Decomposition of special Jacobi sets

by

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Abstract. In [4], Jacobi sets $R$ are introduced and studied using the Weyl group of the usual real system of roots $\mathcal{R}$. In this paper, we study a special class of Jacobi sets using their combinatorial structure without recourse to the Weyl group of $\mathcal{R}$ and show that they can be decomposed as a sum of a classical root system and a nil root system. Finally, we give an example of a Lie algebra whose root system is a special Jacobi set.

0. Introduction. In Winter [7, 10], symmetrizers are introduced and studied, and in Winter [9], Lie root systems are also introduced. These variations of root systems occur naturally in the study of classical and symmetric Lie algebras respectively. For a classical Lie algebra $L$ of characteristic $p$, with Cartan subalgebra $H$, we have a root system called $R(L, H)$. This system is not a root system in Euclidean space due to $p$ torsion. In Winter [7, 10] it has been shown that certain structures of $R(L, H)$ alone lead to the identification of $R(L, H)$ with a root system in Euclidean space, and this enabled direct classification of classical Lie algebras of characteristic $p$ along the same lines as in the theory of complex semisimple Lie algebras [6, 7].

In Hailat [4], Jacobi sets $R$ are introduced and studied using the Weyl group of the usual real system of roots $\mathcal{R}$. Jacobi sets are symmetrizers with a Jacobi condition. In this paper we continue our study of Jacobi sets using their combinatorial structure. This paper is one in a series of papers [2, 3, 4, 5, 7, 10], whose objectives are to classify root systems. The root systems under study are beginning to fill out a pattern which thus far is only partially exposed, the exposed part being the root systems of classical Lie algebras (systems of black roots), the root systems of symmetric Lie algebras (systems of black and white roots) and systems where black and grey roots appear in root systems of Lie algebras over algebraic number fields and other not algebraically closed fields (for more about the colors of roots see Hailat [2]).

In §1 we introduce all the definitions and the preliminaries. In §2 we decompose a root $x$ into its two parts: regular $s$ and nil $n$, that is, $x = s + n$. Consequently, in §3 we decompose $R$ as $R = S \oplus R_n$, where $S$ is a regular symmetrizer and $R_n$ is nil.

1980 Mathematics Subject Classification: Primary 17B05, 17B20, 20H15, 20B25, 20F05; Secondary 05B25.

This research is supported in part by Yarmouk University and by (DAAD).
1. Definitions and preliminaries. Let \( V \) be a vector space over a field of characteristic \( p > 0 \) and let \( R \) be a finite subset of \( V \). We regard \( R \) as a groupoid with \( a + b \in R \) only for certain \( a, b \in R \). For \( a \in V \) the relation \( \{ b, b+a \mid b + a \in R \} \) generates an equivalence relation on \( R \). The corresponding equivalence class of \( b \in R \) is the string \( R_0(b) = \{ b, b+a, b+2a, \ldots \} \). We call \( R_0(b) \) the orbit of \( b \) with length \( q = r \). The orbit \( R_0(a) \) is bounded if \( R_0(a) \neq b + Za \). An automorphism of the set \( R \subseteq V \) is a bijection \( r : R \to R \) such that \( a + b \in R \) if and only if \( r(a) + r(b) \in R \), in which case \( r(a+b) = r(a) + r(b) \), for all \( a, b \in R \). We call an automorphism \( r \) which stabilizes all \( a \)-orbits \( R_0(a) \in R \) a symmetry of \( R \) at \( a \). If \( R_0(a) \) is bounded we define the Cartan integer \( a^* = r - q \), and the reflection \( r_a \) by \( c\{r(a) = c - a^* \} a \) if \( R_0(a) \) is bounded. The element \( aR \in R \) is unbounded if there exists \( b \in R \) such that \( R_0(a) \neq b + Za \). We say, more specifically, that \( R \) is unbounded at \( b \) if \( R_0(a) = b + Z \). The condition following Winter [7, 10], a symmetry set \( R \) has a unique maximally refined decomposition where the subsets \( D_1, \ldots, D_n \) are symmetries and \( R = D_1 \cup \ldots \cup D_n \) with \( D_i \cap D_j = \emptyset \) for \( i \neq j \) such that \( a_1, \ldots, a_n \in R \) if and only if \( a_1, \ldots, a_n \in D_i 
nes \) for some \( i = 1, \ldots, n \). The symmetries \( D \subseteq \ldots \subseteq D_n \) are the irreducible components of \( R \) and \( R \) is irreducible if \( n = 1 \). Thus for any symmetry set \( R, R \) is the inner direct sum of its irreducible components, that is, \( R = D_1 \oplus \ldots \oplus D_n \).

The following results are needed in the paper:

1.3. PROPOSITION (Hallat [2]). Let \( R \) be a symmetry set. If \( a \neq 0 \), then \( a \) is bounded and \( a^* \in \text{Hom}(R, Z) \).

1.4. THEOREM (Hallat [5]). Let \( R \) be an unbounded element in a Jacobi set \( R, \) and let \( b, b+ta \in R \) for some \( t \neq 0 \). Then \( b + t(a) \in R \) for all \( t \in Z \).

1.5. THEOREM (Hallat [5]). Let \( R \subseteq Z_p \) be an unbounded Jacobi set. Then \( R = Z_p \oplus \ldots \oplus Z_p \) (inner direct sum).

1.6. THEOREM (Hallat [5]). Let \( R \subseteq Z_p \) be a symmetrieset (Jacobi set). Then the set \( R_0 = Ker \wedge = \{ a \in R | a = 0 \} \) is a nil symmetry set (Jacobi set).

1.7. THEOREM (Winter [14]). Let \( R \) be a symmetry set such that \( R_0(a) = \{ b \in R | a \neq 0 \} \) and \( R_0(b) = \{ r(a) | a \in R, \alpha = 0, \alpha \neq 0 \} \) are bounded. Then \( \alpha \) is a bijection onto \( R_0(b) \).

(1) \( \alpha^* \alpha = \alpha \), for all \( a, b \in R, a \neq 0, \alpha \neq 0 \).

(2) \( (a, b) \in R, \alpha \neq 0 \), there exists \( c \in R \) such that the closure mapping \( R_0(c) \) bijectively onto \( R_0(b) \).

(3) the closure mapping \( R \to R \) is an isomorphism (of groupoids) if and only if it is bijective.

1.8. THEOREM (Hallat [3]). Let \( R \) be a Jacobi set. Then all regular subsymmetries of \( R \) are isomorphic to \( R \) and, therefore, are root systems in the sense of Bourbaki [1] with 0 added.

Note that the symmetries \( R \) is a system of root in the sense of Bourbaki [1] with 0 added by Theorem 2.3 of Winter [14].

We say a symmetrieset \( S \) is regular if there exists a regular subset \( \subseteq \in R \) such that \( S = Z_p \times R \) and this in case we write \( S = S(x) \).

2. The decomposition: \( x = s+n \). Let \( R \) be a Jacobi set and let \( S \subseteq S(x) \) be a regular subsymmetryset of \( R \). We show, in this section, that any element in \( R \) can be written as a sum of two elements, the first of which is in \( S \) and the second in \( R_0 \). But, to do this, we need the following proposition which is Proposition 2.5 of [3].

2.1. PROPOSITION. Let \( R \) be a Jacobi set and let \( S \subseteq S(x) \) be a regular subsymmetryset of \( R \). Then \( S \subseteq R_0 \). \( \subseteq \) \( \subseteq \) \( \subseteq \) \( \subseteq \).

We now have our decomposition theorem.

2.2. THEOREM. Let \( R \) be a Jacobi set and let \( S \subseteq S(x) \) be a regular subsymmetryset of \( R \). If \( x \in R \) then \( x = s+n \) for some \( s \in S \) and for some \( n \in R_0 \).
Proof. If \( x \in S \text{ then set } s = x, n = 0; \) also, if \( x \in R_0 \text{ then set } s = 0, n = x. \) Since \( R = R_0 \cup R_1, \) the only remaining case for \( x \) is to be in \( R_1 \). Let \( x \in R_1 \). Since \( S = R \) by Theorem 1.8, there exists \( a \in S \) such that \( x = a \). Then \( s = 0. \) Thus \( x^s(a) = s^x = (x^a)(a) = (a) \). By Theorem 1.7, this implies that \( r(s)(a) = -2a, \) so that \( a = -x. \) Set \( s = a, n = x - a. \) We have \( n \in R_0 \text{ since } n = (x-a) = 0. \)

Note that the above decomposition is unique since \( S \cap R_0 = \{0\}. \) We call \( s \) the regular part of \( x \) and \( n \) the nil part of \( x. \)

3. Decomposition of special Jacobi sets. In this section we decompose a special Jacobi set \( R (a) \) (Jacobian set with the condition \( R_0 = R_1 \)) as a sum of a regular symmetrized \( S \) and a nil symmetry set \( R_0. \) But first we construct some special Jacobi sets as an example motivating further studies.

Let \( G \) be a finite subgroup of a vector space \( V \) over a field \( F \) of characteristic \( p \) and let \( S \) be a classical root system over \( F. \) We may regard \( S \subseteq W \) where \( W \) is a vector space.

Now consider \( G \otimes S = \{g \otimes a \mid g \in G, a \in S\} \subseteq V \otimes W. \) Introduce the homomorphism \( r_{g\otimes a}(h \otimes b) = (h \otimes b) - (g \otimes a)(h \otimes b)(g \otimes a) \) by specifying \( (g \otimes a) \in \text{Hom}(V \otimes W, Z_j) \) as follows:

\[
(g \otimes a)(h \otimes b) = a(h),
\]

\[
(g \otimes a)(h \otimes b) = a(h), (h \neq 0).
\]

Note that \( (G \otimes S)_{\text{reg}}(h \otimes b) = (h \otimes b) - (g \otimes a)(h \otimes b)(g \otimes a) \) where \( S_0(a) = [a, a, a, \ldots, a]. \) Then \( b = a = (h \otimes b)(g \otimes a) \), so that \( (h \otimes b)(g \otimes a) = 0. \) This implies that \( G \otimes S \) is a symmetric set. Let \( R = G \otimes S. \) Since \( G \) is a group \( R_0(g) = h \otimes Z \) implies that \( g \) is an unbounded root for all \( n \in G. \) Therefore, \( \bar{g} = 0 \) for every \( g \in G \) by Proposition 1.3. Also, since \( S \) is a classical root system then \( \bar{a} = 0 \) for all \( a \in S \) (every nonzero element in \( S \) is bounded). Now let \( g_1 \otimes a, g_2 \otimes a, g_3 \otimes a \) be elements of \( R \) such that \( g_1 \otimes a \neq g_2 \otimes a. \) Then \( a, b, c \not\in (a, b, c) \) from the definition of \( G \otimes S. \) If \( (g_1 \otimes a)(g_2 \otimes a)(g_3 \otimes a) \neq 0 \) then \( (a + b + c) \neq 0 \) so that \( a = -b, c = 0. \) It follows that:

\[
(g_1 \otimes a)(g_2 \otimes a)(g_3 \otimes a) = (g_1 + g_2)(g_3)(a + 0) = (g_1 + g_2)(a + 0) \in R.
\]

Also, if \( (g_1 \otimes a)(g_2 \otimes a)(g_3 \otimes a) \neq 0 \) and \( (g_2 \otimes a)(g_3 \otimes a) \neq 0 \) then \( (a + b) \neq 0 \) and \( c \neq 0. \) It follows that \( S \subseteq S \) or \( b + c \subseteq S \) since \( R \) is a classical root system. This last conclusion reflects the condition for the complex semisimple Lie algebra \( L = L_0 \) that \( [L_1, L_1, L_1] = 0. \) Therefore, \( [g_1 \otimes a, g_2 \otimes a, g_3 \otimes a] = (g_1 + g_2 + g_3) \otimes (a + c) \in R \) or \( (g_1 \otimes a)(g_2 \otimes a)(g_3 \otimes a) = (g_1 + g_2 + g_3) \otimes (a + c) \in R. \) Note that \( R_0 = G \otimes S. \) The above discussion implies the following proposition.

3.1. Proposition. Let \( G \) be a finite subgroup of a vector space \( V \) over a field \( F \) of characteristic \( p \) and let \( S \) be a classical root system over \( F. \) Then \( R = G \otimes S \) is a special Jacobi set.

Now Proposition 2.2 of Winter [8] and Proposition 3.1 imply that the root system \( G \otimes S \) of the generalized classical Albert–Zassenhaus (GCZ) \( L_{\text{GZ}} \) is a special Jacobi set. Also, the root system of the GCZ \( L_{\text{GZ}} \) is of the form given by Theorem 3.10 below where \( G \cong R_0 \) and \( S \cong R_1. \)

The following theorem, which is Theorem 2.3 of [4], is needed in this section.

3.2. Theorem. Let \( R \) be a special Jacobi set and let \( a \in R_1, a \neq 0 \) be such that \( a \neq 0 \) and \( a + x \in S \). Then \( x = \bar{y} \) if and only if \( a + \bar{y} \in S. \)

A number of lemmas will precede the proof of the main theorem. The same set up is involved in all of them. To avoid repetition the notation will now be fixed for this section.

Let \( R \) be a special Jacobi set and let \( S \) be a regular subsymmetry of \( R. \) Also, let \( x, y, x + y, a R \) such that \( a \in R_0 \) and \( x \neq 0. \)

3.3. Lemma. If \( x, y \in S \) then \( y + x \in S. \)

Proof. Suppose that \( x, y \in S \) and \( y + x \notin S. \) Then

\[
R_k(x + a) \neq y - r(x + a), \ldots, y + q(x + a), \quad r, q > 0.
\]

Note that this orbit is bounded by Proposition 1.3 since \( x + a \not\in \bar{x} \neq 0. \) It follows that

\[
R_k(x + a) = (y - r x, \ldots, y + q x) \notin S.
\]

But \( S \subseteq R \) by Theorem 1.8, therefore \( y - r x, \ldots, y + q x \in S. \) This implies that

\[
R_k(x + a) = (y - r x, \ldots, y + q x) \not\in S.
\]

Note that \( (x + a)^*(x) = (x + a)^*(x) = x + a \not\in S \) by Theorem 1.7, so that \( r_k(x) = x - 2(x + a) = x - 2a. \) Since \( y + x \notin S \) for all integers \( i \) by Theorem 1.4. Therefore, \( x + y - (x + a) = y - x \not\in S. \)

Note that \( R_k(x + a) = (x + a, \ldots, x + y) \in S \) for some \( k \).

This implies that \( x + y, x + y, x + (k + 1) \otimes y \in S, \) since \( S \subseteq R \) is finite, and \( y, x \in S - 0. \) Therefore

\[
R_k(x + a) = (y + x, \ldots, y + (k + 1)x + y) \notin S.
\]

Now we summarize the above results as follows:

\[
r_k(x) = x - 2a, \quad r_k(y) = y - (y - q) x, \quad r_k(y) = y + (k + 1) x + y.
\]

But \( r_k(x + y) = r_k(x) + r_k(y). \) Hence

\[
(y + (k + 1) x + y) \otimes x = (y + (y - q) x) + (y + (y - q) x).
\]

This follows from \( r_k = r_k - (r + 2) x \otimes x \in R. \) But \( r_k(y) = r_k(y) = y + (k + 2) x + y \otimes x \in S. \) Hence \( y + (k + 2) x \in S \) and \( r_k(x + y) = y + (k + 2) x + (y + (k + 1) x + y) \in S. \) Choose \( a \in Z_j \) such that \( i(k + 2) = k + 1. \) Then

\[
(y + (k + 2) x) + (x + a) \in R \text{ by Theorem 1.4, a contradiction to the assumption that the last root to the right in } R_k(x + a) = (y + (k + 1)x + y) \text{. Therefore } y + x \not\in S.
5. Lemma. If \( x \in R_1 - \{0\}, y \in R_0 \) or \( x \in R_0 - \{0\}, y \in R_1 \) then \( y + x \in R \).

Proof. In either case we have \( y + x \in R_0 \) by Theorem 3.2 since \( x + y \in R \) and \( x + z \in R \).

3.5. Lemma. If \( x, y \in R_0 \) then \( y + x \in R \).

Proof. Since \( R_0 = R_0 = \mathbb{Z}_p \cup \cdots \cup \mathbb{Z}_p \) and \( x + a \in R \) and \( y + x \in R \) it follows that \( x, a \in \mathbb{Z}_p \) for some \( i \) and \( x, y \in \mathbb{Z}_p \) for some \( j \). This implies that \( i = j \) and, therefore, \( y, a \in \mathbb{Z}_p \). Thus \( y + x \in \mathbb{Z}_p \).

3.6. Lemma. If \( x \in S - \{0\}, y \in R_1 - S \) then \( y + x \in S \).

Proof. Let \( x \in S - \{0\} \) and \( y \in R_1 - S \). Then \( y = x + n \in S + R_0 \) by Theorem 2.2. But \( x + (s + n) = x + y \in R \). Hence \( x + n \in S \). Suppose first that \( x + n \in R_0 \). Then \( s + x \in R_0 \) by Lemma 3.3 since \( x + n \in R_0 \). But \( s + x = (s + x) + y = n \). Therefore \( a \in R_0 \) if \( y = (x + s + n) \). If \( y = x \) then we are done. But if \( y = (x + s + n) \) then \( n = a \). This implies that \( a = (y - x) = x - (x + n) \). \( y + x \in S \) by Theorem 3.2 since \( n + x \in S \). Suppose, second, that \( x + n \in R \). Then \( x + n \in R_0 \). By Theorem 3.2 since \( x + n \in S \). It follows that \( y + x \in S \) since \( y = n \).

3.7. Lemma. If \( x \in R_1 - S, y \in S \), then \( y + x \in R_0 \).

Proof. Let \( x \in R_1 - S \) and \( y \in S \). Then \( y = x + n \in S + R_0 \). This implies that \( x + n \in R_0 \) by Theorem 3.2 since \( x + n \in S \). But \( y + x = (y + x) \). Therefore \( y = x \). Hence \( s + y \in R_0 \) by Theorem 3.2 since \( x + n \in R_0 \). If \( s + y \in R_0 \) then \( y + x \in R_0 \). By Lemma 3.3 since \( s + y \in S \). (Note that \( n + x \in S \) by Theorem 3.2 since \( n + x \in S \).)

3.8. Lemma. If \( x, y \in R_1 - S \) then \( y + x \in S \).

Proof. Let \( x, y \in R_1 - S \). Then \( y = x + n \in S + R_0 \) and \( x = (x + n) \). This implies that \( x + n \in S + R_0 \) since \( x + n \in S + R_0 \). By Lemma 3.2 and, therefore, \( x + n \in S + R_0 \) since \( x + n \in S + R_0 \). Now if \( x + n \in S + R_0 \) then \( x + n \in S \). This implies that \( y + x \in S \). If \( y + x \in S \) then \( y + x \in S \).

3.9. Lemma. Let \( R \) be a special Jacobi set and let \( x, y, z \in R \). Then \( x + y \in R_0 \) and \( x + y \in R_0 \). If \( x + y \in R_0 \) then \( y + x \in R_0 \).

Proof. If \( x = y \) then \( y + x \in R_0 \) by the previous theorem. If \( x = y \) then \( x = (y + x) \). This implies that \( y = (y + x) \). By Theorem 1.4. Choose \( i = p - 1 \). Then \( y + x \in R_0 \).

3.10. Theorem. Let \( R \) be an irreducible special Jacobi set and let \( S \) be any regular subalgebra set of \( R \). Then \( R = S \oplus R_0 \) (direct sum).

Proof. Let \( R \) be an irreducible special Jacobi set and let \( S \) be any regular subalgebra set of \( R \). Then \( R \subseteq S \oplus R_0 \) by Theorem 2.2. For the other inclusion, let \( x \in S \) and \( y \in S \). Since \( R \) is irreducible then there exists a chain of roots \( c_0 = c_1, c_2, \ldots, c_n \in S \) such that \( c_i + c_{i+1} \in R \) for all \( i = 0, 1, \ldots, n-1 \). Now, we proceed by induction on \( n \). If \( k = 1 \) then \( s + n \in c_k + c_1 \). Suppose this is true for \( k - 1 \).