A'}K_F \\ 0 & F \end{pmatrix}$$

where $A' = \begin{pmatrix} A & M' \\ 0 & T \end{pmatrix}$, ${}_{A'}K_F = \begin{bmatrix} M'' \\ N'' \end{bmatrix}$. If $\bar{A} = A'/\{a \in A';\ aK = 0\}$, then the ring

(2.8) $$\Omega_R^+ = \begin{pmatrix} \bar{A} & {}_{\bar{A}}K_F \\ 0 & F \end{pmatrix}$$

is called an sp-*reduced form of* R.

THEOREM 2.9. *Let R be faithfully right B-traced ring of the form* (1.1) *which is simply* sp-*reducible. Then*

(a) *The* sp-*reduced form Ω_R^+ of R is a right multipeak ring and there exists an* sp-*reduction functor*

(2.10) $$\mathfrak{n}_+\colon \operatorname{mod}_{\mathrm{ic}}(R)_B \to \operatorname{mod}_{\mathrm{sp}}(\Omega_R^+)$$

which is an equivalence of categories.

(b) *If B is a right peak ring then Ω_R^+ is a right peak ring. If, in addition, ${}_iN_B \cong E_B(P'_*)^{t_i}$ then ${}_iM_S \cong P_*^{t_i}$ for $i \in I_A$, where P'_* and P_* are peaks of B and S. If this holds for all $i \in I_A$ then ${}_AM'_T = 0$ and $\Omega_R^+ = \Omega_R$.*

Proof. Since ${}_{\bar{A}}K$ is faithful and Ω_R is obviously right S-traced, Ω_R^+ is a right multipeak ring. Furthermore, we know from [20; Lemma 4.15] that an Ω_R-module $Y = (Y'_A, Y''_S, \psi)$ is in $\operatorname{mod}_{\mathrm{ic}}(\Omega_R)_F$ if and only if Y''_S is socle projective and the map $\bar{\psi}\colon Y'_A \to \operatorname{Hom}_S({}_AM_S, Y''_S)$ adjoint to ψ is injective. Let us define a functor

$$\mathfrak{n}'_+\colon \operatorname{mod}_{\mathrm{ic}}(R)_B \to \operatorname{mod}_{\mathrm{ic}}(\Omega_R)_B$$

by the formula $\mathfrak{n}'_+(X'_A, X''_B, \varphi) = (X'_A, (\mathfrak{n}\omega)^{-1}X''_B, \varphi')$ (see (2.4)), where $\varphi'\colon X' \otimes_A M_S \to (\mathfrak{n}\omega)^{-1}X''_B$ is the map adjoint to the composition

$$X'_A \xrightarrow{\bar{\varphi}} \operatorname{Hom}_B({}_AN_B, X''_B) \overset{\sigma}{\cong} \operatorname{Hom}_S({}_AM_S, (\mathfrak{n}\omega)^{-1}X''_B)$$

and $\sigma^{-1}(f_i)_{i \in I_A} = (\mathfrak{n}\omega(f_i))_{i \in I_A}$. The functor $\mathfrak{n}'_+$ is defined on maps in a natural way. Since (2.4) is an equivalence, by the remark above X_R is in $\operatorname{mod}_{\mathrm{ic}}(R)_B$ iff $\mathfrak{n}'_+(X_R)$ is in $\operatorname{mod}_{\mathrm{ic}}(\Omega_R)_F$ and therefore $\mathfrak{n}'_+$ is an equivalence of categories. Since in view of [20; Lemma 2.8] we have $\operatorname{mod}_{\mathrm{ic}}(\Omega_R)_F = \operatorname{mod}_{\mathrm{sp}}(\Omega_R^+)$, $\mathfrak{n}'_+$ defines the required functor $\mathfrak{n}_+$ and (a) follows. Since the statement (b) is an easy consequence of definitions and (2.5), the theorem is proved.

COROLLARY 2.11. *If the ring B in* (1.1) *has the form $T = T_1 \times \ldots \times T_r$, where*

(2.12) $$T_j = \left.\begin{bmatrix} F & F & \cdots & F \\ & F & \cdots & F \\ & & \ddots & \vdots \\ & & & F \end{bmatrix}\right\} n_j$$

with division ring F *and* ${}_iN_B$ *is injective for any* $i \in I_A$ *then* R *is simply* sp-*reducible with* $S = T$, $\mathrm{mod}_{\mathrm{sp}}(T) = \mathrm{pr}(T)$, $\pi\omega$: $\mathrm{pr}(T) \to \mathrm{inj}(T)$ *carries over the* T_j-*module* $(\underbrace{0, \ldots, 0}_{i-1}, F, \ldots, F)$ *into* $(\underbrace{F, \ldots, F}_{i}, 0, \ldots, 0)$ *and*

$$\Omega_R = \begin{pmatrix} A & {}_AM_T \\ 0 & T \end{pmatrix}.$$

If in addition ${}_iN_B = E_{T_1}(P'_*)^{s_{i1}} \oplus \ldots \oplus E_{T_t}(P'_*)^{s_{it}}$ *for all* $i \in I_A$ *then* $\Omega_R^+ = \Omega_R$ *and* ${}_iMT_j = (0, \ldots, 0, F)^{s_{ij}}$ *for all* $i \in I_A$ *and* $j = 1, \ldots, t$.

Proof. It is well known that any indecomposable in $\mathrm{mod}_{\mathrm{sp}}(T)$ is projective and therefore $A_{\mathrm{sp}}(T) \cong T$. Hence R is simply sp-reducible and applying (2.5), Theorem 2.9 and the definition of $\pi\omega$ we get the required results.

As a consequence we have the following reduction of Kleiner [6]

COROLLARY 2.13. *Let* I *and* J *be finite posets and let*

$$R = F(I \lhd J) = \begin{pmatrix} FI & N \\ 0 & FJ \end{pmatrix}$$

be as in Example 1.3. *If* J *is linearly ordered and* m *is the maximal element in* J *then* R *is simply* sp-*reducible,* $\Omega_R^+ \cong FL$ *where* L *is the poset disjoint union of* I *and* J *with additional relations* $i \prec m$ *for* $i \in I$ *and* π_+ *induces an equivalence of categories*

$$\mathrm{mod}_{\mathrm{ic}}(R)_{FJ} \cong (L - \{m\})\text{-sp}, \quad \mathrm{adj}(R) \cong (L - \{m\})\text{-sp}/[(J - \{m\})\text{-sp}].$$

Remark 2.14. One can generalize Corollary 2.13 as follows. Replace J by a disjoint union H of linearly ordered sets $J_1, \ldots, J_t$ with maximal elements $m_1, \ldots, m_t$ and let U be the disjoint union of the posets $I, J_1, \ldots, J_t$ with additional relations $i \prec m_1, \ldots, i \prec m_t$ for $i \in I$. Then $\Omega_R^+ \cong FU$ and π_+ induces an equivalence of categories $\mathrm{mod}_{\mathrm{ic}}(F(I \lhd H))_{FH} \cong \mathrm{mod}_{\mathrm{sp}}(FU)$ and the right hand category consists of F-linear representations of U with the socle concentrated over the unique maximal elements $m_1, \ldots, m_t$ of U.

EXAMPLE 2.15. Let R be the path F-algebra of the commutative quiver of Fig. 6, i.e.

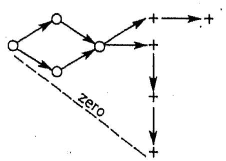


Fig. 6

$$R = \begin{bmatrix} F & F & F & F & F & F & 0 & F & F \\ & F & 0 & F & F & F & F & F & F \\ & & F & F & F & F & F & F & F \\ & & & F & F & F & F & F & F \\ & & & & F & F & F & 0 & 0 \\ & & & & & F & F & 0 & 0 \\ & & & & & & F & 0 & 0 \\ & & & & & & & F & F \\ & & & & & & & & F \end{bmatrix}.$$

Then R is simply sp-reducible and Ω_R^+ is given by the poset of Fig. 7, i.e.

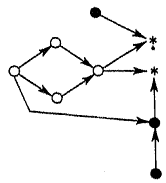


Fig. 7

$$\Omega_R^+ = \begin{bmatrix} F & F & F & F & 0 & F & F & 0 & F \\ & F & 0 & F & 0 & 0 & F & 0 & F \\ & & F & F & 0 & 0 & F & 0 & F \\ & & & F & 0 & 0 & F & 0 & F \\ & & & & F & F & F & 0 & 0 \\ & & & & & F & F & 0 & 0 \\ & & & & & & F & 0 & 0 \\ & & & & & & & F & F \\ & & & & & & & & F \end{bmatrix}.$$

Let us show also a typical indirect application of Theorem 2.9.

EXAMPLE 2.16. Let $I \lhd J$ be the poset of Fig. 8 where J consists of $+$-points and consider the FJ-traced ring $F(I \lhd J)$. We claim that $\mathrm{mod}_{\mathrm{ic}}(R)_{FJ}$ is of tame type [1].

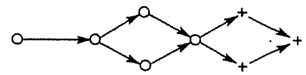
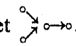


Fig. 8

To see this, we note first that one can apply Theorem 2.9 to the B-traced algebra

$$R^{\mathrm{op}} = \begin{pmatrix} A & N \\ 0 & B \end{pmatrix}$$

with $A = FJ^{\mathrm{op}}$ and $B = FI^{\mathrm{op}}$ because ${}_jN_B \cong E(P'_*)$ for all $j \in J$ and $B \cong A_{\mathrm{sp}}(S)$ where $S = FH$ and H is the poset . It follows from Theorem 2.9 that $\Omega_{R^{\mathrm{op}}}^+$ is the path algebra of the poset of Fig. 9. Thus $\mathrm{mod}_{\mathrm{ic}}(R^{\mathrm{op}})_B \cong \mathrm{mod}_{\mathrm{sp}}(\Omega_{R^{\mathrm{op}}}^+) \cong U$-sp, where U is the

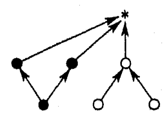
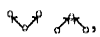


Fig. 9

poset , which according to [9] is of tame type. Since in view of (1.2) and a duality D we have a diagram

$$\begin{array}{ccc} \mathrm{mod}_{\mathrm{ic}}(R)_{FJ} & & \mathrm{mod}_{\mathrm{ic}}(R^{\mathrm{op}})_B \\ \downarrow \theta & & \downarrow \theta \\ \mathrm{adj}(R) & \overset{D}{\underset{\simeq}{\to}} & \mathrm{adj}(R^{\mathrm{op}}) \end{array}$$

and the functors θ preserve the representation type, our claim follows.

Together with sp-reduction an important role in applications is played by a ti-reduction defined below.

DEFINITION 2.17. Let

$$R=\begin{pmatrix} A & N \\ 0 & B \end{pmatrix}$$

be a faithfully right B-traced Artin algebra. We call R *simply* ti-*reducible* if ${}_AN$ is injective and $A \cong A_{\mathrm{sp}}(S)$ where S is an sp-representation-finite right multipeak algebra.

If R is simply ti-reducible we have a commutative diagram

(2.18)
$$\begin{array}{ccccc} \mathrm{mod}_{\mathrm{ti}}(S^{\nabla}) & \xrightarrow{\nabla_+} & \mathrm{mod}_{\mathrm{sp}}(S) & \xrightarrow{\omega} & \mathrm{pr}(A) \\ \downarrow D & & \downarrow D & & \downarrow D \\ \mathrm{mod}_{\mathrm{sp}}(S^{\nabla\mathrm{op}}) & \xrightarrow{\nabla_-} & \mathrm{mod}_{\mathrm{ti}}(S^{\mathrm{op}}) & \xrightarrow{\omega'} & \mathrm{inj}(A^{\mathrm{op}}), \end{array}$$

where D is the standard duality and ω, ω' are the Yoneda equivalences and ∇_+, ∇_- are the reflection functors defined in [20; Definition 2.13], [17; Proposition 2.6]. We put $\bar{\omega}=\bar{\omega}\nabla_+$, $\bar{\omega}'=\omega'\nabla_-$.

We associate to R a B-traced algebra

(2.19)
$${}_R\Omega=\begin{pmatrix} S^{\nabla} & M \\ 0 & B \end{pmatrix},$$

where ${}_{S^{\nabla}}M_B=\bar{\omega}'^{-1}({}_AN)$ with the natural B-module structure. Since S is a right multipeak algebra, S and S^{∇} have the forms

$$S=\begin{pmatrix} T & {}_TN'_F \\ 0 & F \end{pmatrix}, \quad S^{\nabla}=\begin{pmatrix} F & {}_F\tilde{N}'_T \\ 0 & T \end{pmatrix},$$

where F is a product of division rings, $\tilde{N}'=D({}_FN'_T)$ and $\mathrm{soc}(S_S)$ is a direct sum of summands of $(0, F)$. Consider F-traced algebras

(2.20)
$${}_R\Omega=\begin{bmatrix} F & {}_F\tilde{N}'_T & {}_FM''_B \\ & T & {}_TM'_B \\ & & B \end{bmatrix}, \quad {}_R\Omega^{\Delta}=\begin{bmatrix} T & {}_TM'_B & {}_TN'_F \\ & B & {}_B\tilde{M}''_F \\ & & F \end{bmatrix}=\begin{pmatrix} B' & {}_{B'}K_F \\ 0 & F \end{pmatrix},$$

where ${}_{S^{\nabla}}M_B={}_FM'_B\oplus{}_TM''_B$, $B'=\begin{pmatrix} T & M' \\ 0 & B \end{pmatrix}$, ${}_{B'}K_F=\begin{bmatrix} {}_TN'_F \\ {}_B\tilde{M}''_F \end{bmatrix}$ and $\tilde{M}''=D(M)$. The algebra

(2.21)
$$\Omega_R^-=\begin{pmatrix} \bar{B} & {}_{\bar{B}}K_F \\ 0 & F \end{pmatrix}$$

with $\bar{B}=B'/\{b\in B';\ bK=0\}$ will be called the ti-*reduced form of* R.

THEOREM 2.22. *Let R be a simply* ti-*reducible Artin algebra. Under the notation above we have*:

(a) *Ω_R^- is a right multipeak Artin algebra and $\bar{\omega}$ induces a* ti-*reduction functor*

(2.23)
$$\bar{\omega}_*\colon \mathrm{mod}^{\mathrm{pg}}(R)^A\to\mathrm{mod}_{\mathrm{sp}}(\Omega_R^-)$$

which is an equivalence of categories. If A is a right peak algebra then Ω_R^- is also a right peak algebra.

(b) *The functor*

$$\chi_S\colon \mathrm{mod}_{\mathrm{sp}}(S)\to\mathrm{mod}_{\mathrm{sp}}(\Omega_R^-)$$

given by $(Y'_T, Y''_F, \psi\colon Y'\otimes_TN'_F\to Y''_F)\mapsto(Y'_T\oplus Y'\otimes_TM'_B, Y''_F, \psi_0)$, where ψ_0 is the composed map

$$Y'\otimes_TN'_F\oplus Y'\otimes_TM'\otimes_BM_F\xrightarrow{(1,c)}Y'\otimes_TN'_F\xrightarrow{\psi}Y''_F$$

is a fully faithful embedding and induces an equivalence

$$\mathrm{mod}_{\mathrm{sp}}(\Omega_R^-)/[\mathrm{Im}\,\chi_S]\cong\mathrm{adj}_B^A(R).$$

Here $c\colon {}_TM'\otimes_B\tilde{M}_F\to{}_TN'_F$ is a T-F-bimodule map defining the multiplication in ${}_R\Omega^{\Delta}$.

Proof. (a) It is not difficult to check that the diagram (2.18) induces a natural isomorphism

$$\bar{\omega}\colon P\otimes_AN_B\to\bar{\omega}^1(P)\otimes_{S^{\nabla}}M_B$$

for any P in $\mathrm{pr}(A)$. As in the proof of Theorem 2.9 we show that the functor $\bar{\omega}_*\colon \mathrm{mod}^{\mathrm{pg}}(R)^A\to\mathrm{mod}^{\mathrm{pg}}({}_R\Omega)^F$ given by the formula $\bar{\omega}_*(X'_A, X''_B, \varphi)=(\bar{\omega}^{-1}(X'_A), X''_B, \varphi\bar{\omega}^{-1})$ with X' in $\mathrm{pr}(A)$ is an equivalence of categories. Since by [20; Lemmas 2.8, 2.14] there are equivalences

$$\mathrm{mod}^{\mathrm{pg}}({}_R\Omega)^F\xrightarrow{\nabla_+}\mathrm{mod}_{\mathrm{ic}}({}_R\Omega^{\Delta})_F=\mathrm{mod}_{\mathrm{ic}}(\Omega_R^-)=\mathrm{mod}_{\mathrm{sp}}(\Omega_R^-),$$

by composing them with $\bar{\omega}_*$ we get the required equivalence.

(b) It is easy to check that an Ω_R^--module Y is in $\mathrm{Im}\,\chi_S$ if and only if $\bar{\omega}_*^{-1}(Y)$ is of the form $(P, 0, 0)$, where P is in $\mathrm{pr}(A)$. Since the functor ${}_A\Theta\colon \mathrm{mod}^{\mathrm{pg}}(R)^A\to\mathrm{adj}_B^A(R)$ is full dense and $\mathrm{Ker}_A\Theta=[(P, 0, 0);\ P\in\mathrm{pr}(A)]$ [20; Corollary 2.10], it follows that $\mathrm{Ker}_A\Theta\bar{\omega}^{-1}=[\mathrm{Im}\,\chi_S]$ and (b) follows.

COROLLARY 2.24. *If R is an Artin algebra* (1.1), *$A\cong T_1\times\ldots\times T_t$, where T_j is of the form* (2.12) *and ${}_AN$ is injective then R is simply* ti-*reducible and ${}_R\Omega$ has the form $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$.*

(b) *If R is as in Corollary* 2.13 *with I and J interchanged then $\bar{\omega}_*$ induces an equivalence of categories* $\mathrm{mod}^{\mathrm{pg}}(F(J\lhd I))\cong(J-\{m\}\,\dot{\cup}\,I)$-sp.

3. A chain reduction for socle projective poset representations. Throughout this Section I will be a finite poset with the set $\max(I)=\{*_1,\ldots,*_t\}$ of all maximal elements. We put $\mathring{I}=I-\max(I)$ and $c^{\nabla}=\{i\in I;\ i\prec c\}$ for $c\in I$.

Given a division ring F the *path F-algebra FI* is a right multipeak ring [18] (with peaks corresponding to $*_1,\ldots,*_t$) or equivalently, FI is faithfully right traced over $B=F(*_1,\ldots,*_t)\cong F\times\ldots\times F$. The category $\mathrm{mod}_{\mathrm{ic}}(FI)_B=\mathrm{mod}_{\mathrm{sp}}(FI)$ of finite-dimensional socle projective right FI-modules is the category of *socle projective F-linear representations of I*, i.e. systems $X=(X_i, {}_j\varphi_i)_{i\prec j}$ of finite-dimensional F-vector spaces X_i connected by F-linear maps ${}_j\varphi_i\colon X_i\to X_j$, $i\prec j$, satisfying the following conditions:

(i) ${}_t\varphi_{jj}\varphi_i = {}_t\varphi_u$ for $i \prec j \prec t$,

(ii) $\bigcap_{j=1}^{+} \operatorname{Ker}_{*j}\varphi_i = 0$ for all $i \in \check{I}$.

If $t = 1$ then $\operatorname{mod}_{sp}(FI) \cong \check{I}$-sp. We are going to show a class of posets I for which the study of $\operatorname{mod}_{sp}(FI)$ is reducible to the study of ξI-spaces for some poset ξI associated to I.

DEFINITION 3.1. A point $c \in I$ is called an *upper chain point* if

(C1) c^{∇} is a chain $C = \{c_1 \to c_2 \to \ldots \to c_r = c\}$.

(C2) $\max c_i^{\Delta} = \max c_j^{\Delta}$ for all $i \neq j$.

(C3) The F-vector space category $\mathbf{H}_F^c = \operatorname{Hom}_{FI_c}(J(P_c), \operatorname{mod}_{sp}(FI_c))$ with $I_c = I - c^{\nabla}$ is of finite poset type, i.e. $\mathbf{H}^c(\bar{y}, \bar{y}) \cong F$ and $\dim_F|\bar{y}| = \dim|\bar{y}|_F = 1$ for every nonzero indecomposable object

$$\bar{y} = \operatorname{Hom}_{FI_c}(J(P_c), y)$$

of $\mathbf{H}^c$. Here P_c is the projective right FI-module having the space F over all points $j \succ c$ and zero otherwise.

If c is an upper chain point we denote by $I(\mathbf{H}^c)$ the finite poset consisting of indecomposable objects $\bar{y}$ in $\mathbf{H}^c$, where y runs through a fixed set of pairwise nonisomorphic representatives of isoclasses of indecomposables in $\operatorname{mod}_{sp}(FI_c)$ and we put

(3.2) $$\bar{x} \prec \bar{y} \Leftrightarrow \mathbf{H}^c(\bar{y}, \bar{x}) \neq 0.$$

We associate to I the poset disjoint union

(3.3) $$\xi_c I = C - \{c\} \,\dot{\cup}\, I(\mathbf{H}^c)$$

with additional relations $i \prec \bar{y} \Leftrightarrow \operatorname{Hom}_{FI_c}(r(P_i), y) = 0$ for $i \in \check{C} = C - \{c\}$ and $\bar{y} \in I(H^c)$, where $r(P_i)$ is the restriction of P_i to $I_c = I - c^{\nabla}$.

THEOREM 3.4. *If F is a division ring and I is a finite poset with an upper chain point c then under the notation above there exists an equivalence of categories*

(3.5) $$\operatorname{mod}_{sp}(FI)/[\operatorname{mod}_{sp}(FI_c)] \cong \xi_c I\text{-sp}/[\check{C}\text{-sp}]$$

which is induced by the functors in (3.11) *below. The categories* $\operatorname{mod}_{sp}(FI)$ *and* $\xi_c I$-sp *have the same representation type and*

(3.6) $$\# \operatorname{mod}_{sp}(FI) = \# \xi_c I\text{-sp} + \# \operatorname{mod}_{sp}(FI_c) - |c^{\nabla}|,$$

where $\#$ *means the number of isoclasses of indecomposable objects and* $I_c = I - c^{\nabla}$.

Proof. Since c is an upper chain point, FI has an induced triangular form

(3.7) $$FI = \begin{pmatrix} FC & L \\ 0 & FI_c \end{pmatrix},$$

where L is the FC-FI_c-bimodule defined by ${}_iL_j = F$ for $i \prec j$, $i \in C$, $j \in I_c$ and ${}_iL_j = 0$ otherwise. Note that ${}_iL_{FI_c} \cong r(P_i)$ for all $i \in C$ and ${}_cL_{FI_c} = J(P_c)$.

Consider the FC-moduled category $\mathbf{H}_{FC} = \operatorname{Hom}_{FI_c}({}_{FC}L_{FI_c}, \operatorname{mod}_{sp}(FI_c))$ [20]. Note

that $c_i \prec c_j$ induces an FI_c-embedding ${}_{c_i}L_{FI_c} \supsetneq {}_{c_j}L_{FI_c}$ and therefore the left FC-module ${}_{FC}L$ is the system of F-FI_c-bimodule embeddings

(3.8) $${}_{c_1}L_{FI_c} \supsetneq {}_{c_2}L_{FI_c} \supsetneq \ldots \supsetneq {}_cL_{FI_c}.$$

In view of (C2) the FI_c-modules ${}_{c_j}L_{FI_c}$ have equal socles and therefore given an indecomposable module y in $\operatorname{mod}_{sp}(FI_c)$ we have

$$\bar{y} =: \operatorname{Hom}_{FI_c}({}_{FC}L_{FI_c}, y) \neq 0 \quad \text{if and only if} \quad y \neq 0,$$

or equivalently if $\bar{y} \in J := I(\mathbf{H}^c)$. Moreover, given $\bar{y}$ in J the right FC-module $|\bar{y}|_{FC}$ is the system induced by (3.8)

(3.9) $$\operatorname{Hom}_{FI_c}({}_{c_1}L, y) \subsetneq \operatorname{Hom}_{FI_c}({}_{c_2}L, y) \subsetneq \ldots \subsetneq \operatorname{Hom}_{FI_c}({}_cL, y) = \bar{y}$$

of at most one-dimensional subspaces of $\bar{y}$. It follows that $|\bar{y}|_{FC}$ is FC-projective and the left FC-module ${}_{FC}\tilde{H} = D(|\bar{Y}|_{FC})$ is injective, where $D(-) = \operatorname{Hom}_F(-, F)$ and $\bar{Y} = \bigoplus_{\bar{y} \in J} \bar{y}$. The left action of the F-algebra $E = \mathbf{H}(\bar{Y}, \bar{Y})$ on $|\bar{Y}|_{FC}$ induces a right E-module structure on ${}_{FC}\tilde{H}_E$. By our assumption and remarks above E is isomorphic to the path F-algebra $B = FJ$. Then $\tilde{H}$ is an FC-B-bimodule and the right B-traced F-algebra associated to $\mathbf{H}_{FC}$ [20; 3.14] has the form

(3.10) $$R = T_{\mathbf{H}} = \begin{pmatrix} FC & {}_{FC}\tilde{H}_B \\ 0 & B \end{pmatrix}.$$

R is simply ti-reducible because we know that ${}_{FC}\tilde{H}$ is injective and $\operatorname{mod}_{sp}(FC) = \operatorname{pr}(FC)$ implies $FC \cong A_{sp}(FC)$. Therefore in Definition 2.17 we can take FC for the rings A and S and we have $\omega = \omega' = \mathrm{id}$.

We claim that in our situation the F-algebra ${}_R\Omega^d$ (2.20) is isomorphic to the path F-algebra $F(C \wedge J)$ of the poset $C \wedge J$ obtained from $\xi_c I$ by adding the unique maximal element c. Since $B = FJ$ and $A = FC$, it remains to show that

(*) $$\dim_F({}_{\bar{y}}M_F) \leqslant 1, \quad \dim_F({}_iM'_{\bar{y}}) \leqslant 1$$

(**) $${}_iM'_{\bar{y}} \neq 0 \Leftrightarrow i \prec \bar{y} \text{ in } \xi_c I.$$

For this purpose we fix $\bar{y} \in J$ and consider the left T-module ${}_TM'_{\bar{y}}$, where T is the path F-algebra of $c_1 \to c_2 \to \ldots \to c_{r-1}$ in (2.20). Since $(FC)^{\nabla}$ is the path algebra of $c \to c_1 \to \ldots \to c_{r-1}$ and $|\bar{y}|_{FC}$ is indecomposable projective given by the sequence (3.9) with $\dim|\bar{y}|_F = 1$ (by (C3)), it follows that ${}_{(FC)^{\nabla}}M_{\bar{y}} = \bar{\omega}'^{-1}{}_{FC}\tilde{H}_{\bar{y}} = \bar{\omega}'^{-1}D(|\bar{y}|_{FC})$ is indecomposable in $\operatorname{mod}_{sp}(FC) = \operatorname{pr}(FC)$ and therefore (*) follows. Moreover it follows from the definition of $\bar{\omega}'$ that if $|\bar{y}|_{FC} = (\underbrace{0, \ldots, 0}_{p(\bar{y})}, F, \ldots, F)$ then ${}_{(FC)^{\nabla}}M_{\bar{y}} = (\underbrace{F, \ldots, F}_{p(\bar{y})+1}, 0, \ldots, 0)$.

Since $|\bar{y}|_{FC}$ is given by the sequence (3.9) and ${}_iL_{FI_c} \cong r(P_i)$ for $i \in C$, we have

$$i \prec \bar{y} \Leftrightarrow i \leqslant p(\bar{y}) \Leftrightarrow {}_iM_{\bar{y}} \neq 0 \Leftrightarrow {}_iM'_{\bar{y}} \neq 0$$

and (**) follows. Consequently $\Omega_R^- = {}_R\Omega^d \cong F(C \wedge J)$ is a right peak algebra.

By [20; Theorem 4.20, Corollary 2.10] and Theorem 2.22 there are full dense functors

$$(3.11)\qquad \begin{array}{c} \operatorname{mod}_{sp}(FI) \xrightarrow{\operatorname{adj}_{ic}^{FC}} \operatorname{adj}_B^{FC}(R) \\ \uparrow {}_{FC}\Theta \\ \operatorname{mod}^{pg}(R)^{FC} \xrightarrow[\simeq]{\bar{\omega}} \operatorname{mod}_{sp}(\Omega_{\bar{R}}) \xrightarrow[\approx]{\lambda} \xi_c I\text{-sp} \end{array}$$

preserving the representation type and $\operatorname{Ker}\mathbf{ad}_{ic}^{FC} = [\operatorname{mod}_{sp}(FI_c)]$, $\operatorname{Ker}_{FC}\Theta(\lambda\bar{\omega}_*)^{-1} = [\lambda(\operatorname{Im}\chi_{FC})] = [\check{C}\text{-sp}]$. Then the functors $\mathbf{ad}_{ic}^{FC}$ and ${}_{FC}\Theta(\lambda\bar{\omega})^{-1}$ induce the required equivalence and the proof is complete.

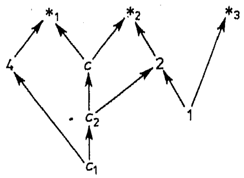


Fig. 10

EXAMPLE 3.12. Let I be the poset of Fig. 10 and let $I' = I - *_1^{\nabla}$. Then the AR-quiver of $\operatorname{mod}_{sp}(FI')$ has the form

$$\begin{array}{c} P_{*_3} \\ \downarrow \\ P_{*_2} \to P_2 \to P_1 \to E(P_{*_2}) := X \\ \downarrow \\ E(P_{*_3}). \end{array}$$

It follows that $\mathbf{H}_F^c$ is of poset type and $I(\mathbf{H}^c) = \left\{\begin{array}{l} \bar{P}_4 \to \bar{P}_{*_1} \\ \bar{X} \to \bar{P}_1 \to \bar{P}_2 \to \bar{P}_{*_2} \end{array}\right\}$, and the ring $R = T_{\mathbf{H}}$ (3.10) is the path algebra of the quiver of Fig. 11 with two zero relations marked by the dotted arrows. Then $\xi_c I$ is the poset of Fig. 12.

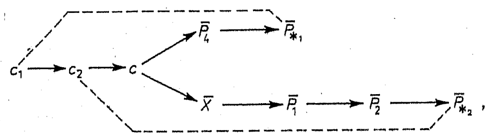


Fig. 11

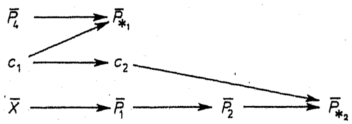


Fig. 12

Since $\xi_c I$ is of finite type by [6], it follows from (3.5) that $\operatorname{mod}_{sp}(FI)$ is of finite type.

COROLLARY 3.13. *Let I be a finite poset with $\max(I) = \{*_1, \ldots, *_t\}$ and let c be a maximal element in $\check{I} = I - \max(I)$ such that c^{∇} is a chain contained in $*_1^{\nabla} \cap \ldots \cap *_t^{\nabla}$ and $I - c^{\nabla}$ is a disjoint union of pairwise unrelated posets $I_1, \ldots, I_t$ of width $\leqslant 2$ and $*_j \in I_j$ for $j = 1, \ldots, t$. Then c is an upper chain point and*

$$(3.14)\qquad \xi_c I \cong (c^{\nabla} - \{c\}) \mathbin{\dot\cup} \hat{I}_1 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} \hat{I}_t$$

where $\hat{I}_j = (I_j - \{*_j\}) \cup \{\bar{*}_j\} \cup \{s \vee t;\ s, t \text{ incomparable in } I_j\}$. The order in I_j extends to the order in $\hat{I}_j$ by adding the following relations:

$$\bar{*}_j \prec u \quad \text{for all } u \in \hat{I}_j,$$
$$u \prec s \vee t \Leftrightarrow u \prec s \text{ or } u \prec t,$$
$$s \vee t \prec u \Leftrightarrow s \prec u \text{ and } t \prec u,$$
$$s \vee t \prec s' \vee t' \Leftrightarrow \text{each of } s, t \text{ is less than or equal to one of } s', t'.$$

The set $\xi_c I$ is equipped with the disjoint union order completed by relations $i \prec s$, whenever $i \in c^{\nabla} - \{c\}$, $s \in \check{I}_1 \cup \ldots \cup \check{I}_t$ and $i \prec s$ in I. Then we have the equivalence (3.5) and the formula (3.6) holds.

Proof. Since $w(I_j) \leqslant 2$, $\operatorname{mod}_{sp}(FI_j) \cong \check{I}_j$-sp is of finite type and every indecomposable in I_j-sp has a simple socle [0, 10]. It follows that the FI_j-modules $Q^{(u)}$, $Q^{(s)} \cap Q^{(t)}$, $u, s, t \in \check{I}_j \cup \{\bar{*}_j\}$, s, t incomparable, furnish all indecomposable $\check{I}_j$-spaces, where $Q^{(u)}$ has zero spaces over all points $j \preccurlyeq u$ and the space F over points $i \in I_j - u^{\nabla}$ [16; Section 6]. They form a poset isomorphic to I_j under the correspondence $Q^{(u)} \mapsto u$, $Q^{(s)} \cap Q^{(t)} \mapsto t \vee s$. Since $J(P_c)$ is the direct sum of simple projective modules $P_{*_1}, \ldots, P_{*_t}$,

$$\mathbf{H}_F^c = \operatorname{Hom}_{FI_1}(P_{*_1}, \check{I}_1\text{-sp}) \times \ldots \times \operatorname{Hom}_{FI_t}(P_{*_t}, \check{I}_t\text{-sp}).$$

By our observation above $\mathbf{H}_F^c$ is of finite poset type and there is a poset isomorphism $I(\mathbf{H}^c) \cong \hat{I}_1 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} \hat{I}_t$. It follows that c is an upper chain point and it remains to show that given s in $(I_j - \{*_j\}) \cup \{\bar{*}_j\}$ and $i \in c^{\nabla} - \{c\}$ we have $i \prec s$ in I_j if and only if $i \prec \bar{Q}^{(s)}$ (in the notation of (3.2)). If $i \prec s$ then $\operatorname{Hom}_{FI_j}(P_i|_{I_j}, Q^{(s)}) = 0$ because P_i has the space F and $Q^{(s)}$ has zero space over the point i. Hence $i \prec \bar{Q}^{(s)}$ by (3.2). Conversely, if $i \prec s$ does not hold then $i \prec t$ for all $t \in I_j - s^{\nabla}$ and by the definition of $Q^{(s)}$ we have $P_i|_{I_j} \subseteq Q^{(s)}$. Then in view of (3.2), $i \prec \bar{Q}^{(s)}$ does not hold. The remaining part follows from Theorem 3.4.

Remark 3.15. (a) Theorem 3.4 remains valid when we replace the condition (**C1**) by the following weaker property of c:

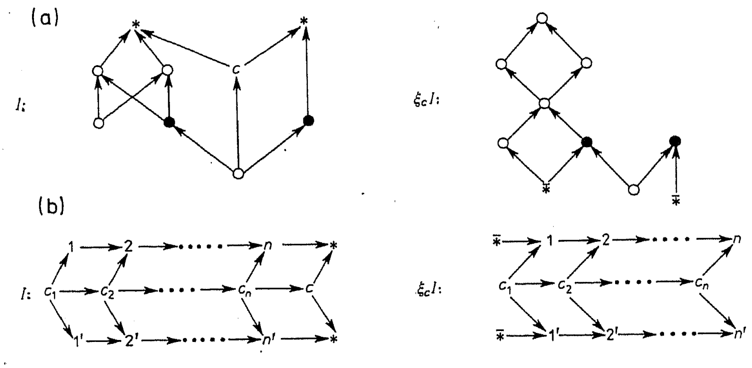
(**C1'**) c^{∇} is dual to the poset C' of all indecomposable J-spaces of some poset J of width $\leqslant 2$ and $|\bar{y}|_{FC}$ is projective for all $\bar{y}$ in $I(\mathbf{H}^c)$.

In this case we consider the subposet $C_0 \cong J$ of C consisting of points corresponding to indecomposable projectives in J-sp and we take for $\xi_c I$ the poset (3.3) with $\check{C}$ and $\check{C}_0$ interchanged. Then there is the equivalence (3.5) with $\check{C}$ and $\check{C}_0$ interchanged, whereas the formula (3.6) remains unchanged.

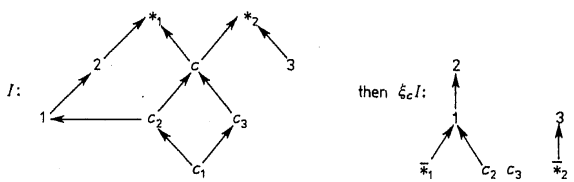
In the case J is a chain we are in our original position of Definition 3.1.

(b) It would be interesting to know how the AR-quiver of $\mathrm{mod}_{sp}(R)$ depends on the AR-quiver of $\mathrm{mod}_{sp}(\Omega_R^-)$.

EXAMPLE 3.16.



EXAMPLE 3.17. If



and the Remark above applies because c^∇ is dual to the poset of indecomposable J-spaces, where $J = \{c_2, c_3\}$.

Remark 3.18. Theorem 3.4 can be extended to an arbitrary multipeak algebra $\hat{A}$ [18; (1.3)] having a smooth upper chain point s in $(I_{\hat A}, \mathbf{d})$ in the following sense:

(S1) $d_{st} = d'_{st} \leqslant 1$ for all $t \in \max(I_{\hat A}) = \{1_*, \dots, r_*\}$.

(S2) c_{jsm} is surjective for all $m \in \max(I_{\hat A})$ and j in s^∇.

(S3) $s^\nabla = \{j \in_{\hat A};\ d_{js} \neq 0\}$ is a homogeneous chain $C = \{s_1 \to s_2 \to \dots \to s_r = s\}$, i.e. $d_{ij} = d'_{ij} = 1$ for $i \neq j$ in C.

(S4) The vector space category $\mathbf{H}^s_F = \mathrm{Hom}_T(J(P_s), \mathrm{mod}_{sp}(T))$ has finitely many indecomposable objects, where $T = \hat{A}_{I_{\hat A} - c}$ [18; 1.13].

In this case $\hat{A} = \begin{pmatrix} S & {}_SL_T \\ 0 & T \end{pmatrix}$, $S = \hat{A}_C$, and the corresponding triangular reduction [20; Theorem 4.20] with $T_{\mathbf{H}} = \begin{pmatrix} S & {}_S\tilde{H}_E \\ 0 & E \end{pmatrix}$, $E = E(\mathbf{H})$, together with Theorem 2.22 applied to $R = T_{\mathbf{H}}$ yields the diagram (3.11) with FI and $\hat{A}$ interchanged and λ omitted. Hence we get an equivalence of categories

$$\mathrm{mod}_{sp}(\hat{A})/[\mathrm{mod}_{sp}(T)] \cong \mathrm{mod}_{sp}(\Omega^-_{T_{\mathbf{H}}})/[\mathrm{mod}_{sp}(S)]. \tag{3.19}$$

COROLLARY 3.20. *Let I be a two-peak poset with $\max(I) = \{*, +\}$ and suppose that the algebra FI is* sp*-representation-finite. Then $C = *^\nabla \cap +^\nabla$ is a chain, $c = \max(C)$ is an upper chain point, there is an equivalence* (3.5) *and the study of $\mathrm{mod}_{sp}(FI)$ reduces to the study of $\xi_c I$-*sp.

Proof. First we note that FL is sp-representation-infinite if L is one of the two-peak posets of Fig. 13. It follows that I does not contain L as a full two-peak subposet and C is a chain. We shall show that the vector space category $\mathbf{H}^c$ in (3.1) is of poset type. For this purpose we note that the posets $*^\nabla$, $+^\nabla$ are sp-representation-finite and therefore indecomposable objects in $*^\nabla$-sp and in $+^\nabla$-sp have endomorphism rings isomorphic to F [6, 7]. Hence $\mathbf{H}^c(\bar{y}, \bar{y}) \cong F$ for all $\bar{y} \in \mathrm{ind}(\mathbf{H}^c)$. This implies that $\dim|\bar{y}|_F = 1$ because otherwise the category $\mathrm{adj}(R)$ in (3.11) contains all representations of $\circ\rightrightarrows\circ$ without simple summands and therefore $\mathrm{mod}_{sp}(FI)$ is of infinite type; a contradiction. Consequently, c is an upper chain point and by Theorem 3.4 the corollary follows.

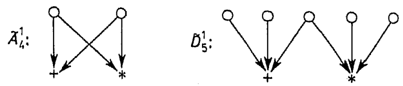


Fig. 13

Remark 3.21. It is shown in [25; Theorem 4.32] that given a two-peak poset I with $\max(I) = \{*, +\}$ the algebra FI is sp-representation-finite if and only if the posets $*^\nabla$, $+^\nabla$ do not contain as full subposets the posets (1, 1, 1, 1), (2, 2, 2), (1, 3, 3), (N, 4), (1, 2, 5) of Kleiner [6, 7] and I does not contain as a full two-peak subposet the forms $\tilde{\mathbf{A}}^1_4$, $\tilde{\mathbf{D}}^1_5$ above and the ones of Fig. 14. This also results from 3.20.

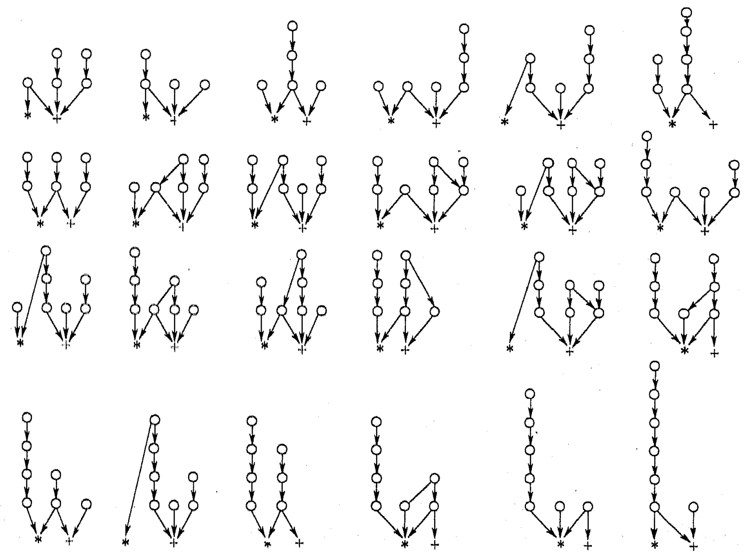


Fig. 14.

4. Waist reflection functors.

Definition 4.1. Let $R = \begin{pmatrix} A & N \\ 0 & B \end{pmatrix}$ be a faithfully right B-traced ring. We say that R *has a right multipeak waist* if

(W1) B is a left multipeak ring with peak idempotents $e_{v_1}, \ldots, e_{v_b}$.

(W2) N_B is a top injective B-module.

We say that R *has a right multiarrow waist* if R has a right multipeak waist and

(W3) A is a right multipeak ring with peak idempotents $e_{u_1}, \ldots, e_{u_a}$.

(W4) ${}_AN_v := {}_ANe_v$ is a top injective A-module, where $e_v = e_{v_1} + \ldots + e_{v_b}$.

(W5) The division rings A_{u_j}, B_{v_t} in (1.4) are isomorphic and the A_{u_j}-B_{v_t}-bimodules are either zero or one-dimensional on each side for all j, t. In the case $a = b = 1$ we say that R *has a right arrow waist* $u \to v$. Finally, we say that R *has a left peak waist* (*multiarrow waist*) if R^{op} has a right peak waist (multiarrow waist).

Lemma 4.2. *R has a right multiarrow waist if and only if R has the matrix form like* (1.4)

$$(4.3)\qquad R = \left[\begin{array}{cccccc|cccc} & \vdots & & \vdots & & & \vdots & & \vdots & \\ \cdots & A_i & \cdots & {}_iA_u & & & {}_iN_v & \cdots & {}_iN_t & \cdots \\ & & \ddots & \vdots & & & \vdots & \ddots & \vdots & \\ 0 & \cdots & 0 & \cdots 0 & A_u & & {}_uN_v & \cdots & {}_uN_t & \cdots \\ \hline & & & & & & B_v & \cdots & {}_vB_t & \cdots \\ & & & & & & 0 & \ddots & \vdots & \\ & & 0 & & & & \vdots & & & \\ & & & & & & 0 & \cdots & {}_rB_t & \cdots \\ & & & & & & \vdots & & \vdots & \end{array}\right],$$

where $A_u = e_uAe_u = A_{u_1} \times \ldots \times A_{u_a}$, $e_u = e_{u_1} + \ldots + e_{u_a}$, $B_v = e_vBe_v = B_{v_1} \times \ldots \times B_{v_b}$, A_{u_i} and B_{v_j} are division rings, ${}_iA_u = e_iAe_u$, ${}_vB_{t_*} = e_vBe_{t_*}$, ${}_\alpha N_\beta = e_\alpha Ne_\beta$, and ${}_uA_j = {}_iB_v = 0$ for all i, j.

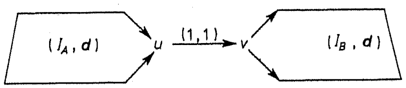
(i) *The maps $\bar{c}_{iju}\colon {}_iA_j \to \mathrm{Hom}({}_jA_u, {}_iA_u)$, $\bar{c}_{vt_*s_*}\colon {}_{t_*}B_{s_*} \to \mathrm{Hom}({}_vB_{t_*}, {}_vB_{s_*})$ adjoint to c_{iju} and $c_{vt_*s_*}$ are injective for all i, j, t_*, s_*.*

(ii) *The lengths $l({}_iA_u)_{A_u}$ and $l_{B_v}({}_vB_{t_*})$ are finite, ${}_{u_i}N_{v_t}$ is either zero or is one-dimensional on each side and $A_{u_i} \cong B_{v_t}$ for all i and t.*

(iii) *The maps $c_{ivt_*}\colon {}_iN_v \otimes {}_vB_{t_*} \to {}_iN_{t_*}$, $c_{iuv}\colon {}_iA_u \otimes {}_uN_v \to {}_iN_v$ are surjective.*

Proof. It follows from [17; Proposition 2.2] that **(W1)** together with **(W3)** and **(W5)** is equivalent to (i)–(ii), whereas by [17; Proposition 2.4], **(W2)** together with **(W4)** is equivalent to (iii).

It follows from Lemma 4.2 that if R has a right arrow waist then the value scheme $(I_R, \boldsymbol{d})$ of R has the form

(4.4) 

The central arrow corresponds to the one-dimensional A_u-A_v-bimodule ${}_uN_v$ which according to Lemma 4.2(iii) generates the A-B-bimodule ${}_AN_B$. This is our motivation

for the name "arrow waist". In the multiarrow case $u \to v$ is replaced by arrows $u_i \to v_t$ corresponding to nonzero bimodules ${}_{u_i}N_{v_t}$.

Now suppose that R has a right multiarrow waist $u \to v$, I_B is finite and B is an artinian PI-ring with identity. Let $B = \begin{pmatrix} B_v & L \\ 0 & B' \end{pmatrix}$ where $L = J(e_vB)$ and $B' = (1_B - e_v)B(1_B - e_v)$ is the ring obtained from the matrix form of B in (4.3) by omitting its vth row and vth column. Set

$$B^\Delta = \begin{pmatrix} B' & {}_{B'}\tilde{L}_{B_v} \\ 0 & B_v \end{pmatrix},$$

where $\tilde{L} = \mathrm{Hom}_{B_v}({}_{B_v}L_{B'}, B_v)$ and let

$$\nabla_+\colon \mathrm{mod}_{\mathrm{ti}}(B) \to \mathrm{mod}_{\mathrm{sp}}(B^\Delta)$$

be the reflection functor in [17; Prop. 2.6], [20; Def. 2.13]. Since ${}_iN_B$ is in $\mathrm{mod}_{\mathrm{ti}}(B)$ then

$$N^\nabla_{B^\Delta} = \bigoplus_{i \in I_A} \nabla_+({}_iN_B)$$

has a natural left A-module structure. We form two rings

$$(4.5)\qquad \delta_vR = \begin{pmatrix} A & {}_AN^\nabla_{B^\Delta} \\ 0 & B^\Delta \end{pmatrix}, \quad \delta_vR = \begin{pmatrix} A' & {}^-N^\nabla \\ 0 & B^\Delta \end{pmatrix}$$

called the *right waist reflection forms of* R, where $A' = (1_A - e_u)A(1_A - e_u)$ and ${}^-N^\nabla = \sum_{i \neq u_1, \ldots, u_a} {}_iN^\nabla$. Note that if R has an arrow waist then the value scheme of δ_vR is obtained from (4.4) by omitting the point u and by reflecting $(I_B, \boldsymbol{d})$ at v as shown in Fig. 15.

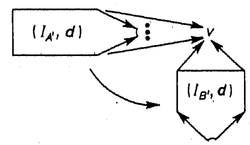


Fig. 15

It follows from the definition of ∇_+ that ${}_AN^\nabla$ has the form $[{}_AN'_{B'}, {}_AN_v]$ and

$$(4.6)\qquad \delta_vR = \begin{pmatrix} A & {}_AN'_{B'} & {}_AN_v \\ & B' & \tilde{L} \\ & & B_v \end{pmatrix}.$$

Then a right δ_vR-module is a system

$$(4.7)\qquad Z = (Z'_A, Z''_{B'}, Z''_v, \psi', \psi'', {}_v\psi),$$

where $\psi'\colon Z' \otimes_A N'_{B'} \to Z''_{B'}$, $\psi''\colon Z'' \otimes \tilde{L} \to Z''_v$, ${}_v\psi\colon Z' \otimes_A N_v \to Z''_v$ satisfy obvious associativity conditions. We call Z *A-complete* (resp. *A_u-complete*) if ${}_v\psi$ (resp. ${}_v\psi_u$) is surjective. Let $\operatorname{mod}^A_{\mathrm{sp}}(\hat{\delta}_v R)$ be the full subcategory of $\operatorname{mod}_{\mathrm{sp}}(\hat{\delta}_v R)$ consisting of A-complete modules.

LEMMA 4.8. *If R has a right multiarrow waist then Z is A-complete if and only if Z is A_u-complete. If, in addition, R has a right arrow waist then the restriction functor*

$$r_\delta\colon \operatorname{mod}^A_{\mathrm{sp}}(\hat{\delta}_v R) \to \operatorname{mod}_{\mathrm{sp}}(\delta_v R) \tag{4.9}$$

defined by the formula $r_\delta(Z) = (\check{Z}'_{A'}, \bar{Z}'', Z''_v, \check{\psi}', \psi'', {}_v\check{\psi})$, where $\check{Z}' = \bigoplus_{j\neq u} \bar{Z}' e_j$ and $\check{\psi}'$, ${}_v\check{\psi}$ are the corresponding restrictions of ψ' and ${}_v\psi$, is an equivalence of categories.

Proof. In order to prove the first statement we consider the commutative diagram

$$\begin{array}{ccc} \bar{Z}'_i \otimes_i A_u \otimes_u N_v & \xrightarrow{{}_u\psi'_i \otimes 1} & \bar{Z}'_u \otimes_u N_v \\ \downarrow{\scriptstyle 1\otimes c_{iuv}} & & \downarrow{\scriptstyle {}_v\psi_u} \\ \bar{Z}'_i \otimes_i N_v & \xrightarrow{{}_v\psi_i} & Z''_v. \end{array}$$

where $1\otimes c_{iuv}$ is surjective by Lemma 4.2 (iii). It follows that $\operatorname{Im}{}_v\psi_i \subseteq \operatorname{Im}{}_v\psi_u$ and hence if Z is A-complete then $Z''_v = \operatorname{Im}{}_v\psi = \operatorname{Im}{}_v\psi_u + \sum_{i\neq u} \operatorname{Im}{}_v\psi_i = \operatorname{Im}{}_v\psi_u$ and therefore Z is A_u-complete. Since the converse implication is trivial, the first statement is proved. Hence, if Z is socle projective and R has a right arrow waist then Z is A-complete if and only if ${}_v\psi_u$ is bijective because it is surjective, ${}_v\bar{\psi}_u$ is injective and ${}_uN_v$ is one-dimensional on each side. It follows that Z is uniquely determined by $r_\delta(Z)$ and therefore r_δ is an equivalence of categories.

Following an idea explained in Example 1.3 we define a *right waist reflection functor*

$$\hat{\delta}_v\colon \operatorname{adj}^A_B(R) \to \operatorname{mod}^A_{\mathrm{sp}}(\hat{\delta}_v R) \tag{4.10}$$

by the formula $\hat{\delta}_v(X_R) = (X'_A, \nabla_+(X''_B), \tilde{\varphi})$, where $\tilde{\varphi}$ is the map adjoint to the composed monomorphism

$$X'_A \xrightarrow{\bar{\varphi}} \operatorname{Hom}_B({}_AN_B, X''_B) \xrightarrow[\sim]{\nabla_+} \operatorname{Hom}_{B^\nabla}(N^\nabla, \nabla_+(X''_B)).$$

Since N_B is top injective, $X' \otimes_A N_B$ is top injective and therefore X''_B is in $\operatorname{mod}_{\mathrm{ti}}(B)$ because it is the image of $X' \otimes_A N_B$ under φ. Moreover, since φ is surjective, ${}_v\varphi$: $X'_A \otimes_A N_v \to X''_v := X'' e_v$ is surjective and therefore the module $Z = \hat{\delta}_v(X_R)$ is A-complete because ${}_v\tilde{\varphi}$ is just the map ${}_v\psi$ in Z (4.7).

If, in addition, R has a right arrow waist then we have a *reduced right waist reflection functor*

$$\delta_v = r_\delta \hat{\delta}_v\colon \operatorname{adj}^A_B(R) \to \operatorname{mod}_{\mathrm{sp}}(\delta_v R) \tag{4.11}$$

THEOREM 4.12. *Let $R = \begin{pmatrix} A & N \\ 0 & B \end{pmatrix}$ be a faithfully right B-traced ring such that B is an artinian PI-ring with identity. If R has a right multipeak waist then the waist reflection forms $\hat{\delta}_v R$ and $\delta_v R$ of R are right multipeak rings and the functor $\hat{\delta}_v$ (4.10) is an equivalence of categories. If, in addition, R has a right arrow waist then δ_v (4.11) is an equivalence too.*

Proof. It is clear that $\hat{\delta}_v$ is full and faithful because ∇_+ is an equivalence of categories. In order to prove that $\hat{\delta}_v$ is dense take Z in $\operatorname{mod}^A_{\mathrm{sp}}(\hat{\delta}_v R)$ of the form (4.7). Then $U = (\bar{Z}''_B, Z''_v, {}_v\psi)$ is in $\operatorname{mod}_{\mathrm{sp}}(B^\Delta)$ and we put $X'_A = \bar{Z}'_A$, $X''_B = \nabla(U)$ and $\bar{\varphi}$ is the composed map

$$\bar{Z}'_A \xrightarrow{(\bar{\psi}', {}_v\bar{\psi})} \operatorname{Hom}_{B^\Delta}({}_AN'_B \oplus {}_AN_v, U) \cong \operatorname{Hom}_B({}_AN_B, X''_B).$$

Since Z is socle projective, $(\bar{\psi}', {}_v\bar{\psi})$ is injective and therefore $\bar{\varphi}$ is injective. In order to prove that X_R is adjusted it remains to show that φ is surjective. To see this, first note that since Z is A-complete, ${}_v\psi$ is surjective. Next, for any $t_* \in I_B$ we consider the commutative diagram

$$\begin{array}{ccc} X'_A \otimes {}_AN_v \otimes {}_vB_t & \xrightarrow{1\otimes c} & X'_A \otimes {}_AN_{t_*} \\ \downarrow{\scriptstyle {}_v\varphi\otimes 1} & & \downarrow{\scriptstyle {}_{t_*}\varphi} \\ X''_v \otimes {}_vB_{t_*} & \xrightarrow{{}_{t_*}\varphi''_v} & X''_{t_*}. \end{array}$$

Since Be_v is a left peak of B and X''_B is top injective, it follows from [17; Prop. 2.4] that ${}_{t_*}\varphi''_v$ is surjective. Moreover, ${}_v\varphi = {}_v\psi$ is surjective. Hence ${}_{t_*}\varphi$ is surjective for any t_* and therefore φ is surjective. Consequently $\hat{\delta}_v$ is dense and it is an equivalence of categories. The remaining statement follows from Lemma 4.8. and [20; Lemma 4.15].

Now suppose that R has a left multipeak waist u, I_A is finite and A is an Artin algebra. Then A has the form

$$A = \begin{pmatrix} A' & K \\ 0 & A_u \end{pmatrix}$$

with $K = J(Ae_u)$ and by [20; Lemma 2.14] there is a commutative diagram

$$\begin{array}{ccc} \operatorname{mod}_{\mathrm{sp}}(A) & \xrightarrow[\sim]{\nabla} & \operatorname{mod}_{\mathrm{ti}}(A^\nabla) \\ \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle D} \\ \operatorname{mod}_{\mathrm{ti}}(A^{\mathrm{op}}) & \xrightarrow[\sim]{\nabla} & \operatorname{mod}_{\mathrm{sp}}(A^{\nabla\mathrm{op}}) \end{array} \tag{4.13}$$

where D is the standard duality and

$$A^\nabla = \begin{pmatrix} A_u & \tilde{K} \\ 0 & A' \end{pmatrix},$$

$\tilde{K} = DK$. Since ${}_AN_{t_*}$ is in $\operatorname{mod}_{\mathrm{ti}}(A^{\mathrm{op}})$ for all t_* then

$${}_{A^\nabla}N_B = \bigoplus_{t_*\in I_B} \nabla_+({}_AN_{t_*}), \qquad {}_{A^\nabla}N^-_{B'} = \bigoplus_{t\neq v_1,\ldots,v_b} \nabla_+({}_AN_{t_*}) \tag{4.14}$$

have natural structures of right modules over B and B', respectively. We form two *left waist reflection forms of R*

$$\hat{\delta}^u_- R = \begin{pmatrix} A^\nabla & {}^\nabla N \\ 0 & B \end{pmatrix} = \begin{pmatrix} A_u & \tilde{K} & {}_uN_B \\ & A' & {}_{A'}N''_B \\ & & B \end{pmatrix}, \qquad \delta^u_- R = \begin{pmatrix} A^\nabla & {}^\nabla N^- \\ 0 & B' \end{pmatrix}. \tag{4.15}$$

A module $Y = (Y'_u, \bar{Y}'_{A'}, \bar{Y}''_B, f_u, f', f'')$ in $\operatorname{mod}_{\mathrm{ti}}(\hat{\delta}^u_- R)$ is called *B-complete* if the map $\bar{f}_u$:

$Y'_u \to \mathrm{Hom}_B({}_uN_B, \bar{Y}''_B)$ adjoint to f_u is injective. We denote by $\mathrm{mod}^B_{\mathrm{ti}}(\hat{\delta}^u_- R)$ the full subcategory of $\mathrm{mod}_{\mathrm{ti}}(\hat{\delta}^u_- R)$ consisting of B-complete modules. As in Lemma 4.8 one can prove that if R has a left arrow waist then the restriction functor

(4.16) $$r^-_\delta : \mathrm{mod}^B_{\mathrm{ti}}(\hat{\delta}^u_- R) \to \mathrm{mod}_{\mathrm{ti}}(\delta^u_- R)$$

is an equivalence of categories.

We define two *left waist reflection functors*

(4.17) $$\hat{\delta}^u_- : \mathrm{adj}^A_B(R) \to \mathrm{mod}^B_{\mathrm{ti}}(\hat{\delta}^u_- R), \qquad \delta^u_- = r^-_\delta \hat{\delta}^u_- : \mathrm{adj}^A_B(R) \to \mathrm{mod}_{\mathrm{ti}}(\delta^u_- R)$$

by the formula $\hat{\delta}^u_-(X_R) = (\nabla_-(X'_A), X''_B, \varphi g)$, where $g: \nabla_-(X'_A) \otimes {}_A{}^{\nabla}N_B \to X' \otimes {}_AN_B$ is the natural B-isomorphism induced by the diagram (4.13). Note that since $\bar{\varphi}$ is injective and ${}_AN$ is top injective, X'_A is in $\mathrm{mod}_{\mathrm{sp}}(A)$.

Similarly to Theorem 4.12 one can prove the following.

THEOREM 4.18. *Suppose that* $R = \begin{pmatrix} A & N \\ 0 & B \end{pmatrix}$ *has a left multipeak waist u and A is an Artin algebra. Then $\hat{\delta}^u_- R$ and $\delta^u_- R$ are left multipeak rings and the waist reflection functor $\hat{\delta}^u_-$ (4.17) is an equivalence of categories. If, in addition, R has a left arrow waist then $\hat{\delta}^u_- R$ and $\delta^u_- R$ are left peak rings and the reflection functor δ^u_- is an equivalence of categories. If R is a finite-dimensional algebra over a field and R has a right as well as a left arrow waist then $(\delta_v R)^\nabla \cong \delta^u_- R$ and there is a commutative diagram*

(4.19) $$\begin{array}{ccc} \mathrm{adj}^A_B(R) & \xrightarrow{\delta_v} & \mathrm{mod}_{\mathrm{sp}}(\delta_v R) \\ & \searrow^{\delta^u_-} & \downarrow{\scriptstyle \nabla_-} \\ & & \mathrm{mod}_{\mathrm{ti}}(\delta^u_- R) \end{array}$$

5. Applications.

5.1. *Representations of a pair of posets with zero relations.* Let I and J be finite posets and let F be a division ring. We fix two sequences $a_1, \ldots, a_r$ and $b_1, \ldots, b_r$ of nonmaximal elements in I and J respectively. Consider $I \lhd J$ (1.3) as a commutative quiver and consider the set Σ of zero relations in $I \lhd J$ generated by γ_j: $a_j \to b_j$ for $j \leq r$. Then the bound quiver algebra $R = F(I \lhd J)/(\Sigma)$ has a B-traced form

$$R = \begin{pmatrix} A & N \\ 0 & B \end{pmatrix},$$

where $A = FI$, $B = FJ$ and ${}_iN_j = F$ if $i \prec j$ has no factorization through some γ_t and ${}_iN_j = 0$ otherwise. It we put $\mathbf{K} = \mathrm{pr}(FI)$, $\mathbf{L} = \mathrm{pr}(FJ^{\mathrm{op}})$ and ${}_{\mathbf{K}}\tilde{\mathfrak{N}}_{\mathbf{L}}$: $\mathbf{K}^{\mathrm{op}} \times \mathbf{L} \to \mathcal{A}b$ is a bimodule defined by $\tilde{\mathfrak{N}}(eA, e'B^{\mathrm{op}}) = D(eNe')$ for idempotents e and e' then objects of $\mathrm{Mat}({}_{\mathbf{K}}\tilde{\mathfrak{N}}_{\mathbf{L}}) \cong \mathrm{mod}^{\mathrm{pr}}_{\mathrm{in}}(R)^A_B$ [20; Prop. 5.14] can be interpreted as matrix representations of the pair I, J with zero relations $\gamma_1, \ldots, \gamma_r$ (see [16; Part B, Lemma 7.16]). By (1.2) the matrix problem is reduced to the study of $\mathrm{adj}^A_B(R)$.

Suppose that J has a unique minimal element v and construct the disjoint union poset

$$\delta_v(I, J) = I \dot\cup J - \{v\}$$

with additional relations generated by $a_j \prec b_j$ for $j = 1, \ldots, r$. Then R has a right peak waist, $\delta_v R \cong F(\delta_v(I, J))$ and by Theorem 4.12 the right waist reflection functor induces an equivalence

$$\hat{\delta}_v : \mathrm{adj}^{FI}_{FJ}(R) \to \delta_v(I, J)\text{-}\mathrm{sp}_I,$$

where $\delta_v(I, J)$-sp_I consists of *I-complete spaces* (M, M_t) in the sense that $M = \sum_{i \in I} M_i$. In the notation of Example 1.3, $\hat{\delta}_v$ is given by $\hat{\delta}_v(X_i, {}_s\varphi_t) = (M, M_t)$, where $M = X_v$, $M_i = \mathrm{Im}_v\varphi_i$ for $i \in I$ and $M_j = \mathrm{Ker}_j\varphi_v$ for $j \succ v$. If, in addition, I has a unique maximal element u then R has a right as well as a left arrow waist corresponding to $u \to v$ and (4.11) induces an equivalence of categories δ^u_v: $\mathrm{adj}^{FI}_{FJ}(R) \to \delta^u_v(I, J)$-sp, where $\delta^u_v(I, J) = I \dot\cup J - \{u, v\}$.

5.2. *A peak triangular reduction.* Suppose that F is a division ring, Q is a finite quiver without oriented cycles and $\mathfrak{b}$ is an ideal of FQ generated by some F-linear combinations of paths in Q of length ≥ 2. We put

$$i \prec j \text{ in } Q \Leftrightarrow \text{there is a nonzero path } i \to j \text{ in } Q$$

and we denote by

$$\max(Q) = \{*_1, \ldots, *_t\}$$

the set of all maximal elements in Q. Moreover, we suppose that the bound quiver algebra

$$\Lambda = F(Q, \mathfrak{b}) := FQ/\mathfrak{b}$$

is a right multipeak algebra, i.e. for every combination $w = \sum r_i w_i \notin \mathfrak{b}$ of paths w_i: $p \to q$ in Q with $r_i \in F$ there is $* \in \max(Q)$ and a path u: $q \to *$ such that $uw \notin \mathfrak{b}$.

We say that $s \in Q_0$ is an *upper triangle point* in $(Q, \mathfrak{b})$ if for any $i \prec s$ in Q_0 and a path β: $i \to *$, $* \in \max(Q)$, which does not belong to $\mathfrak{b}$ there are paths β': $i \to s$, β'': $s \to *$ such that $\beta''\beta' - \beta \in \mathfrak{b}$. Given such a point s we associate to it a triangular form

(5.2a) $$\Lambda = \begin{pmatrix} A & {}_AL_T \\ 0 & T \end{pmatrix}$$

of Λ, where $A = Fs^\nabla/\mathfrak{b} \cap Fs^\nabla$, $T = F(Q - s^\nabla)/\mathfrak{b} \cap F(Q - s^\nabla)$, ${}_iL_t = {}_i\Lambda_t = e_i\Lambda e_t$ and e_i is the idempotent corresponding to the trivial path at i. Next we associate to Λ the A-moduled category

$$\mathbf{H}_A = \mathrm{Hom}_T({}_AL_T, \mathrm{mod}_{\mathrm{sp}}(T))$$

in the sense of [20] and the vector space category

$$\mathbf{H}^s_F = \mathrm{Hom}_T(J(P_s), \mathrm{mod}_{\mathrm{sp}} T))$$

where $P_s = e_s\Lambda$. Note that $J(P_s) \cong {}_sL_T = \bigoplus_{t\in Q_0 - s^\nabla} e_s\Lambda e_t$. Given y in $\mathrm{mod}_{\mathrm{sp}}(T)$ we put

$$\bar{y} = \mathrm{Hom}_T(J(P_s), y), \qquad \bar{\bar{y}} = \mathrm{Hom}_T({}_\Lambda L_T, y).$$

We suppose for simplicity of the presentation that $\mathrm{ind}(H^s) = \{y_1, \ldots, y_m\}$ is finite and we put

$$t_s(\Lambda) = \begin{pmatrix} A & {}_A\tilde{H}_E \\ 0 & E \end{pmatrix}$$

where $E = \mathbf{H}^s(\bar{Y}, \bar{Y}) \cong \mathbf{H}(\bar{\bar{Y}}, \bar{\bar{Y}})$ (see 2° below), $Y = y_1 \oplus \ldots \oplus y_m$, ${}_A\tilde{H}_E = D\,\mathrm{Hom}_T({}_AL_T, Y) = D({}_E|\bar{\bar{Y}}|_A)$, $D(-) = \mathrm{Hom}_F(-, F)$ and $|\bar{\bar{Y}}|_A$ denotes $\mathrm{Hom}_T({}_AL_T, Y)$ considered as a right A-module in a natural way. The shape of $(Q, \mathfrak{b})$ and of the bound quiver of $t_s(\Lambda)$ is illustrated in Figure 16.

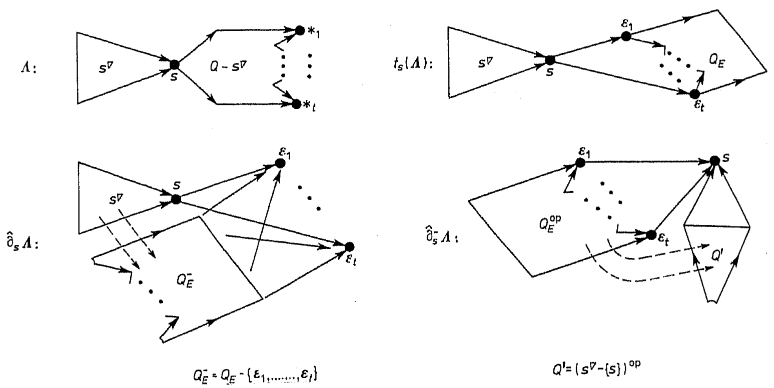


Fig. 16

THEOREM. *Let $(Q, \mathfrak{b})$ be a bound quiver as above, let $s \in Q_0$ be an upper triangle point in $(Q, \mathfrak{b})$ and let $*_1, \ldots, *_r$ be all points in $\max(Q)$ which are connected with s by nonzero paths in $(Q, \mathfrak{b})$. Then $\Lambda = F(Q, \mathfrak{b})$ has the triangular form* (5.2a) *and in the notation above we have*:

(a) *The algebra $t_s(\Lambda)$ has a right as well as a left multipeak waist, A is a right peak algebra with the peak idempotent e_s, and E is a left multipeak algebra with left peak idempotents $\varepsilon_1, \ldots, \varepsilon_r$ corresponding to the modules $E_{*_j} = E_T(e_{*_j}T)$, $j = 1, \ldots, r$, in $\{y_1, \ldots, y_m\}$.*

(b) *$\hat{\partial}_s\Lambda := \hat{\partial}t_s(\Lambda)$* (4.5) *is a right multipeak algebra with peak idempotents $\varepsilon_1, \ldots, \varepsilon_r$, $\hat{\partial}_s^-\Lambda := \delta_- t_s(\Lambda)^\infty$* (4.15) *is a right peak algebra with the peak idempotent e_s* (see Fig. 16) *and the triangular reduction functor* (5.2b) *below together with the waist reflection functors* (4.10) *and* (4.17) *induce equivalences of categories*

$$\mathrm{mod}_{\mathrm{sp}}(\Lambda)/[\mathrm{mod}_{\mathrm{sp}}(T)] \cong \mathrm{adj}_E^A(t_s(\Lambda)) \xrightarrow[\sim]{\hat{\partial}} \mathrm{mod}_{\mathrm{sp}}^A(\hat{\partial}_s\Lambda)$$
$$\xrightarrow[\sim]{D\hat{\partial}_-} (\mathrm{mod}_{\mathrm{sp}}^{E^{\mathrm{op}}}(\hat{\partial}_s^-\Lambda))^{\mathrm{op}}$$

where the right hand categories consist of complete socle projective modules defined below (4.15).

Proof. We proceed in several steps.

1° *A is a right peak algebra and ${}_A\tilde{H}$ is in $\mathrm{mod}_{\mathrm{u}}(A^{\mathrm{op}})$.* The first statement easily follows from the definition of an upper triangle point. For the second one it is enough to show that $|\bar{\bar{Y}}|_A = \mathrm{Hom}_T({}_AL_T, Y)$ is socle projective. For this purpose we note that $|\bar{\bar{Y}}|_A = (|Y|_i, {}_j\varphi_i)_{i,j \preccurlyeq s}$ where $|Y|_i = \mathrm{Hom}_T({}_iL_T, Y)$ and ${}_j\varphi_i$: $|Y|_i \otimes {}_iA_j \to |Y|_j$ is the map induced by the multiplication ${}_iA_j \otimes {}_jL_T \to {}_iL_T$ and ${}_iA_j = e_i\Lambda e_j$ is generated over F by cosets modulo $\mathfrak{b}$ of paths $i \to j$ in Q. By [17; Proposition 2.2] it is sufficient to show that given a nonzero T-homomorphism f: ${}_iL_T \to Y$ there is a path γ: $i \to s$, $\gamma \notin \mathfrak{b}$, such that ${}_s\varphi_i(f \otimes \bar{\gamma}) = f\bar{\gamma}_* \neq 0$, where γ_*: ${}_sL \to {}_iL$ is the left F-homomorphism induced by γ. Since f is nonzero and Y is socle projective, the restriction f_{*_j}: ${}_iL_{*_j} \to Y_{*_j}$ of f to some peak space ${}_iL_{*_j} = e_i\Lambda e_{*_j}$ of ${}_iL_T$ is nonzero. Since ${}_iL_{*_j}$ is generated over F by cosets $\bar{\beta}$ of paths β: $i \to *_j$, $\beta \notin \mathfrak{b}$, there is some β such that $f_{*_j}(\bar{\beta}) \neq 0$. If we choose β': $i \to s$ and β'': $s \to *_j$ as in the definition of an upper triangle point and we put $\gamma = \beta'$ then $f\bar{\gamma}_* \neq 0$ because $(f\bar{\gamma}_*)(\bar{\beta}'') = f(\bar{\beta}''\bar{\beta}') = f(\bar{\beta}) \neq 0$ by our choice of β. Then 1° follows.

2° *There is an equivalence of categories $\mathrm{ind}(\mathbf{H}) \cong \mathrm{ind}(\mathbf{H}^s)$ and the ring homomorphism $\mathbf{H}(\bar{Y}, \bar{Y}) \to E$, $\bar{\beta} \mapsto \bar{\bar{\beta}}$, is bijective.* For, note that by 1° given any indecomposable module y in $\mathrm{mod}_{\mathrm{sp}}(T)$ the right A-module $|\bar{\bar{y}}|_A$ is socle projective and $|\bar{\bar{y}}|_s = \mathrm{Hom}_T({}_sL_T, y) = |\bar{y}|$ is the socle space of $|\bar{\bar{y}}|_A$. Hence $\bar{y} \neq 0$ if and only if $\bar{\bar{y}} \neq 0$, and given $f \in \mathrm{Hom}_T(y, y')$ we have $\bar{\bar{f}} \neq 0$ if and only if $\bar{f} \neq 0$.

3° It follows from 2° that the algebra $t_s(\Lambda)$ is isomorphic to the ring

$$T_{\mathbf{H}} = \begin{pmatrix} A & {}_A\tilde{H}_{E(\mathbf{H})} \\ 0 & E(\mathbf{H}) \end{pmatrix}$$

of the A-moduled category $\mathbf{H}_A$ [20; 3.14] and by [20; Theorem 4.20] there is a full and dense adjustment functor [20; 4.23]

(5.2b) $$\mathbf{ad}_{(s)}^A\colon \mathrm{mod}_{\mathrm{sp}}(\Lambda) \to \mathrm{adj}_E^A(t_s(\Lambda))$$

such that $\mathrm{Ker}\,\mathbf{ad}_{(s)}^A = [\mathrm{mod}_{\mathrm{sp}}(T)]$. Moreover, $\mathbf{ad}_{(s)}^A$ induces an equivalence of categories

(5.2c) $$\mathrm{mod}_{\mathrm{sp}}(\Lambda)/[\mathrm{mod}_{\mathrm{sp}}(T)] \cong \mathrm{adj}_E^A(t_s(\Lambda)).$$

4° *E is a left multipeak algebra with peak idempotents $\varepsilon_1, \ldots, \varepsilon_r$ as in the Theorem. Moreover, $\tilde{H}_E$ is top injective.* In order to prove the first statement suppose that $\bar{\beta}$: $\bar{y}_j \to \bar{y}_i$ is a nonzero map. Then there is h: ${}_sL_T \to y_j$ such that $\beta h \neq 0$. Since y_i is socle projective, there exist $*_q$ and a nonzero T-homomorphism g: $y_i \to E_{*_q}$ such that $g\beta h \neq 0$. Since $e_{*_q}T$ is simple projective, it can be embedded into ${}_sL_T$ and therefore E_{*_q} is a summand of $E_T({}_sL_T)$. Finally, by the choice of g we have $\bar{g}\bar{\beta} \neq 0$. It follows from [18; p. 22] that E is a left multipeak algebra.

We prove the second statement by showing that ${}_EH_j = \mathrm{Hom}_T({}_jL_T, Y)$ is socle projective for all $j \preccurlyeq s$. By [18; p. 24] it is sufficient to show that for any nonzero $h \in {}_yH_j = \mathrm{Hom}_T({}_jL_T, y)$ there exist q and $\bar{f} \in \mathbf{H}^s(\bar{y}, \bar{E}_{*_q}) = \varepsilon_q E_y$ such that $fh \neq 0$. To see this, let q be such that there is a nonzero T-map g: $\mathrm{Im}\,h \to E_{*_q}$ and take for f an extension of g. This finishes the proof of the Theorem.

One can also prove the following interesting fact.

PROPOSITION. *Let Λ, s and $t_s(\Lambda)$ be as above and suppose that for every $*_j$ there is at most one path β: $s \to *_j$ not in $\mathfrak{b}$ and $\dim_s \tilde{H}\varepsilon_j \leqslant 1$ for all $j = 1, \ldots, r$. Then ${}_s\tilde{H}_E = P'_1 + \ldots + P'_r$ for some $P'_j \cong \varepsilon_j E$. In particular, ${}_s\tilde{H}_E$ is projective if $r = 1$. If, in addition, for every α: $i \to s$ there is β: $s \to *_q$ such that $\beta\alpha \notin \mathfrak{b}$ then ${}_\Lambda\tilde{H}\varepsilon_j \cong \Lambda e_s$ for all $j = 1, \ldots, r$.*

The rings $\hat{\partial}_s\Lambda$, $\hat{\partial}_s^- \Lambda$ and the equivalences in Theorem (b) have a nature of the differentiation procedure [10], [17; Section 5]. We call them *peak triangular reductions*.

5.2d. Suppose that we are in the situation as in the Theorem above. Moreover, we suppose that $\mathfrak{b} \cap Fs^\nabla$ is generated by all commutativity relations $w - w'$: $i \to j$, $i, j \prec s$, and that the vector space category $\mathbf{H}^s_F$ is of finite poset type (see 3.1).

It follows that s^∇ is a poset with a unique maximal element s and the relation (3.2) is a partial order in $I(\mathbf{H}^s) = \{\bar{y}_1, \ldots, \bar{y}_m\} = \operatorname{ind}(\mathbf{H}^s)$ and $\bar{E}_{*_1}, \ldots, \bar{E}_{*_r}$ are all minimal elements in $I(\mathbf{H}^s)$, where $E_{*_j} = E_T(e_{*_j}T)$. Let us construct the poset

$$t_s(Q, \mathfrak{b}) = s^\nabla \lhd I(\mathbf{H}^s)$$

(see 1.3) and let Ω_s be the set of zero relations in $t_s(Q, \mathfrak{b})$ defined by the formula

$$i \to \bar{y}_j \in \Omega_s \Leftrightarrow \operatorname{Hom}_T(e_i \Lambda e_T, y_j) = 0, \quad i \preccurlyeq s,$$

where $e_T = \sum_{j \in Q_0 - s^\nabla} e_j$ and $T = e_T \Lambda e_T$ is as in the definition of $t_s(\Lambda)$.

PROPOSITION. *Under the assumption and notation above there exists an F-algebra isomorphism*

$$t_s(\Lambda) \cong Ft_s(Q, \mathfrak{b})/(\Omega_s)$$

and a duality of categories

$$\operatorname{mod}_{sp}(\Lambda)/[\operatorname{mod}_{sp}(T)] \cong \hat{\delta}_s(I^{op}, s^{\nabla op})\text{-}\mathrm{sp}_{I^{op}}$$

where $I = I(\mathbf{H}^s)$ and $\hat{\delta}_s(I^{op}, s^{\nabla op})$ is the disjoint union of the posets I^{op} and $(s^\nabla - \{s\}^{op})$ with additional relations $\bar{y}_j \prec i$ iff $i \to \bar{y}_j$ is in Ω_s (see 5.1).

Proof. It follows from our assumptions that the rings A and E appearing in $t_s(\Lambda)$ are isomorphic to Fs^∇ and FI respectively, and $\begin{pmatrix} F & {}_s\tilde{H}_E \\ 0 & E \end{pmatrix} \cong F(\{s\} \lhd I)$. Given $i \prec s$ in s^∇ we fix a unique nonzero path ${}_s\gamma_i$: $i \to s$ modulo $\mathfrak{b}$. Since ${}_iL_T = e_i \Lambda e_T$ is generated by all paths w: $i \to t$ modulo $\mathfrak{b}$, where t runs through all vertices in $Q_0 - s^\nabla$, from the assumption that s is an upper triangle point it follows that the induced T-homomorphism ${}_s\bar{\gamma}_i$: ${}_sL_T \to {}_iL_T$, $(\beta$: $s \to t) \mapsto \beta_{s\gamma_i}$, is surjective and therefore the induced map ${}_s\bar{\bar{\gamma}}_i = \operatorname{Hom}_T({}_s\bar{\gamma}_i, y_j)$: $\operatorname{Hom}_T({}_iL_T, y_j) \to \operatorname{Hom}_T({}_sL_T, y_j)$ is injective, whereas its dual $D({}_s\bar{\bar{\gamma}}_i)$: ${}_s\tilde{H}_{\bar{y}_j} \to {}_i\tilde{H}_{\bar{y}_j}$ is surjective for $j = 1, \ldots, m$. Since $\dim {}_s\tilde{H}_{\bar{y}_j} = 1$ for $j = 1, \ldots, m$, we have $\dim {}_iH_{\bar{y}_j} \leqslant 1$ for all $i \preccurlyeq s$ and $\bar{y}_j \in I$, and therefore

$$D({}_s\bar{\bar{\gamma}}_i) \text{ is bijective} \Leftrightarrow {}_iA_s \otimes {}_s\tilde{H}_{\bar{y}_j} \to {}_i\tilde{H}_{\bar{y}_j} \text{ is bijective}$$
$$\Leftrightarrow \dim {}_i\tilde{H}_{\bar{y}_j} = 1$$
$$\Leftrightarrow (i \to \bar{y}_j) \notin \Omega_s.$$

It follows that the natural F-algebra homomorphism $Ft_s(Q, \mathfrak{b}) \to t_s(\Lambda)$ induces the required isomorphism. The remaining part follows from 5.1.

Remark. It is easy to see that $\mathbf{H}^s$ is of finite poset type in case Λ is sp-representation-finite.

EXAMPLE. Consider the algebra $\Lambda = F(Q, \mathfrak{b})$, where $(Q, \mathfrak{b})$ is the poset $I_{\mathfrak{e}}^{*+}$ with one zero relation defined in Example 1.3′. If we set $s = 8$ then $T = F(I_{\mathfrak{e}}^{*+} - \{1,2,3,4,8\})$, $J(P_s) = e_* \Lambda \oplus e_+ \Lambda$, $I(\mathbf{H})^{op}$ is isomorphic to the Auslander–Reiten quiver of $\operatorname{mod}_{sp}(T)$ and the poset $L = \hat{\delta}_s(I(\mathbf{H})^{op}, s^{\nabla op})$ has the form of Fig. 17. It follows from [9] that L is of tame type and according to the Proposition above $\operatorname{mod}_{sp}(\Lambda)$ is also of tame type.

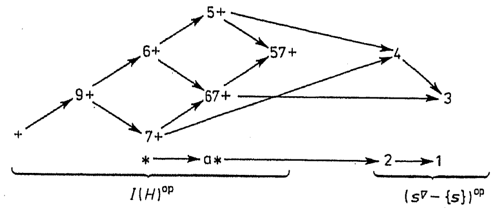


Fig. 17

Remark. Let I be a two-peak poset with $\max(I) = \{*, +\}$. We consider I as a bound quiver $(Q, \mathfrak{b})$ with $Q_0 = I$ and $\mathfrak{b}$ generated by all commutativity relations in I. Suppose that $\operatorname{mod}_{sp}(\Lambda)$ is of tame type, where Λ is the path algebra FI of I and F is algebraically closed.

We are going to show that in this case the study of $\operatorname{mod}_{sp}(\Lambda)$ can be reduced to the study of L-spaces for some L provided $I - *^\nabla \cap +^\nabla$ is of finite type. For this purpose we note that by our assumption I does not contain as a full two-peak subposet the two-peak posets of Fig. 18. It follows that $I_0 = *^\nabla \cap +^\nabla$ is a full subposet of a garland (see Fig. 19). If I_0 is linearly ordered we can reduce the study of $\operatorname{mod}_{sp}(\Lambda)$ to the study of $\xi_c I$-spaces by applying Theorem 3.4. If $w(I) = 2$ and I_0 has a unique maximal element s we easily show that $s \prec j$ for all $j \in (*^\nabla \cup +^\nabla) - I_0$ and s is an upper triangle point in I.

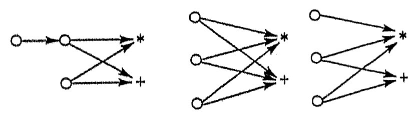


Fig. 18

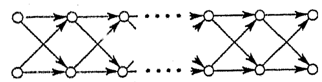


Fig. 19

Moreover, the assumption that $\mathrm{mod}_{sp}(\Lambda)$ is of tame type implies that the vector space category $\mathbf{H}^s$ is of poset type and therefore the Proposition above applies. The case $|\max(I_0)| = 2$ is more difficult but it can be reduced to the above one.

By applying [9] together with the method above we can get a critical list for two-peak posets I to be sp-representation-tame (compare with Remark 3.21). We shall discuss this problem in a subsequent paper.

5.3. *Differentiations.* We are going to show that the differentiation procedures for posets I [10] and for right peak rings [19] can be obtained in two steps: the triangular adjustment applied to $\mathrm{mod}_{sp}(\Lambda)$ and then by applying the arrow waist reflection functor. This procedure admits also a suitable generalization [17; § 6], [11], [20; Remark 4.28].

Let us explain this in the case where Λ is a right peak Artin algebra having a maximal element s in $(I_\Lambda, \boldsymbol{d})$, i.e. s is maximal in I_Λ-$\max(I_\Lambda)$ and the conditions (S1), (S2) in Remark 3.18 are satisfied. Then Λ has the form

$$\Lambda = \begin{pmatrix} S & {}_SL_T \\ 0 & T \end{pmatrix},$$

where $S = e\Lambda e$, $L = e\Lambda(1-e)$, $T = (1-e)\Lambda(1-e)$ and $e = \sum_{j \in s^\nabla} e_j$. Here e_j is the primitive idempotent corresponding to j and $s^\nabla = \{j \in I_\Lambda;\ d_{js} \neq 0\}$. Consider the S-moduled category $\mathbf{H}_S = \mathrm{Hom}_T({}_SL_T, \mathrm{mod}_{sp}(T))$. Since ${}_SL_T \simeq P_*$, the ring $E = \boldsymbol{E}(\mathbf{H})$ of the category $\mathbf{H}$ [20] is isomorphic to $\Lambda_{sp}(T)$ (2.1) (in general it has no identity). By [20; Theorem 4.20] there is a full dense functor

$$\mathbf{ad}^S_{sp}\colon \mathrm{mod}_{sp}(\Lambda) \to \mathrm{adj}^S_E(R)$$

such that $\mathrm{Ker}\,\mathbf{ad}^S_{sp} = [\mathrm{mod}_{sp}(T)]$, where $R = T_H = \begin{pmatrix} S & {}_S\tilde{H}_E \\ 0 & E \end{pmatrix}$, $\tilde{H} = \bigoplus_y D|\tilde{y}|_S$ and y runs through all representatives of isoclasses of indecomposables in $\mathrm{mod}_{sp}(T)$ (we use the notation in 5.2). It is easy to check that S is a right peak algebra with the peak idempotent e_s, E is a left peak algebra with the peak idempotent corresponding to $E_T(P_*)$ and similarly to 5.2 one can show that the E-traced algebra R has a right arrow waist $s \to v$ which is also a left arrow waist. Then according to Theorem 4.12 the waist reflection functor (4.11) $\delta_v\colon \mathrm{adj}^S_E(R) \to \mathrm{mod}_{sp}(\delta_v R)$ is an equivalence of categories. The ring $\partial_s\Lambda := \delta_v R$ is a right peak algebra and the functor

$$\partial_s = \boldsymbol{\delta}_v \mathbf{ad}^S_{sp}\colon \mathrm{mod}_{sp}(\Lambda) \to \mathrm{mod}_{sp}(\partial_s\Lambda)$$

is full dense and $\mathrm{Ker}\,\partial_s = [\mathrm{mod}_{sp}(T)]$. It is easy to check that $\partial_s\Lambda$ is isomorphic to the differential algebra Λ'_s of Λ and ∂_s is the differential functor $\boldsymbol{\Phi}_s$ in [17; Section 5]. The poset differentiation [10] we get by applying the above to the ring $\Lambda = FI^*$, where F is a division ring, I is a finite poset and I^* is an enlargement of I by a unique maximal element. Then any maximal element $s \in I$ is smooth for Λ and if $I - s^\nabla$ is of width $\leqslant 2$ then a straightforward analysis shows that $\partial_s\Lambda \cong F(I'_s)^*$, where I'_s is the differential of I in the sense of Nazarova–Roiter [10]. Moreover, the functor ∂_s induces a full and dense functor $\partial_s\colon I\text{-sp} \to I'_s\text{-sp}$ such that $\mathrm{Ker}\,\partial_s = [(I - s^\nabla)\text{-sp}]$ (see [16]).

The procedure extends naturally to the case where s is smooth nonmaximal (see [20; Example 4.30]). Using this idea we define in [19] a differentiation of a right peak ring with respect to a suitable pair of elements.

5.4. *A preprojective component in* $\mathrm{adj}^A_B(R)$. Suppose that $R = \begin{pmatrix} A & N \\ 0 & B \end{pmatrix}$ is a faithfully right B-traced Artin algebra. We know from [20] that there are almost split sequences in $\mathrm{adj}^A_B(R)$. Since one can easily show that $\mathrm{adj}^A_B(R)$ has enough projectives, as in [5, 8] one can introduce the notions of a complete preprojective component and a preinjective component in $\mathrm{adj}^A_B(R)$. It is clear that if $\mathrm{adj}^A_B(R)$ has a preprojective component then R is schurian, i.e. A_i, B_{t_*} are division rings (4.3).

PROPOSITION. *Let* $R = \begin{pmatrix} A & N \\ 0 & B \end{pmatrix}$ *be a faithfully right B-traced schurian Artin algebra and let* $(I_R, \boldsymbol{d})$ *be the value scheme of R* [7, 17, 20]. *If R has a right arrow waist* $u \to v$, $(I_R, \boldsymbol{d})$ *has no subschemes of the form of Fig. 20 and* $d_{iv}d'_{iv} \leqslant 3$, $d_{vt_*}d'_{vt_*} \leqslant 3$ *for all* $i \in I_A - \{u\}$, $t_* \in I_B - \{v\}$ *then there exist a preprojective component and a preinjective component in* $\mathrm{adj}^A_B(R)$.

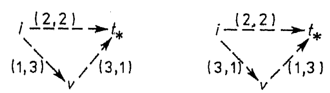


Fig. 20

Proof. We know from Theorem 4.12 that (4.11) is an equivalence of categories. It follows from our assumption that the valued poset $(\bar{I}, \bar{\boldsymbol{d}})$ of $\delta_v R$ has $\bar{d}_{i*}\bar{d}'_{i*} \leqslant 3$ for all $i \in \bar{I}$ and does not contain peak subposets of the form $\circ - - \to \circ \xrightarrow{(e,e')} *$, $ee' = 3$. Then by [8], $\mathrm{mod}_{sp}(\delta_v R)$ has a preprojective component as well as a preinjective component and the proposition follows.

COROLLARY. *Suppose that* $R = \begin{pmatrix} A & N \\ 0 & B \end{pmatrix}$ *is a faithfully right B-traced Artin algebra having a right arrow waist and such that* $\mathrm{adj}^A_B(R)$ *is of finite type. Then the Auslander–Reiten valued translation quiver of* $\mathrm{adj}^A_B(R)$ *is simply connected if and only if R is schurian.*

Proof. Since R is schurian if and only if $\delta_v R$ is schurian, in view of the equivalence (4.11) the corollary is a consequence of [8].

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ERRATA

Page, line	For	Read
111_4	$r_{(x+\alpha)}$	$r_{(x+\alpha)^\wedge}$
123_1	θ	$\boldsymbol{\Theta}$
124^{11}	$\bar{\omega} = \bar{\omega}\nabla_+,\ \bar{\omega}' = \omega'\nabla_-$	$\bar{\boldsymbol{\omega}} = \boldsymbol{\omega}\nabla_+,\ \bar{\boldsymbol{\omega}}' = \omega'\nabla_-$
135, first diagram	$v\varphi^{\oplus}1$	${}_v\varphi \oplus 1$
	$t_*\varphi''_v$	${}_{t_*}\varphi''_v$