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The least number of fixed points of bimaps

by

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Abstract. A bimap $\varphi\colon X\to X$ on a topological space X is a continuous multifunction for which the image of each point consists of either one or two points. The Nielsen number of a bimap φ is at most one. Nevertheless we show that it is still the optimal lower bound for the number of fixed points of bimaps on a compact triangulable manifold of dimension $\leqslant 3$, i.e. that there exists a bimap in the bihomotopy class of φ which has exactly $N(\varphi)$ fixed points.

1. Introduction. It is the purpose of this paper to construct bimaps in a given bihomotopy class which have minimal fixed point sets, where a *bimap* $\varphi\colon X\to X$ on a topological space X is a continuous (i.e. both upper and lower semicontinuous) multifunction for which the image of each point consists of either one or two points. Hence bimaps can be regarded as the simplest possible multifunctions, and include single-valued maps as a special case. They are also the special case which $\{1, n\}$-valued multifunctions and symmetric product maps have in common.

The construction of minimal fixed point sets for single-valued maps uses Nielsen fixed point theory. For a single-valued map $f\colon X\to X$ the *Nielsen number* $N(f)$ is a lower bound for the number of fixed points for all maps $g\colon X\to X$ in the homotopy class of f, and for most polyhedra it is an optimal lower bound, i.e. there exists a map g homotopic to f which has precisely $N(f)$ fixed points. The construction of such a map g uses essentially geometric methods from Nielsen fixed point theory, but also a suitable fixed point index which is obtained in a more algebraic manner. (See e.g. [1] and [2].) To adapt this construction to bimaps, we mix the more geometric methods suited to $\{1, n\}$-valued multifunctions with the more algebraic methods suited to symmetric product maps, and use the fixed point index for bimaps from [9] as well a Nielsen number.

Fixed point sets of bimaps $\varphi\colon X\to X$ are equivalent to fixed point sets of symmetric product maps $f\colon X\to X_2$. A Nielsen number $N(f)$ for a symmetric product map $f\colon X\to X_n$ was defined by S. Masih [4], who proved that $N(f)$ is a lower bound for the number of fixed points for all symmetric product maps in the homotopy class of f.

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Recently D. Miklaszewski [5] has shown that this Nielsen number has, for $n \geqslant 2$, quite different values than in the case $n = 1$, i.e. in the case of single-valued maps: it is zero if the Lefschetz number $L(f) = 0$, and one if $L(f) \neq 0$. Hence $N(f)$ is a rather trivial lower bound for the number of fixed points of $f\colon X \to X_2$, and (equivalently) for the number of fixed points of bimaps.

Nevertheless we will show, in the Minimum Theorem 3.7 which is the main result of this paper, that such a Nielsen number is still optimal for a bimap $\varphi\colon X \to X$, as we will construct a bimap $\psi\colon X \to X$ which is bihomotopic to φ and has precisely $N(\varphi)$ fixed points. But we have to modify the definition of the Nielsen number in [4] and [5] and use the fixed point index from [9] as it is necessary to remove an isolated fixed point of index zero, and it is not known whether this can be done with Masih's index. It is very likely (especially in view of [9], Corollary 5.2 and Example 4.10) that Masih's index and the index in [9] are the same, but details vould have to be checked.

To prove Theorem 3.7 we first define in § 2 the fixed point classes of a bimap $\varphi\colon X \to X$ in a way which is equivalent to Masih's, but stresses the geometric nature of the definition which is needed in later proofs. The result by Miklaszewski [5] shows that φ has at most one fixed point class, and hence the Nielsen number $N(\varphi)$ is $\leqslant 1$, depending on the fixed point index (Corollaries 2.2 and 2.3). Thus it is clear that $N(\varphi)$ is a lower bound for the number of fixed points of φ and is bihomotopy invariant (Propositions 2.4 and 2.5). We have illustrated the value of $N(\varphi)$ with two examples in which results from [9] are used (Examples 2.6 and 2.7).

The main result, Theorem 3.7, is proved in § 3. The proof is modelled on the one of the corresponding results for n-valued multifunctions [7], Theorem 5.2, and as in [7] we restrict our attention to compact triangulable manifolds of dimension $\geqslant 3$ in order to shorten the proof. It is not known whether Theorem 3.7 is still true for lower dimensional manifolds, as it is only known that it is false for single-valued maps on surfaces with respect to single-valued (rather than bi-valued) homotopies. An interesting open problem is the possibility of the extension of Theorem 3.7 to $\{1, n\}$-valued multifunctions and to symmetric product maps $f\colon X \to X_n$ for $n > 2$.

We have not repeated definitions, explanations of terminology and results from [9], and in § 3 we have also used terminology and results from [7]. The paper can therefore not be understood without at least a superficial knowledge of these papers. We also assume that the reader is familiar with the basic results of Nielsen fixed point theory for single-valued maps.

2. The Nielsen number of a bimap. In order to construct bimaps with a minimal fixed point set we need the concepts of a fixed point class and its index for a bimap $\varphi\colon X \to X$ on a compact polyhedron X. The restriction to compact polyhedra is necessary as we shall use the properties of the fixed point index from [9] which were proved in this setting.

We call a bimap $\alpha\colon I \to X$, with $I = [0,1]$, a *bipath* in X, and say that two bipaths $\alpha_0, \alpha_1\colon I \to X$ are *bihomotopic* (written $\alpha_0 \sim \alpha_1$) if there exists a bihomotopy $\{\alpha_t\}\colon I \to X$ which leaves end points fixed, i.e. we require that $\alpha_t(0) = \alpha_0(0) = \alpha_1(0)$ and $\alpha_t(1) = \alpha_0(1) = \alpha_1(1)$ for all $t \in I$. Two fixed points x, x' of a bimap $\varphi\colon X \to X$ are called *φ-equivalent* if there exists a (single-valued) path $p\colon I \to X$ from $p(0) = x$ to $p(1) = x'$ so that the bipath $\varphi \circ p\colon I \to X$ is bihomotopic to a bipath $\alpha\colon I \to X$ which has p as a selection. In other words, $\alpha = \{a_1, a_2\}$ must split into two maps $a_k\colon I \to X$ which can be indexed so that $a_1 = p$. Clearly φ-equivalence is an equivalence relation on $\mathrm{Fix}\,\varphi$, and we call the equivalence classes the *fixed point classes* of φ.

As a bimap $\varphi\colon X \to X$ induces a symmetric product map $f\colon X \to X_2$, an equivalence relation on $\mathrm{Fix}\,\varphi = \mathrm{Fix}\, f$ was defined by S. Masih [4], § 4, as follows: a point $z \in X \times X$ is called *admissible* with respect to $x \in \mathrm{Fix}\, f$ if $q(z) = f(x)$ and $p_1(z) = x$, where $q\colon X \times X \to X_2$ is the quotient map and $p_1\colon X \times X \to X$ is the projection onto the first factor. Two points $x, x' \in \mathrm{Fix}\, f$ are called *f-equivalent* if there exist points $z, z' \in X \times X$ which are admissible with respect to x, x' and a path $c\colon I \to X \times X$ from z to z' so that $q \circ c$ is homotopic to $f \circ p_1 \circ c$ with end points fixed. We show that for bimaps Masih's definition coincides with ours.

THEOREM 2.1. *Let $\varphi\colon X \to X$ be a bimap which induces the symmetric product map $f\colon X \to X_2$. Then $x, x' \in \mathrm{Fix}\,\varphi$ are φ-equivalent if and only if they are f-equivalent.*

Proof. (i) If x, x' are φ-equivalent, then there exists a path $p\colon I \to X$ from x to x' so that $\varphi \circ p$ is bihomotopic to a bipath $\alpha = \{p, a_2\}$. Let $z = (p(0), a_2(0)) \in X \times X$, $z' = (p(1), a_2(1)) \in X \times X$ and $c\colon I \to X \times X$ be the path given by $c(s) = (p(s), a_2(s))$. Then z, z' are admissible with respect to x, x', and the bihomotopy from $\varphi \circ p$ to α induces a homotopy from $f \circ p = f \circ p_1 \circ c$ to $q \circ c$. So x and x' are f-equivalent.

(ii) If x, x' are f-equivalent, then there exist $z, z' \in X \times X$ which are admissible with respect to x, x' and a path $c\colon I \to X \times X$ from z to z' so that $q \circ c$ is homotopic to $f \circ p_1 \circ c$. If $\alpha = p_1 \circ q^{-1} \circ q \circ c$ and $p = p_1 \circ c$, then $\varphi \circ p \sim \alpha$, and α has p as a selection.

Recently D. Miklaszewski [5] has shown, using Masih's definition of fixed point classes, that any two fixed points of a symmetric product map $f\colon X \to X_2$ (or, more generally, $f\colon X \to X_n$ for all $n \geqslant 2$) on a path-connected space X are f-equivalent. From this rather surprising result we obtain

COROLLARY 2.2. *If $\varphi\colon X \to X$ is a bimap on a path-connected space X, then either $\mathrm{Fix}\,\varphi = \emptyset$ or φ has one fixed point class.*

If $\varphi\colon X \to X$ is a bimap on a compact polyhedron X with $\mathrm{Fix}\,\varphi \neq \emptyset$, then we take any open set U of X with $\mathrm{Fix}\,\varphi \subset U$ and define the *index of the fixed point class* $F = \mathrm{Fix}\,\varphi$ of φ by $\mathrm{Ind}(F) = \mathrm{Ind}(\varphi, U)$, where Ind is the fixed point index of a bimap from [9]. It follows from the additivity of the index [9], Theorem 4.6, that the definition is independent of U, and so we have, in particular, $\mathrm{Ind}(F) = \mathrm{Ind}(\mathrm{Fix}\,\varphi) = \mathrm{Ind}(\varphi, X)$. As usual we call F an *essential fixed point class* of φ if $\mathrm{Ind}(F) \neq 0$, and define the *Nielsen number* $N(\varphi)$ as the number of essential fixed point classes of φ. We also define, as in [9], that $\mathrm{Ind}(\varphi, X) = 0$ if $\mathrm{Fix}\,\varphi = \emptyset$. Thus Corollary 2.2 implies

COROLLARY 2.3. *If $\varphi\colon X \to X$ is a bimap on a compact connected polyhedron, then*

$$N(\varphi) = \begin{cases} 0 & \text{if } \mathrm{Ind}(\varphi, X) = 0, \\ 1 & \text{if } \mathrm{Ind}(\varphi, X) \neq 0. \end{cases}$$

It is clear from the definition of $N(\varphi)$ and from the homotopy invariance of $\mathrm{Ind}(\varphi, X)$ [9], Theorem 4.7, that $N(\varphi)$ has the following two basic properties which are typical for Nielsen-type numbers.

PROPOSITION 2.4 (lower bound). *If $\varphi\colon X\to X$ is a bimap on a compact connected polyhedron with $N(\varphi)\neq 0$, then φ has at least one fixed point.*

PROPOSITION 2.5 (homotopy invariance). *If $\varphi_0, \varphi_1\colon X\to X$ are bimaps on a compact connected polyhedron which are bihomotopic, then $N(\varphi_0)=N(\varphi_1)$.*

Here are two examples where it is easy to compute $N(\varphi)$.

EXAMPLE 2.6. Let $\varphi=\{f_1, f_2\}\colon X\to X$ be a bimap on a compact connected polyhedron which splits into two (not necessarily distinct) maps. If $L(f)$ denotes the Lefschetz number of the map $f\colon X\to X$, then [9], Example 4.10 implies

$$\mathrm{Ind}(\varphi, X)=\mathrm{ind}(f_1, X)+\mathrm{ind}(f_2, X)=L(f_1)+L(f_2),$$

and so Corollary 2.3 shows that

$$N(\varphi)=\begin{cases}0 & \text{if } L(f_1)=-L(f_2),\\ 1 & \text{otherwise.}\end{cases}$$

EXAMPLE 2.7. Let $\varphi\colon S^n\to S^n$ be a bimap of the n-sphere S^n of degree d, which means that the induced symmetric product map $f\colon X\to X_2$ is of degree d. (See the definition in [9], § 4.) If $\psi=\{\mathrm{id}, g\}\colon S^n\to S^n$ is a bimap which splits into the identity map and a map g of degree $d-1$, then [9], Lemmas 2.1 and 2.3 show that φ and ψ are bihomotopic. Hence it follows from Proposition 2.5 and Example 2.6 that

$$N(\varphi)=\begin{cases}0 & \text{if } d=2 \text{ and } n \text{ odd, or } d=-2\ n \text{ even},\\ 1 & \text{otherwise.}\end{cases}$$

3. Bimaps with minimal fixed points sets. We now want to show, in Theorem 3.7, that although $N(\varphi)$ is at most one, it still can be realized by a bimap ψ in the bihomotopy class of φ if φ is a bimap on a compact connected triangulable manifold (with or without boundary) of dimension $\geqslant 3$. The proof will proceed along the lines of the proof of the corresponding theorem for 2-valued multifunctions [7], Theorem 5.2, and make use of the fact that we can homotope φ to a bimap which is 2-valued at all of its fixed points [9], Theorem 5.2. As in the n-valued case [7], Remark 5.3, it is very likely that the minimum theorem can be extended to bimaps of polyhedra which satisfy the assumptions of [2], Theorem 5.3, but the proof would likely be quite a bit longer.

The proof of [7], Theorem 5.2, uses as a tool the Coincidence Lemma 2.2 of [7] which was obtained with the help of [7], Lemma 2.1. We establish the equivalent lemmas for bimaps. As in [7] we write $B^m(r)$ for the m-ball $\{x\in R^m|\ |x|\leqslant r\}$, choose a metric d of the manifold M, and let $\bar{d}(f, g)$ denote the distance in the sup metric between two maps $f, g\colon M\to M$.

LEMMA 3.1. *Let $0<k<m$, $-1\leqslant l<k$ and $r>0$. If σ^l is a face of σ^k, then every bimap*

$$\chi\colon (\dot\sigma^k, \dot\sigma^k-\bar\sigma^l, \bar\sigma^l)\to(B^m(r), B^m(r)-0, 0)$$

has an extension to a bimap

$$\bar\chi\colon (\bar\sigma, \bar\sigma^k-\bar\sigma^l, \bar\sigma^l)\to(B^m(r), B^m(r)-0, 0).$$

Proof. If $l=-1$, then χ is of the form $\chi\colon \dot\sigma^k\to B^m(r)-0$, and hence induces a symmetric product map $v\colon \dot\sigma^k\to Y_2$, where $Y=B^m(r)-0$. Thus it follows from [3], (1.4), that $\pi_{k-1}(Y_2)=\pi_{k-1}(S_2^{m-1})=0$ for $k<m$, and so v extends to a symmetric product map $\bar v\colon \bar\sigma^k\to Y_2$ which induces the desired bimap $\bar\chi\colon \bar\sigma^k\to B^m(r)-0$.

If $l\geqslant 0$, then the proof is analogous to the proof of [7], Lemma 2.1, for $l\geqslant 0$ with v and $w=v\circ q^{-1}$ replaced by the bimaps χ and $\chi\circ q^{-1}$.

We need only a special case of [7], Lemma 2.2, for bimaps.

LEMMA 3.2 (Coincidence Lemma). *Let M be a compact manifold of dimension $\geqslant 3$, let $I=[0,1]$, $P=I\times I$, $P_1=\mathrm{Bd}(I\times I)$ and $P_0=(\mathrm{Bd}\, I)\times I$. Given a bimap $\varphi\colon P\to \mathrm{Int}\, M$ and a map $g\colon P\to \mathrm{Int}\, M$ so that*

$$g(x)\in\varphi(x) \text{ for all } x\in P_0, \quad g(x)\notin\varphi(x) \text{ for all } x\in P_1-P_0,$$

there exists a bimap $\varphi'\colon P\to\mathrm{Int}\, M$ so that

$$\varphi'(x)=\varphi(x) \text{ for all } x\in P_1 \text{ and } g(x)\notin\varphi'(x) \text{ for all } x\in P-P_0.$$

Proof. The proof is similar to the proof of [7], Lemma 2.2, but some modifications have to be made as φ can be either 1- or 2-valued.

As $\varphi(P)\cup g(P)$ is compact, we can choose finite collections $\{V_j\}$, $\{U_j\}$ of open sets in M so that $\varphi(P)\cup g(P)\subset\bigcup V_j$, each $V_j\subset U_j\subset \mathrm{Cl}\,U_j\subset\mathrm{Int}\, M$, and so that for each index j there exists a homomorphism $h_j\colon (\mathrm{Cl}\,U_j, \mathrm{Cl}\,V_j)\to(B^m(1), B^m(1/2))$. Let $\lambda>0$ be the Lebesgue number of the cover $\{V_j\}$ and determine $\delta_2, \varepsilon_1, \delta_1, \varepsilon_0$ with $0<\delta_k<1/2$ and $0<\varepsilon_0\leqslant\varepsilon_1\leqslant\lambda/3$ as in the proof of [7], Lemma 2.2. Using subdivisions we can assume that for each simplex σ of P the diameter diam $g(\bar\sigma)<\varepsilon_0/4$ and, if $\varphi|\bar\sigma$ is not 2-valued, then diam $\varphi(\bar\sigma)<\varepsilon_0/4$. If $\varphi|\bar\sigma$ is 2-valued, then $\varphi|\bar\sigma=\{f_1, f_2\}$ splits into two maps, and we can assume that diam$f_i(\bar\sigma)<\varepsilon_0/4$ for $i=1, 2$. Finally, we can assume that P_0 is full in P. Let $C=\{x\in P|\ g(x)\in\varphi(x)\}$ be the coincidence set of φ and g. We shall define φ' inductively on the 0-, 1- and 2-simplexes of P.

Let first $\sigma^0=x$ be a 0-simplex of $(P-P_1)\cap C$. If $\varphi(x)=g(x)$ is 1-valued, then we select $\varphi'(x)\in\bigcup V_j$ arbitrarily as a point with $0<d(\varphi'(x), g(x))<\varepsilon_0$. If $\varphi(x)=\{g(x), y_2\}$ is 2-valued, then we select $y_1\in\bigcup V_j$ arbitrarily with $0<d(y_1, g(x))<\varepsilon_0$, and put $\varphi'(x)=\{y_1, y_2\}$. If $\sigma^0=x$ is a 0-simplex not contained in $(P-P_1)\cap C$, then we put $\varphi'(x)=\varphi(x)$. Thus φ' is defined on the 0-skeleton of P.

Now let σ^1 be a 1-simplex of $P-P_1$ with $\bar\sigma^1\cap C\neq\emptyset$. If $\varphi|\bar\sigma^1$ is 2-valued, then $\varphi|\bar\sigma^1=\{f_1, f_2\}$ splits into two maps. If $g(x)=f_i(x)$ for some $x\in\bar\sigma^1$, where $i=1, 2$, then

$$\mathrm{diam}\ [f_i(\bar\sigma^1)\cup g(\bar\sigma^1)\cup\varphi'(\dot\sigma^1)]<\varepsilon_0/4+\varepsilon_0/4+2\varepsilon_0\leqslant\lambda,$$

so $f_i(\bar\sigma^1)\cup g(\bar\sigma^1)\cup\varphi'(\dot\sigma^1)\subset V_j$ for some index j. We change $f_i\colon \bar\sigma^1\to V_j$ to a map $f_i'\colon \bar\sigma^1\to V_j$ with $\bar d(f_i, f_i')<\varepsilon_1$, $f_i'(x)=f_i(x)$ for $x\in P_1$ and $f_i'(x)\neq g(x)$ for $x\in P-P_0$ as in the proof of [7], Lemma 2.2. If $g(x)\neq f_i(x)$ for all $x\in\bar\sigma^1$, we put $f_i'=f_i$. Then we define $\varphi|\bar\sigma^1=\{f_1', f_2'\}$.

If $\varphi|\bar{\sigma}^1$ is not 2-valued, then

$$\operatorname{diam}[\varphi(\bar{\sigma}^1)\cup g(\bar{\sigma}^1)\cup\varphi'(\dot{\sigma}^1)]<\lambda,$$

so $\varphi(\bar{\sigma}^1)\cup g(\bar{\sigma}^1)\cup\varphi'(\dot{\sigma}^1)\subset V_j$ for some index j. We now proceed as in the proof of [7], Lemma 2.2, with φ instead of f and Lemma 3.1 instead of [7], Lemma 2.1, to change $\varphi|\bar{\sigma}^1$ to a bimap $\varphi'|\bar{\sigma}^1\colon \bar{\sigma}^1\to\operatorname{Int}M$ with Hausdorff distance $\varrho(\varphi(x),\varphi'(x))<\varepsilon_1$ for all $x\in\bar{\sigma}^1$, so that $\varphi'(x)=\varphi(x)$ for $x\in\bar{\sigma}^1\cap P_1$ and $g(x)\notin\varphi'(x)$ for $x\in\bar{\sigma}^1\cap(P-P_0)$.

If $\bar{\sigma}^1\cap(P-P_1)\cap C=\varnothing$, we define $\varphi'|\bar{\sigma}^1=\varphi|\bar{\sigma}^1$, and have thus constructed a bimap φ' on the 1-skeleton of P with $\bar{d}(\varphi,\varphi')<\varepsilon_1$ which satisfies Lemma 3.2. Then we construct φ' on all 2-simplexes of P in the same way to obtain the desired bimap $\varphi'\colon P\to\operatorname{Int}M$.

As in [7] we use the concept of a special homotopy, introduced by Boju Jiang [2]. If $\varphi_0,\varphi_1\colon A\to X$ are two bimaps from a subspace A of X into X and $\Phi\colon A\times I\to X$ is a bihomotopy from φ_0 to φ_1, we call Φ a *special bihomotopy* if $\operatorname{Fix}\varphi_t=\operatorname{Fix}\varphi_0$ for all $t\in I$, and $\varphi_t(x)=\varphi_0(x)$ for all $x\in\operatorname{Fix}\varphi_0$ and $t\in I$. Two bimaps $\varphi_0,\varphi_1\colon A\to X$ which have the same fixed point set are called *specially bihomotopic* if there exists a special bihomotopy from φ_0 to φ_1.

The next lemma reduces to [2], Lemma 2.1, if φ is 1-valued, and is a special case of [7], Lemma 3.2, if φ is 2-valued. Its proof is analogous to the proof of [2], Lemma 2.1, and is omitted.

Lemma 3.3 (Special bihomotopy extension). *Let A be a subspace of X and let A and X be* ANR's. *If $\varphi_0\colon X\to X$ is a bimap and $\Phi_A\colon A\times I\to X$ a special bihomotopy of $\varphi_0|A$, then Φ_A can be extended to a special bihomotopy $\Phi\colon X\times I\to X$ of φ_0.*

We call a bipath $\alpha\colon I\to M$ *special* with respect to the arc $Q=q(I)$ in M if

$$q(0)\in\alpha(0),\quad q(1)\in\alpha(1)\quad\text{and}\quad q(s)\notin\alpha(s)\quad\text{for}\quad 0<s<1.$$

Two special bipaths $\alpha_0,\alpha_1\colon I\to M$ are called *specially bihomotopic* if there exists a bihomotopy $\{\alpha_t\}\colon I\to M$ so that every bipath $\alpha_t\colon I\to M$ is special with respect to q. The relation between special bihomotopies of bimaps and bipaths is given in

Lemma 3.4. *Let $Q=q(I)$ be an arc in M from x_1 to x_2, let $\varphi\colon M\to M$ be a bimap with $\operatorname{Fix}\varphi\cap Q=\{x_1,x_2\}$ and let $\alpha\colon I\to M$ be a bipath from x_1 to x_2. Then the bimap $\varphi|Q\colon Q\to M$ is specially bihomotopic to the bimap $\alpha\circ q^{-1}\colon Q\to M$ if and only if the bipath $\alpha\colon I\to M$ is specially bihomotopic to the bipath $\varphi\circ q\colon I\to M$.*

The very easy proof is omitted. (See the corresponding Lemma 4.2 in [7].)

The next lemma is crucial for the proof of Theorem 3.7. Its proof is similar to the proof of [7], Lemma 3.3, but easier.

Lemma 3.5. *Let M be a compact manifold of dimension $\geqslant 3$. If two bipaths α_0, $\alpha_1\colon I\to\operatorname{Int}M$ are special with respect to an arc $q\colon I\to M$ and are bihomotopic, then they are specially bihomotopic with respect to q.*

Proof. Let $\{\alpha_t\}\colon I\to M$ be the given bihomotopy. Using a collaring argument we can assume that if $\operatorname{Bd}M\neq\varnothing$, then $\alpha_t(s)\in\operatorname{Int}M$ for all $(s,t)\in I\times I$. The special bihomotopy $\{\alpha'_t(s)\}=\{\varphi'(s,t)\}$ can then be obtained from the Coincidence Lemma 3.2 with $\varphi(s,t)=\alpha_t(s)$ and $g(s,t)=q(s)$ for all $(s,t)\in I\times I$.

In the single-valued case the uniting of two fixed points in the same fixed point class can be carried out by reducing the general case to that of a small deformation. The next lemma, which corresponds to [7], Lemma 3.3, contains the situation to which the uniting of two fixed points of a bimap can be reduced. Due to [9], Theorem 5.2, we will only have to deal with fixed points at which the bimap is 2-valued.

Lemma 3.6. *Let M be a compact manifold of dimension $\geqslant 2$, let $x_1,x_2\in\operatorname{Int}M$ be two isolated fixed points of the bimap $\varphi\colon M\to M$, and let $Q=q(I)$ be an arc from x_1 to x_2 with $\operatorname{Fix}\varphi\cap Q=\{x_1,x_2\}$. Assume that $\varphi|Q$ is specially bihomotopic to a bimap $\alpha\circ q^{-1}\colon Q\to M$, where $\alpha=\{p_1,p_2\}$ is a bipath which splits so that $p_1(0)=x_1$, $p_1(1)=x_2$ and $p_1(s)\neq p_2(s)$ for $0\leqslant s\leqslant 1$. Then there exists an $\varepsilon>0$ so that $\bar{d}(p_1,q)<\varepsilon$ implies that φ is bihomotopic to a bimap $\varphi'\colon M\to M$ relative $M-N$, where N is a closed tubular neighbourhood of Q with $\operatorname{Fix}\varphi\cap N=\{x_1,x_2\}$, and*

(i) $\operatorname{Fix}\varphi'=\operatorname{Fix}\varphi-\{x_1\}$,

(ii) *φ' is 2-valued at x_2.*

Proof. Lemma 3.3 shows that φ is specially bihomotopic to a bimap $\psi\colon M\to M$ with $\psi|Q=\alpha\circ q^{-1}$. As $\psi|Q$ is 2-valued, there exists a closed tubular neighbourhood N of Q so that $\psi|N$ is 2-valued, and hence $\psi|N=\{f_1,f_2\}$ splits into two distinct maps $f_i\colon N\to M$ [6], Lemma 2.1, with $\chi=\bar{d}(f_1,f_2)>0$. We index the f_i so that $f_i(x)=p_i\circ q^{-1}(x)$ for $i=1,2$ and $x\in Q$. Then [7], Lemma 3.2, shows that there exists an $\varepsilon>0$ so that $\bar{d}(p_1,q)<\varepsilon$ implies that f_1 is γ-homotopic relative $\operatorname{Bd}N$ to a map $g_1\colon N\to M$ with $\operatorname{Fix}g_1=\operatorname{Fix}f_1-\{x_1\}$. If we define the bimap $\varphi'\colon M\to M$ by

$$\varphi'(x)=\begin{cases}\{g_1(x),f_2(x)\} & \text{if } x\in N,\\ \varphi(x) & \text{if } x\in M-N,\end{cases}$$

then φ' satisfies Lemma 3.6.

We are now ready for the proof of the Minimum Theorem, which is done as usual in three steps. First the bimap is approximated by a fix-finite one, and by using [9], Theorem 5.2, we can also ensure that the bimap is 2-valued at each of its fixed points. Secondly, two fixed points in the same fixed point class are united, and this will be done in the proof of Theorem 3.7 with the help of Lemma 3.6. Thirdly, isolated fixed points of index zero are removed, and [9], Theorem 5.3, shows that this can be done.

Theorem 3.7 (Minimum Theorem). *Let M be a compact connected triangulable manifold of dimension $\geqslant 3$. Then every bimap $\varphi\colon M\to M$ is bihomotopic to a bimap $\psi\colon M\to M$ which is fixed point free if $N(\varphi)=0$ and has one fixed point if $N(\varphi)\neq 0$.*

Proof. According to [9], Theorem 5.2, we can assume that φ is fix-finite, that all its fixed points lie in maximal simplexes and that φ is 2-valued on $\operatorname{Fix}\varphi$. If $\operatorname{Bd}M\neq\varnothing$, then a collaring argument allows us to assume that $\varphi(M)\subset\operatorname{Int}M$. In view of Corollary 2.3 and [9], Theorem 5.3 it is sufficient to show that if x_1 and x_2 are two fixed points of φ, then φ is bihomotopic to a bimap $\varphi'\colon M\to M$ which is 2-valued on $\operatorname{Fix}\varphi'=\operatorname{Fix}\varphi-\{x_1\}$.

So let x_1 and x_2 be two isolated fixed points of $\varphi\colon M\to\operatorname{Int}M$. As they are in the same fixed point class, there exists a path $p\colon X\to M$ from x_1 to x_2 so that $\varphi\circ P\sim\alpha$, where $\alpha = \{p, p_2\}$ is a bipath which has p as a selection. We homotope p with end points fixed to a path q so that $q(I) = Q$ is an arc in $\operatorname{Int} M$ with $\operatorname{Fix}\varphi = \{x_1, x_2\}$. Then $\{q, p_2\} \sim \{p, p_2\} \sim \varphi \circ p \sim \varphi \circ q$. For any $\varepsilon > 0$ let

$$p_\varepsilon(s) = q(s - \delta \sin \pi s) \quad \text{for } 0 \leqslant s \leqslant 1,$$

where $0 < \delta = \delta(\varepsilon) < 1$ is selected so that $\bar{d}(p_\varepsilon, q) < \varepsilon$. It follows from an easy general position argument that the path p_2 is homotopic to a path p_2': $M \to \operatorname{Int} M$ with $p_2'(s) \notin Q$ for $0 < s < 1$. Then $\alpha_0 = \varphi \circ q$ and $\alpha_1 = \{p_\varepsilon, p_2'\}$ are two bipaths which are special with respect to q, and as

$$\alpha_0 \sim \{q, p_2\} \sim \{p_\varepsilon, p_2'\} = \alpha_1,$$

they are bihomotopic. According to Lemma 3.5 they are specially bihomotopic, and hence the bimaps $\varphi | Q$ and $\alpha_1 \circ q^{-1}$: $Q \to M$ are specially bihomotopic. Therefore Lemma 3.6 states that we can choose $\varepsilon > 0$ so that φ is bihomotopic to a bimap φ' with the necessary properties.

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Making the hugeness of $\varkappa$ resurrectable after $\varkappa$-directed closed forcing

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Abstract. We consider generalizations, to the context of huge cardinals, of Laver's result ([7]) on the indestructibility of supercompactness.

§ 0. Introduction. Questions of which large cardinals are preserved by which forcing notions often arise. Most large cardinal properties are preserved by small forcing notions. In particular, the inaccessibility, weak compactness, measurability, supercompactness, or hugeness, of a cardinal $\varkappa$ is preserved by any forcing notion of cardinality less than $\varkappa$ (see [8]).

In one sense, large forcing notions trivially preserve large cardinals, where, by large, we do not mean large in cardinality, but large in closure. In particular, the inaccessibility (respectively weak compactness, measurability, λ-supercompactness, hugeness with target λ) of a cardinal $\varkappa$ is preserved by any forcing notion which is $\varkappa$-closed (respectively $\varkappa^+$-closed, $\varkappa^+$-closed, $(\lambda^{\varkappa})^+$-closed, λ^+-closed). This is so since a forcing notion adds no new subsets of the ground model of cardinality less than its degree of closure.

It is easy to see that all of the large cardinal properties we have mentioned can be destroyed by a forcing notion which is γ-closed, for any $\gamma < \varkappa$ we choose, and has cardinality $\varkappa$. Simply consider the standard forcing notion for adding a function from γ to $\varkappa$. In the extension, $\varkappa$ is not even a cardinal.

More interesting questions arise when we consider preservation of large cardinal properties of $\varkappa$, by forcing notions which are $\varkappa$-closed. These are the types of forcing notions that allow us to manipulate the value of $2^\varkappa$.

Clearly, it is consistent that $\varkappa$-closed forcing can destroy the measurability of $\varkappa$. Consider a model in which $\varkappa$ is measurable, and the GCH holds (the standard model for this is $L[U]$, the collection of all sets constructible from U, where U is any normal ultrafilter on $\varkappa$). The standard forcing notion that makes $2^\varkappa = \varkappa^{++}$ is $\varkappa$-closed. Hence, the GCH below $\varkappa$ is not affected. It follows that, in the

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