6. Open problems.

PROBLEM 1. In Theorem 5 suppose we take F_0, F_1 to be Borel but E an arbitrary second countable metrizable or even a Π^1_1-set. Do the conclusions of Theorem 5 hold in this case?

PROBLEM 2. Does Theorem 7 hold for a Σ^1_1-set A? We do not know the answer even when Z is a convex subset of $\boldsymbol{R}^2$.

PROBLEM 3. Can Theorem 8 be extended for Π^1_1-sets A? We do not know the answer even when $Z = \boldsymbol{R}$.

A question related to Problem 3 is the following:

PROBLEM 4. Let C_0 and C_1 be two disjoint Π^1_1-sets in $E \times X$ such that for every $e \in E$, the sections $C_0(e)$ and $C_1(e)$ are closed. Further assume that there is a Borel set B containing C_0 but disjoint from C_1. Do there exist disjoint Borel sets B_0 and B_1 such that $C_0 \subseteq B_0$, $C_1 \subseteq B_1$ and for every $e \in E$, the sections $B_0(e)$ and $B_1(e)$ are closed in X?

References

[1] G. F. Andrus and L. Brown, *Measurable extension theorems*, J. Math. Anal. Appl. 96 (1983), 454–462.
[2] R. Barua, *Structure of hyperarithmetical sets of ambiguous Borel classes*, preprint.
[3] D. Blackwell and C. Ryll-Nardzewski, *Non-existence of everywhere proper conditional distributions*, Ann. Statist. 34 (1963), 223–225.
[4] F. S. de Blasi and J. Myjak, *On the random Dugundji extension theorem*, J. Math. Anal. Appl. 128 (1987), 305–311.
[5] C. Dellacherie, *Ensembles Analytiques: Théorèmes de séparation et applications*, Lecture Notes in Math. 465, Springer-Verlag, Berlin–Heildelberg.
[6] J. Dugundji, *Topology*, Prentice Hall of India, New Delhi 1975.
[7] O. Hanš, *Measurability of extensions of continuous random transformations*, Ann. Statist. 30 (1959), 1152–1157.
[8] A. Louveau, *A separation theorem for Σ^1_1 sets*, Trans. Amer. Math. Soc. 260 (1980), 363–378.
[9] A. Maitra and B. V. Rao, *Generalizations of Castaing's theorem on selectors*, Colloq. Math. 42 (1979), 295–300.
[10] D. E. Miller, *Borel selectors for separated quotients*, Pacific J. Math. 91 (1980), 187–198.
[11] Y. N. Moschovakis, *Descriptive Set Theory*, North-Holland Publishing Company, Amsterdam–New York–Oxford.
[12] J. Saint-Raymond, *Boréliens à coupe K_σ*, Bull. Soc. Math. France 104 (1976), 389–400.
[13] H. Sarbadhikari and S. M. Srivastava, *Random Tietze and Dugundji extension theorems*, J. Math. Anal. Appl., submitted.

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A splitting theorem for $\mathcal{F}$-products

by

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Abstract. Let $\mathcal{F}$ be a non-principal ultrafilter on an infinite index set A. Let $\{M_a\colon a \in A\}$ be a family of left R-modules. We define the $\mathcal{F}$-product of the M_a's to be $\prod^{\mathcal{F}}_{a\in A} M_a = \{(m_a)_{a\in A} \in \prod_{a\in A} M_a\colon \{a \in A\colon m_a = 0\} \in \mathcal{F}\}$. In the present paper, we determine a necessary condition for the $\mathcal{F}$-product of the M_a's to split in the corresponding direct product. This condition will be given in terms of the lattice of ideals which are annihilators of subsets of a certain factor ring of R, and will depend on $\mathcal{F}$.

R will always denote a ring with identity, all modules will be unital, A will always denote an infinite index set and $|X|$ will always denote the cardinality of a set X.

1. Introduction. The question of when the canonical embedding of a direct sum of modules splits in the corresponding direct product has been extensively studied. See, for example, [2], [9], [11], [18], and [19]. Recently modules that are in between direct sums and direct products, called $\varkappa$-products, have been introduced. See, for example, [5], [6], [13], [14] and [17]. In [14], a necessary condition was determined for the canonical embedding of the $\varkappa$-product of modules in the corresponding direct product to split. This result generalized the above-mentioned classical theorems on the sum-product splitting property.

The study of $\varkappa$-products can be done in a more natural setting. Indeed, the $\varkappa$-products are special cases of a larger class of submodules of the direct product, called $\mathcal{F}$-products, where $\mathcal{F}$ is a filter on the index set A. The main objective of this paper is to determine a necessary condition for the $\mathcal{F}$-product of modules to split in the corresponding direct product. This condition will be given in terms of the lattice of ideals which are annihilators of subsets of a certain factor ring of R, and will depend on $\mathcal{F}$.

Let $\operatorname{cpl}(\mathcal{F})$ be the largest cardinal number $\varkappa$ such that $\mathcal{F}$ is $\varkappa$-complete. If $\operatorname{cpl}(\mathcal{F}) = \aleph_0$ (e.g., if $|A| <$ first measurable cardinal number) and if the $\mathcal{F}$-product splits in the corresponding direct product, then a certain factor ring of R has the Ascending Chain Condition (ACC) on annihilators (Theorem 3.5). Under these hypotheses, if R is simple or if $M_a = R$ for every $a \in A$, then R itself has ACC on annihilators (Corollaries 3.6 and 3.7).

The proof of Theorem 3.5 extends a technique that was used in [2] and [14]. Following the classical definition (see, for example, [1], [3], [4], [8], [12] and [16]), we say that $\mathscr{F}$ is a *filter* on A if it is a subset of the power set of A that satisfies the following conditions:

(1) $\emptyset \notin \mathscr{F}$ and $A \in \mathscr{F}$;
(2) If $B_1, B_2 \in \mathscr{F}$, then $B_1 \cap B_2 \in \mathscr{F}$;
(3) If $B \in \mathscr{F}$ and $B \subseteq C \subseteq A$, then $C \in \mathscr{F}$.

Let $\mathscr{F}$ be a filter on A and let $\{M_a: a \in A\}$ be a family of left R-modules. We define an equivalence relation $\sim$ on $\prod_{a\in A} M_a$ as follows. For $(m_a)_{a\in A}$ and $(m'_a)_{a\in A}$ in $\prod_{a\in A} M_a$, we have $(m_a) \sim (m'_a)$ if and only if $\{a \in A: m_a = m'_a\} \in \mathscr{F}$. The equivalence class of $(0, 0, \ldots)$ is called the *$\mathscr{F}$-product* of the M_a's, and is denoted by $\prod^{\mathscr{F}}_{a\in A} M_a$. Clearly, $\prod^{\mathscr{F}}_{a\in A} M_a$ is a submodule of $\prod_{a\in A} M_a$. Also, $(m_a)_{a\in A} \in \prod^{\mathscr{F}}_{a\in A} M_a$ if and only if $\{a \in A: m_a = 0\} \in \mathscr{F}$.

EXAMPLE. Let $\varkappa$ be an infinite cardinal number such that $\varkappa \leqslant |A|$. Then $\mathscr{F} = \{B \subseteq A: |A\backslash B| < \varkappa\}$ is a filter on A and the $\mathscr{F}$-product is the $\varkappa$-product. In particular, if $\varkappa = \aleph_0$, the $\mathscr{F}$-product is the direct sum.

A filter $\mathscr{F}$ on A is an *ultrafilter* if $\mathscr{F}$ is maximal, or equivalently, if whenever $B \subseteq A$, then either $B \in \mathscr{F}$ or $A\backslash B \in \mathscr{F}$. Let $\mathscr{F}$ be an ultrafilter on A and let $\{M_a: a \in A\}$ be a family of left R-modules. The set of equivalence classes under $\sim$, denoted $\prod_{a\in A} M_a/\mathscr{F}$, is a left R-module and is called the *ultraproduct* of the M_a's. Hence, in case $\mathscr{F}$ is an ultrafilter on A, the $\mathscr{F}$-product can be viewed as the kernel of the natural epimorphism $\prod_{a\in A} M_a \to \prod_{a\in A} M_a/\mathscr{F}$. The ultraproduct is a very useful construction in Algebra, Model Theory, Topology and Set Theory. See, for example, [1], [3], [4], [7], [8], [12], [15] and [16].

2. Preliminaries. In this section, we first look at the case where $\mathscr{F}$ is a principal filter, we then establish some properties of non-principal filters that will be referred to in proving the main theorem in Section 3. Some of these results have straightforward proofs that will be omitted.

DEFINITION 2.1. Let $\mathscr{F}$ be a filter on A and let $B \in \mathscr{F}$. We define

$$\mathscr{F}_B = \{C: C \subseteq B \text{ and } C \in \mathscr{F}\} = \{B \cap C: C \in \mathscr{F}\}.$$

Note that $\mathscr{F}_B$ is a filter on B. Moreover, if $\mathscr{F}$ is an ultrafilter on A, then $\mathscr{F}_B$ is an ultrafilter on B.

LEMMA 2.2. *Let $\mathscr{F}$ be a filter on A and let $B \in \mathscr{F}$. Then, for any family $\{M_a: a \in A\}$ of left R-modules, we have:*

$$\prod_{a\in A} M_a \simeq \prod^{\mathscr{F}_B}_{a\in B} M_a \oplus \prod_{a\in A\backslash B} M_a.$$

Therefore if $\prod^{\mathscr{F}}_{a\in A} M_a$ is a direct summand of $\prod_{a\in A} M_a$, then $\prod^{\mathscr{F}_B}_{a\in B} M_a$ is a direct summand of $\prod_{a\in B} M_a$.

DEFINITION 2.3. A filter $\mathscr{F}$ is a *principal filter* on A if there exists $B \in \mathscr{F}$ such that $\mathscr{F} = \{C: C \subseteq A \text{ and } B \subseteq C\}$. $\mathscr{F}$ is said to be *generated* by B.

THEOREM 2.4. *Let $\mathscr{F}$ be the principal filter on A generated by B. Then, for any family $\{M_a: a \in A\}$ of left R-modules, we have $\prod^{\mathscr{F}}_{a\in A} M_a \simeq \prod_{a\in A\backslash B} M_a$, and hence $\prod^{\mathscr{F}}_{a\in A} M_a$ is a direct summand of $\prod_{a\in A} M_a$.*

We now turn our attention to the case where $\mathscr{F}$ is a non-principal filter. There are $2^{2^{|A|}}$ non-principal filters on A, and in fact $2^{2^{|A|}}$ non-principal ultrafilters on A (see, for example, Theorem 1.5, p. 108, in [1]). The rest of this section will be devoted to establishing some important properties of non-principal filters on A.

DEFINITION 2.5. Let $\mathscr{F}$ be a filter on A, let $\varkappa$ be an infinite cardinal number. $\mathscr{F}$ is said to be *$\varkappa$-complete* if for every $S \subseteq \mathscr{F}$, with $|S| < \varkappa$, we have $\bigcap S \in \mathscr{F}$. $\mathscr{F}$ is said to be *$\varkappa$-incomplete* if $\mathscr{F}$ is not $\varkappa$-complete. See, for example, [1], [4] and [15].

Note that a filter $\mathscr{F}$ is either principal or $\varkappa$-incomplete for some $\varkappa$.

DEFINITION 2.6. Let $\mathscr{F}$ be a non-principal filter on A. Let $\varkappa$ be the least cardinal number such that $\mathscr{F}$ is $\varkappa$-incomplete. Then $\varkappa$ cannot be a limit ordinal, for, otherwise, $\mathscr{F}$ would be γ^+ complete for all $\gamma < \varkappa$ and hence $\mathscr{F}$ would be closed under all intersections of fewer than $\varkappa$ sets, i.e., $\mathscr{F}$ would be $\varkappa$-complete. Therefore $\varkappa$ is a sucessor cardinal number. We define $\operatorname{cpl}(\mathscr{F})$ to be the predecessor of $\varkappa$. Note that $\operatorname{cpl}(\mathscr{F})$ is the largest cardinal number β such that $\mathscr{F}$ is β-complete.

Recall that a cardinal number $\varkappa$ is *measurable* if there exists a nonprincipal $\varkappa$-complete ultrafilter on $\varkappa$. By [3, Proposition 4.2.7], $\operatorname{cpl}(\mathscr{F})$ is a measurable cardinal number.

The next few results will give some information on $\operatorname{cpl}(\mathscr{F})$ and will be referred to in Section 3.

LEMMA 2.7 (Lemma 1.12, p. 113 in [1]). *Let $\mathscr{F}$ be a non-principal ultrafilter on A. Then, whenever $\beta < \operatorname{cpl}(\mathscr{F})$ and $\{B_\xi: \xi < \beta\}$ is a partition of A, then there exists a unique $\xi_0 < \beta$ such that $B_{\xi_0} \in \mathscr{F}$.*

LEMMA 2.8 (Lemma 1.13, p. 114, in [1]). *Let $\mathscr{F}$ be a non-principal filter on A. Then there exists a sequence $\{B_\alpha: \alpha < \operatorname{cpl}(\mathscr{F})\}$ of elements of $\mathscr{F}$ such that $B_\alpha \supsetneqq B_\beta$ for all $\alpha < \beta < \operatorname{cpl}(\mathscr{F})$, and $\bigcap_{\alpha<\operatorname{cpl}(\mathscr{F})} B_\alpha \notin \mathscr{F}$. Moreover, if $\mathscr{F}$ is an ultrafilter, then we may choose $\{B_\alpha: \alpha < \operatorname{cpl}(\mathscr{F})\}$ such that $\bigcap_{\alpha<\operatorname{cpl}(\mathscr{F})} B_\alpha = \emptyset$.*

COROLLARY 2.9. *Let $\mathscr{F}$ be a filter on A and let $B \in \mathscr{F}$. Then $\operatorname{cpl}(\mathscr{F}) = \operatorname{cpl}(\mathscr{F}_B)$.*

Remark. Let $\varkappa$ be a regular cardinal number such that $\aleph_0 \leqslant \varkappa \leqslant |A|$, and let $\mathscr{F} = \{B \subseteq A: |A\backslash B| < \varkappa\}$. Then $\operatorname{cpl}(\mathscr{F}) = \varkappa$.

3. Main theorems. Using the Axiom of Choice, we can extend any filter $\mathscr{G}$ on A to an ultrafilter $\mathscr{F}$. This extension induces the inclusion $\prod^{\mathscr{G}}_{a\in A} M_a \subseteq \prod^{\mathscr{F}}_{a\in A} M_a$ for any family $\{M_a: a \in A\}$ of left R-modules. If is therefore natural to only consider ultrafilters $\mathscr{F}$ in the study of the embedding of the $\mathscr{F}$-product in the corresponding direct product.

The proof of the main theorem (Theorem 3.5) extends a technique that was used in [2] and [14]. But first we need some definitions and few lemmas.

A *p-functor* is a subfunctor of the forgetful functor $F\colon R\text{-}\mathcal{M}od \to \mathbf{Z}\text{-}\mathcal{M}od$ that commutes with direct products. Let $\{U_\alpha\colon \alpha < \delta\}$ be a well-ordered set of p-functors. Then $\{U_\alpha\colon \alpha < \delta\}$ is a *descending chain* of p-functors if, for all $M \in R\text{-}\mathcal{M}od$, $U_\alpha(M) \supseteq U_\beta(M)$ for all $\alpha < \beta < \delta$.

EXAMPLE. For a subset S of R, define $U(M) = \operatorname{ann}_M(S)$, for $M \in R\text{-}\mathcal{M}od$. Then $U(\text{-})$ is a p-functor.

LEMMA 3.1. *Let $\mathcal{F}$ be a non-principal ultrafilter on A. Let $\{M_a\colon a \in A\}$ be a family of left R-modules and let $\{U_\alpha\colon \alpha < \sigma\}$ be a descending chain of p-functors with $\operatorname{cpl}(\mathcal{F}) \leqslant \sigma$. Suppose that $A = \bigcup_{1 \leqslant \alpha < \sigma} A_\alpha$ for some $A_\alpha \subseteq A$ satisfying:*

(1) *$A_\alpha \neq \emptyset$ for all $\alpha < \sigma$;*
(2) *$A_\alpha \cap A_\beta = \emptyset$ for all $\alpha \neq \beta$, $\alpha, \beta < \sigma$;*
(3) *$\bigcup_{\beta \leqslant \alpha < \sigma} A_\alpha \in \mathcal{F}$ for all $\beta < \sigma$;*
(4) *$U_\alpha(M_a) \supsetneqq U_{\alpha+1}(M_a)$ for all $a \in A_\alpha$ and for all $\alpha < \sigma$.*

Then $\prod_{a\in A}^{\mathcal{F}} M_a$ is not a direct summand of $\prod_{a \in A} M_a$.

Note. Conditions (1), (2) and (3) of Lemma 3.1 imply that $A_\alpha \notin \mathcal{F}$ for all $\alpha < \sigma$. Therefore the requirement that $\sigma \geqslant \operatorname{cpl}(\mathcal{F})$ is not a restriction in view of Lemma 2.7.

Proof. Let φ be the natural isomorphism $\prod_{a\in A} M_a \simeq \prod_{1 \leqslant \alpha < \sigma} (\prod_{a \in A_\alpha} M_a)$. Let $S = \varphi(\prod_{a\in A}^{\mathcal{F}} M_a)$ and $P = \prod_{1 \leqslant \alpha < \sigma} (\prod_{a \in A_\alpha} M_a)$. Note that $((m_a)_{a \in A_\alpha})_{1 \leqslant \alpha < \sigma} \in S$ if and only if $\bigcup_{1 \leqslant \alpha < \sigma} \{a \in A_\alpha\colon m_a = 0\} \in \mathcal{F}$.

Now, suppose to the contrary that $\prod_{a\in A}^{\mathcal{F}} M_a$ is a direct summand of $\prod_{a\in A} M_a$. Then S is a direct summand of P. Let $P = S \oplus Q$ and let $p\colon P \to S$ be the natural projection. By (4), there exists $0 \neq x_{a,\alpha} \in U_\alpha(M_a) - U_{\alpha+1}(M_a)$ for all $a \in A_\alpha$ and all $\alpha < \sigma$. Let $z = [(x_{a,\alpha})_{a \in A_\alpha}]_{\alpha < \sigma}$ and for $\beta < \sigma$, let $z_\beta = [(x_{a,1})_{a \in A_1}, \ldots, (x_{a,\beta})_{a \in A_\beta}(0), \ldots]$.

CLAIM. *$z_\beta \in S$ for all $\beta < \sigma$.*

Let $\beta < \sigma$. Since $x_{a,\alpha} \neq 0$ for all $a \in A_\alpha$ and for all $\alpha < \sigma$, the set of indices where the coordinates of z_β are zero is exactly $\bigcup_{\beta < \alpha < \sigma} A_\alpha$, which is in $\mathcal{F}$ by (3). Hence $z_\beta \in S$.

Now let $u_\beta = z - z_\beta$ for $\beta < \sigma$. There exist s_β, $s \in S$ and q_β, $q \in Q$ such that $z = s + q$ and $u_\beta = s_\beta + q_\beta$.

Let $\beta < \sigma$. Then $z = s + q = z_\beta + u_\beta = z_\beta + (s_\beta + q_\beta)$ and hence, $s = s_\beta + z_\beta$.

Let $\alpha < \beta < \sigma$. Then

$$(x_{a,\beta})_{a \in A_\beta} \in \prod_{a \in A_\beta} U_\beta(M_a) \subseteq \prod_{a \in A_\beta} U_{\alpha+1}(M_a) = U_{\alpha+1}\Big(\prod_{a \in A_\beta} M_a\Big).$$

Hence,

$$u_\alpha = [(0), \ldots, (0), (x_{a,\alpha+1})_{a \in A_{\alpha+1}}, \ldots] \in \prod_{1 \leqslant \beta < \sigma} \Big[U_{\alpha+1}\Big(\prod_{a \in A_\beta} M_a\Big)\Big] = U_{\alpha+1}(P).$$

Now,

$$s_\alpha = p(u_\alpha) \in p[U_{\alpha+1}(P)] \subseteq$$

$$(*) \qquad U_{\alpha+1}(S) = U_{\alpha+1}\Big[\varphi\Big(\prod_{a\in A}^{\mathcal{F}} M_a\Big)\Big] \subseteq \varphi\Big[\prod_{a\in A}^{\mathcal{F}} U_{\alpha+1}(M_a)\Big].$$

Let $a \in A$, then $a \in A_\alpha$ for some unique $\alpha < \sigma$. Let $\varrho_{a,\alpha}\colon P \to M_a$ be the natural projection. Since $s \in S$, there exist $\mu < \sigma$ and $b \in A_\mu$ such that $\varrho_{b,\mu}(s) = 0$. Now, $x_{b,\mu} = \varrho_{b,\mu}(z_\mu) = \varrho_{b,\mu}(s - s_\mu) = -\varrho_{b,\mu}(s_\mu) \in U_{\mu+1}(M_b)$ by $(*)$, which is a contradiction to the choice of $x_{b,\mu}$.

LEMMA 3.2. *Let $\mathcal{F}$ be a non-principal ultrafilter on A. Let $\{M_a\colon a \in A\}$ be a family of left R-modules. If $\prod_{a\in A}^{\mathcal{F}} M_a$ is a direct summand of $\prod_{a \in A} M_a$*, then $\{\operatorname{ann}_R(P_B)\colon B \in \mathcal{F}\}$ has a maximal element, where $P_B = \prod_{a \in B} M_a$.

Proof. Suppose to the contrary that $\{\operatorname{ann}_R(P_B)\colon B \in \mathcal{F}\}$ has no maximal element.

CLAIM 1. *There exists an ordinal number σ and a sequence $(B_\xi)_{\xi < \sigma}$ such that:*

(1) *$B_\xi \in \mathcal{F}$ for all $\xi < \sigma$;*
(2) *$B_\xi \supsetneqq B_{\xi+1}$ for all $\xi < \sigma$;*
(3) *$\operatorname{ann}_R(P_{B_\xi}) \subsetneqq \operatorname{ann}_R(P_{B_{\xi+1}})$ for all $\xi < \sigma$;*
(4) *$\operatorname{ann}(P_{B_\xi}) \subseteq \operatorname{ann}_R(M_a)$ if and only if $a \in B_\xi$, $\xi < \sigma$;*
(5) *$\bigcap_{\xi < \sigma} B_\xi \notin \mathcal{F}$.*

Proof of Claim 1. Let $B_1 = A$. We proceed by induction. Let $\alpha > 1$. Suppose that B_ξ has been defined for all $\xi < \alpha$, satisfying 1–4.

Case 1. α is not a limit ordinal. Then $B_{\alpha-1}$ has been defined. By hypothesis, $\operatorname{ann}_R(P_{B_{\alpha-1}})$ is not maximal. Thus, there exists $B'_\alpha \in \mathcal{F}$ such that $\operatorname{ann}_R(P_{B_{\alpha-1}}) \subsetneqq \operatorname{ann}_R(P_{B'_\alpha})$. Let $B_\alpha = \{a \in A\colon \operatorname{ann}_R(P_{B'_\alpha}) \subseteq \operatorname{ann}_R(M_a)\}$. Then $B'_\alpha \subseteq B_\alpha$ and hence $B_\alpha \in \mathcal{F}$. Also, if $a \in B_\alpha$, then $\operatorname{ann}_R(P_{B_{\alpha-1}}) \subsetneqq \operatorname{ann}_R(P_{B'_\alpha}) \subseteq \operatorname{ann}_R(M_a)$ and hence by induction hypothesis, $a \in B_{\alpha-1}$. Therefore $B_\alpha \subseteq B_{\alpha-1}$. Now,

$$\operatorname{ann}_R(P_{B_\alpha}) = \operatorname{ann}_R(P_{B'_\alpha} \oplus P_{B_\alpha \setminus B'_\alpha}) = \operatorname{ann}_R(P_{B_\alpha}) \cap \operatorname{ann}_R(P_{B_\alpha \setminus B'_\alpha})$$

$$= \operatorname{ann}_R(P_{B'_\alpha}) \cap \Big[\bigcap_{a \in B_\alpha \setminus B'_\alpha} \operatorname{ann}_R(M_a)\Big] = \operatorname{ann}_R(P_{B'_\alpha}).$$

Case 2. α is a limit ordinal. If $\bigcap_{\xi < \alpha} B_\xi \notin \mathcal{F}$, we let $\sigma = \alpha$ and we stop the induction. Otherwise, let $B_\alpha = \bigcap_{\xi < \alpha} B_\xi \in \mathcal{F}$. If $a \in B_\alpha$, then $\operatorname{ann}_R(P_{B_\alpha}) \subseteq \operatorname{ann}_R(M_a)$. Conversely, if $\operatorname{ann}_R(P_{B_\alpha}) \subseteq \operatorname{ann}_R(M_a)$, then $\operatorname{ann}_R(P_{B_\xi}) \subseteq \operatorname{ann}_R(M_a)$ for all $\xi < \alpha$ and hence $a \in B_\xi$ for all

$\xi < \alpha$ by induction hypothesis; i.e., $a \in B_\alpha$. Clearly, there exists a limit ordinal σ such that $B = \bigcap_{\xi<\alpha} B_\xi \notin \mathscr{F}$. Note that $\sigma \geqslant \operatorname{cpl}(\mathscr{F})$. This completes the proof of Claim 1.

CLAIM 2. $A \setminus B = \bigcup_{1 \leqslant \alpha < \sigma} (B_\alpha \setminus B_{\alpha+1})$.

Proof of Claim 2. Let $a \in A \setminus B$. Then $a \notin B_\xi$ for some $\xi < \sigma$. Let α be the least such ordinal number. Then α is not a limit ordinal. For, if α is a limit ordinal, then $a \notin B_\alpha = \bigcap_{\xi<\alpha} B_\xi$, by the construction of B_α in Claim 1, and therefore $a \notin B_\xi$ for some $\xi < \alpha$, which is a contradiction to the choice of α. Hence $a \in B_{\alpha-1} \setminus B_\alpha$. Note that $B_\alpha \setminus B_{\alpha+1} \neq \emptyset$ for all $\alpha < \sigma$. This proves Claim 2.

Now, for $\alpha < \sigma$ and for $M \in R\text{-}\mathscr{M}od$, define $U_\alpha(M) = \operatorname{ann}_M \operatorname{ann}_R(P_{B_\alpha})$. Then $\{U_\alpha: \alpha < \sigma\}$ is a descending chain of p-functors.

CLAIM 3. $U_\alpha(M_a) \supsetneqq U_{\alpha+1}(M_a)$ for all $a \in B_\alpha \setminus B_{\alpha+1}$.

Proof of Claim 3. Suppose to the contrary that $U_\alpha(M_a) = U_{\alpha+1}(M_a)$ for some $a \in B_\alpha \setminus B_{\alpha+1}$. Since $a \in B_\alpha$, we have $M_a = U_\alpha(M_a)$, and hence $M_a = U_{\alpha+1}(M_a) = \operatorname{ann}_{M_a} \operatorname{ann}_R(P_{B_{\alpha+1}})$. Therefore $\operatorname{ann}_R(P_{B_{\alpha+1}}) \subseteq \operatorname{ann}_R(M_a)$, so that $a \in B_{\alpha+1}$ by (4) in Claim 1, which is a contradiction to the choice of a.

Now, let $\mathscr{G} = \mathscr{F}_{A \setminus B}$. (See Definition 2.1.) Then $\mathscr{G}$ is a non-principal ultrafilter on $A \setminus B$. By Lemma 2.2 and by hypothesis, $\prod_{a \in A \setminus B}^{\mathscr{G}} M_a$ is a direct summand of $\prod_{a \in A \setminus B} M_a$. Also, by Corollary 2.9, $\operatorname{cpl}(\mathscr{G}) = \operatorname{cpl}(\mathscr{F})$ and hence $\sigma \geqslant \operatorname{cpl}(\mathscr{G})$. Now, for $\beta < \sigma$, $\bigcup_{\beta \leqslant \alpha < \sigma} B_\alpha \setminus B_{\alpha+1} = B_\beta \setminus B \in \mathscr{G}$. We thus obtain a contradiction to Lemma 3.1.

LEMMA 3.3. *Let $\mathscr{F}$ be a non-principal ultrafilter on A. Let $\{M_a: a \in A\}$ be a family of left R-modules. Let $\{U_\alpha: \alpha < \operatorname{cpl}(\mathscr{F})\}$ be a descending chain of p-functors. Suppose that $\prod_{a \in A}^{\mathscr{F}} M_a$ is a direct summand of $\prod_{a \in A} M_a$. If there exists $B \subseteq A$ such that for all $a \in B$ and for all $\alpha < \operatorname{cpl}(\mathscr{F})$ we have $U_\alpha(M_a) \supsetneqq U_{\alpha+1}(M_a)$, then $B \notin \mathscr{F}$.*

Proof. Suppose to the contrary that $B \in \mathscr{F}$. Then, by Lemma 2.2, $\prod_{a \in B}^{\mathscr{F}_B} M_a$ is a direct summand of $\prod_{a \in B} M_a$. Also, by Corollary 2.9, $\operatorname{cpl}(\mathscr{F}) = \operatorname{cpl}(\mathscr{F}_B)$. Now, by Lemma 2.8, there exists a sequence $(B_\alpha)_{\alpha < \operatorname{cpl}(\mathscr{F})}$ of elements of $\mathscr{F}$ such that $B_\alpha \supsetneqq B_\beta$ for all $\alpha < \beta < \operatorname{cpl}(\mathscr{F})$ and $\bigcap_{\alpha < \operatorname{cpl}(\mathscr{F})} B_\alpha = \emptyset$. We may clearly choose $B_1 = B$. Then $B = \bigcup_{\alpha < \operatorname{cpl}(\mathscr{F})} E_\alpha$, where $E_\alpha = (\bigcap_{\mu<\alpha} B_\mu) \setminus B_\alpha$. For $\alpha < \operatorname{cpl}(\mathscr{F})$, $E_\alpha \neq \emptyset$ and $E_\alpha \notin \mathscr{F}_B$, since $B_\alpha \in \mathscr{F}_B$. Moreover, $E_\alpha \cap E_\beta = \emptyset$ for all $\alpha \neq \beta$ with $\alpha, \beta < \operatorname{cpl}(\mathscr{F})$. If $\beta < \operatorname{cpl}(\mathscr{F})$, then $\bigcup_{\beta \leqslant \alpha < \operatorname{cpl}(\mathscr{F})} E_\alpha = B_\beta \in \mathscr{F}$. Finally, if $a \in E_\alpha$, then $U_\alpha(M_a) \supsetneqq U_{\alpha+1}(M_a)$ by assumption. Therefore the hypotheses of Lemma 3.1 are satisfied and we obtain a contradiction. Hence $B \notin \mathscr{F}$.

At this point, we need to specialize $\operatorname{cpl}(\mathscr{F})$. As was shown in the remark after Corollary 2.9, given a regular cardinal number $\varkappa$ such that $\aleph_0 \leqslant \varkappa \leqslant |A|$, there exists a filter $\mathscr{F}$ on A such that $\operatorname{cpl}(\mathscr{F}) = \varkappa$. The question arises as to whether there exists an ultrafilter $\mathscr{F}$ on A such that $\operatorname{cpl}(\mathscr{F}) = \varkappa$. The answer is trivial for A countable: every filter $\mathscr{F}$ on a countable set A satifies $\operatorname{cpl}(\mathscr{F}) = \aleph_0$. The existence of a non-principal ultrafilter $\mathscr{F}$ on an uncountable set such that $\operatorname{cpl}(\mathscr{F}) > \aleph_0$ is an axiom which in often used in modern Set Theory. In the next few paragraphs, we will state some results dealing with this problem that will be used later; a more detailed exposition of these results and further information on the properties and uses of the definitions that will be given can be found in [1], [4], [7], [10] and [15].

A cardinal number $\varkappa > \aleph_0$ is said to be *ω-measurable* (or *Ulam-measurable*) if there exists a non-principal ultrafilter $\mathscr{F}$ on a set of cardinality $\varkappa$ such that $\operatorname{cpl}(\mathscr{F}) > \aleph_0$. By Theorem 8.31 in [4], a cardinal number $\varkappa$ is ω-measurable if and only if it is greater or equal to the first uncountable measurable cardinal. Hence, the existence of a non-principal ultrafilter $\mathscr{F}$ on a set A such that $\operatorname{cpl}(\mathscr{F}) > \aleph_0$ is equivalent to the existence of uncountable measurable cardinal numbers ($\exists$MC). It cannot be proved in standard set theory ZF that such cardinals exist. Moreover, the consistency of the theory ZF+$\exists$MC cannot be proved in ZF. However, ZF+$\exists$MC is often used in descriptive set theory and even stronger theories, so-called natural extensions of ZF, are used and believed to be consistent.

If uncountable measurable cardinals exist, they are very large. In fact, if $\varkappa$ is the first measurable cardinal number, then it is inaccessible (i.e., it is a regular limit cardinal number) and there are $\varkappa$ inaccessible cardinal numbers less than $\varkappa$ (Theorem 3, p. 26 in [15]). The existence of such cardinal numbers is independent of ZF. Also, the axiom of constructibility, that "V = L" in the sense of Gödel, implies that measurable cardinal numbers do not exist (Theorem 6.9, p. 305 in [1]).

In view of the above, the case $\operatorname{opl}(\mathscr{F}) = \aleph_0$ is a very important case. The following few results deal with this case.

LEMMA 3.4. *Let $\mathscr{F}$ be a non-principal ultrafilter on A such that $\operatorname{cpl}(\mathscr{F}) = \aleph_0$. Let $\{U_n: n = 1, 2, \ldots\}$ be a descending chain of p-functors. Let $\{M_a: a \in A\}$ be a family of left R-modules. If $\prod_{a \in A}^{\mathscr{F}} M_a$ is a direct summand of $\prod_{a \in A} M_a$, then there exist $B \in \mathscr{F}$ and n such that for all $a \in B$ and for all $p \geqslant n$, $U_n(M_a) = U_p(M_a)$.*

Proof. For $p \geqslant 1$, define

$$C_p = \{a \in A: U_q(M_a) \supsetneqq U_{q+1}(M_a) \text{ for some } q \geqslant p\}.$$

If $C_n \notin \mathscr{F}$ for some $n \geqslant 1$, then $B = A \setminus C_n \in \mathscr{F}$. Moreover, for $a \in B$ and $p \geqslant n$, we have $U_n(M_a) = U_p(M_a)$. Therefore, we may assume that $C_p \in \mathscr{F}$ for all $p \geqslant 1$. We will obtain a contradiction. Note that we have $C_p \supseteq C_{p+1}$ for all $p \geqslant 1$. Let $C = \bigcap_{p \geqslant 1} C_p$. Then

$$(*) \qquad C_1 = C \cup \Big[\bigcup_{p \geqslant 1} C_p \setminus C_{p+1}\Big].$$

Step 1. $F = \bigcup_{p \geq 1} C_p \setminus C_{p+1} \notin \mathcal{F}$.

Proof. Suppose to the contrary that $F \in \mathcal{F}$. Since $C_p \in \mathcal{F}$ for all $p \geq 1$, we have $C_p \setminus C_{p+1} \notin \mathcal{F}$ for all $p \geq 1$. Moreover $(C_p \setminus C_{p+1}) \cap (C_q \setminus C_{q+1}) = \emptyset$ for all $p \neq q$. Therefore, by Lemma 2.7, there are $\aleph_0$ non-empty $C_p \setminus C_{p+1}$'s. Hence, we may assume that $C_p \setminus C_{p+1} \neq \emptyset$ for all $p \geq 1$. For $n > 1$,

$$\bigcup_{n \leq p} C_p \setminus C_{p+1} = F \setminus [\bigcup_{1 \leq p \leq n-1} C_p \setminus C_{p+1}] = \bigcap_{1 \leq p \leq n-1} [F \setminus (C_p \setminus C_{p+1})] \in \mathcal{F},$$

since $C_p \setminus C_{p+1} \notin \mathcal{F}$.

Now, for $p \geq 1$ and $a \in C_p \setminus C_{p+1}$, $U_p(M_a) \supsetneqq U_{p+1}(M_a)$. Also, $\mathcal{F}_F$ is a non-principal ultrafilter on F. (See Definition 2.1.) By Corollary 2.9, $\operatorname{cpl}(\mathcal{F}_F) = \operatorname{cpl}(\mathcal{F}) = \aleph_0$. Therefore the hypothese of Lemma 3.1 are satisfied and hence $\prod_{a \in F}^{\mathcal{F}_F} M_a$ is not a direct summand of $\prod_{a \in F} M_a$. But by Lemma 2.2 and by hypothesis, $\prod_{a \in F}^{\mathcal{F}_F} M_a$ is a direct summand of $\prod_{a \in F} M_a$. We thus obtain a contradiction to our assumption.

Step 2. $C = \bigcap_{p \geq 1} C_p \notin \mathcal{F}$.

Proof. Suppose to the contrary that $C \in \mathcal{F}$. For $a \in C$ and for $p \geq 1$, let $\eta(a, p) = \inf\{n\colon n \geq p \text{ and } U_n(M_a) \supsetneqq U_{n+1}(M_a)\}$. Then $\eta(a, p)$ is well defined for all $a \in C$ and all $p \geq 1$. For $p \leq q$, let $B_q^p = \{a \in C\colon \eta(a, p) = q\}$. We construct inductively a sequence (p_i) of natural numbers such that $p_i < p_{i+1}$ and $B_{p_{i+1}}^{p_i+1} \in \mathcal{F}$ for all $i \geq 0$. Let $p_0 = 0$. Now, $C = \bigcup_{1 \leq q} B_q^{p_0+1} = \bigcup_{1 \leq q} B_q^1$.

Claim 1. *$B_q^1 \in \mathcal{F}$ for some $q \geq 1$.*

Proof. Suppose to the contrary that $B_q^1 \notin \mathcal{F}$ for all $q \geq 1$. Since $B_q^1 \cap B_{q'}^1 = \emptyset$ for all $q \neq q'$, there are $\aleph_0$ non-empty B_q^1's, by Lemma 2.7. We may therefore assume that $B_q^1 \neq \emptyset$ for all $q \geq 1$. For $n > 1$,

$$\bigcup_{n \leq q} B_q^1 = C \setminus \bigcup_{1 \leq q \leq n-1} B_q^1 = \bigcap_{1 \leq q \leq n-1} (C \setminus B_q^1) \in \mathcal{F},$$

since $B_q^1 \notin \mathcal{F}$ for all $q \geq 1$. For $a \in B_q^1$, $U_q(M_a) \supsetneqq U_{q+1}(M_a)$. Finally, since $C \in \mathcal{F}$, $\mathcal{F}_C$ is a non-principal ultrafilter on C. By Lemma 2.2 and by hypothesis, $\prod_{a \in C}^{\mathcal{F}_C} M_a$ is a direct summand of $\prod_{a \in C} M_a$. Also by Corollary 2.9, $\operatorname{cpl}(\mathcal{F}_C) = \operatorname{cpl}(\mathcal{F}) = \aleph_0$. Therefore, we obtain a contradiction to Lemma 3.1. Let p_1 be the natural number such that $B_{p_1}^1 \in \mathcal{F}$. Then $p_1 > p_0$. Now, suppose that $p_0, p_1, \ldots, p_n$ have been defined such that $p_i < p_{i+1}$ and $B_{p_{i+1}}^{p_i+1} \in \mathcal{F}$ for all $i \leq n-1$. Since $B_{p_n}^{p_{n-1}+1} = [\bigcup_{q \geq p_n+1} B_q^{p_n+1}] \cap B_{p_n}^{p_{n-1}+1}$, $\bigcup_{q \geq p_n+1} B_q^{p_n+1} \in \mathcal{F}$. A similar argument to the one in the proof of Claim 1 shows that $B_q^{p_n+1} \in \mathcal{F}$ for some $q \geq p_n+1$. Let p_{n+1} be that natural number. Then $p_n < p_{n+1}$.

Now, let $D_n = \bigcap_{i \leq n-1} B_{p_{i+1}}^{p_i+1}$ for $n \geq 1$. Then $D_n \in \mathcal{F}$ for all $n \geq 1$. Also, $D_1 \supseteq D_2 \supseteq \ldots \supseteq D_n \supseteq D_{n+1} \supseteq \ldots$ Let $D = \bigcap_{n \geq 1} D_n$.

Claim 2. $D \notin \mathcal{F}$.

Proof. If $D = \emptyset$, then $D \notin \mathcal{F}$. If $D \neq \emptyset$, let $a \in D$. Then $U_{p_n}(M_a) \supsetneqq U_{p_n+1}(M_a) \supseteq U_{p_{n+1}}(M_a)$ for all $n \geq 1$. Hence, by Lemma 3.3, $D \notin \mathcal{F}$. Then $D_1 \setminus D \in \mathcal{F}$ and $D_1 \setminus D = \bigcup_{n \geq 1} D_n \setminus D_{n+1}$. By Lemma 2.7, there are $\aleph_0$ non-empty $D_n \setminus D_{n+1}$. We may therefore assume that $D_n \setminus D_{n+1} \neq \emptyset$ for all $n \geq 1$. For $m \geq 1$, $\bigcup_{m \leq n} D_n \setminus D_{n+1} = D_m \setminus D \in \mathcal{F}$. Also, for $a \in D_n \setminus D_{n+1}$, $a \in D_n$ and hence $U_{p_n}(M_a) \supsetneqq U_{p_n+1}(M_a) \supseteq U_{p_{n+1}}(M_a)$. Finally, since $D_1 \setminus D \in \mathcal{F}$, $\mathcal{F}_{D_1 \setminus D}$ is a non-principal ultrafilter on $D_1 \setminus D$. By Lemma 2.2 and by hypothesis, $\prod_{a \in D_1 \setminus D}^{\mathcal{F}_{D_1 - D}} M_a$ is a direct summand of $\prod_{a \in D_1 \setminus D} M_a$. Also, by Corollary 2.9, $\operatorname{cpl}(\mathcal{F}_{D_1 \setminus D}) = \operatorname{cpl}(\mathcal{F}) = \aleph_0$. We thus obtain a contradiction to Lemma 3.1. This completes the proof of Step 2.

Now, by (*), $C_1 = C \cup F$ with $C \cap F = \emptyset$. Also, by Steps 1 and 2, $C \notin \mathcal{F}$ and $F \notin \mathcal{F}$. Therefore $C_1 \notin \mathcal{F}$, which is a contradiction to our hypothesis. This completes the proof of the lemma.

Theorem 3.5. *Let $\mathcal{F}$ be a non-principal ultrafilter on A such that $\operatorname{cpl}(\mathcal{F}) = \aleph_0$. Let $\{M_a\colon a \in A\}$ be a family of left R-modules. If $\prod_{a \in A}^{\mathcal{F}} M_a$ is a direct summand of $\prod_{a \in A} M_a$, then there exists $B \in \mathcal{F}$ such that the factor ring $\bar{R} = R/\operatorname{ann}_R(\prod_{a \in B} M_a)$ satisfies the Ascending Chain Condition* (ACC) *on annihilators.*

Note. C. F. Faith, in [9], extensively studied rings with ACC on annihilators.

Proof. For $C \subseteq A$, define $P_C = \prod_{a \in C} M_a$. By Lemma 3.2, $\{\operatorname{ann}_R(P_C)\colon C \in \mathcal{F}\}$ has a maximal element, say $\operatorname{ann}_R(P_B)$ for some $B \in \mathcal{F}$. Let $\bar{R} = R/\operatorname{ann}_R(P_B)$.

Now, let $\operatorname{ann}_{\bar{R}}(B_1) \subseteq \operatorname{ann}_{\bar{R}}(B_2) \subseteq \ldots$ be an ascending chain of annihilators of subsets of $\bar{R}$. We need to show that this chain stops.

Define the descending chain of p-functors $\{V_n\colon n \geq 1\}$ via $V_n(M) = \operatorname{ann}_M \operatorname{ann}_{\bar{R}}(B_n)$ for $M \in \bar{R}\text{-}\mathcal{M}od$. Note that $\prod_{a \in B}^{\mathcal{F}_B} M_a$ and $\prod_{a \in B} M_a$ are $\bar{R}$-modules. Also, by Lemma 2.2, $\prod_{a \in B}^{\mathcal{F}_B} M_a$ is a direct summand of $\prod_{a \in B} M_a$ as R-modules, and hence as $\bar{R}$-modules. By Corollary 2.9, $\operatorname{cpl}(\mathcal{F}_B) = \operatorname{cpl}(\mathcal{F}) = \aleph_0$. Hence, by Lemma 3.4, there exists $C \in \mathcal{F}_B$ and there exists $n \geq 1$ such that for all $a \in C$ and all $p \geq n$, $V_n(M_a) = V_p(M_a)$. Now, since the V_p's commute with direct product, we have $V_n(P_C) = V_p(P_C)$ for all $p \geq n$.

Claim 1 (Claim 1 of Theorem 4 in [14]). *If X is a faithful left $\bar{R}$-module and if $S, T \subseteq \bar{R}$ with $\operatorname{ann}_X \operatorname{ann}_{\bar{R}} S = \operatorname{ann}_X \operatorname{ann}_{\bar{R}} T$, then $\operatorname{ann}_{\bar{R}} S = \operatorname{ann}_{\bar{R}} T$.*

Claim 2. *P_L is a faithful left $\bar{R}$-module for all $L \in \mathcal{F}_B$.*

Proof. $L \subseteq B$ implies that $\operatorname{ann}_R(P_B) \subseteq \operatorname{ann}_R(P_L)$. Since $L \in \mathcal{F}_B$, $\operatorname{ann}_R(P_L) = \operatorname{ann}_R(P_B)$, by the choice of B. Therefore $\operatorname{ann}_{\bar{R}}(P_L) = 0$.

Since $C \in \mathscr{F}$, P_C is a faithful $\bar{R}$-module, and since $\operatorname{ann}_{P_C} \operatorname{ann}_{\bar{R}}(B_n) = V_n(P_C) = V_p(P_C) = \operatorname{ann}_{P_C} \operatorname{ann}_{\bar{R}}(B_p)$ for all $p \geq n$, we have $\operatorname{ann}_{\bar{R}}(B_n) = \operatorname{ann}_{\bar{R}}(B_p)$ for all $p \geq n$, by Claim 2. Hence $\bar{R}$ has ACC on annihilators.

COROLLARY 3.6. *Let R be a simple ring. Let $\mathscr{F}$ be a non-principal ultrafilter on A such that* $\operatorname{cpl}(\mathscr{F}) = \aleph_0$. *Let $\{M_a: a \in A\}$ be a family of left R-modules. If $\prod_{a\in A}^{\mathscr{F}} M_a$ is a direct summand of $\prod_{a\in A} M_a$, then R has* ACC *on annihilators.*

COROLLARY 3.7. *Let $\mathscr{F}$ be a non-principal ultrafilter on A such that* $\operatorname{cpl}(\mathscr{F}) = \aleph_0$. *If $\prod_{a\in A}^{\mathscr{F}} R$ is a direct summand of $\prod_{a\in A} R$, then R has* ACC *on annihilators.*

Note that Theorem 3.5 and Corollaries 3.6 and 3.7 can be applied to any non-principal ultrafilter on any set A with $|A| <$ first measurable cardinal number. In particular, these results can be applied to any non-principal ultrafilter on any set A if there are no measurable cardinals.

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References

[1] J. L. Bell and A. B. Slomson, *Models and Ultraproducts: An Introduction*, North-Holland, Amsterdam 1969.
[2] V. Camillo, *On Zimmermann–Huisgen's splitting theorem*, Proc. Amer. Math. Soc. 94 (1985), 206–208.
[3] C. C. Chang and H. J. Keisler, *Model Theory*, North-Holland, Amsterdam 1973.
[4] W. W. Comfort and S. Negrepontis, *The Theory of Ultrafilters*, Grundlehren Math. Wiss. Springer-Verlag, New York 1974.
[5] J. Dauns, *Subdirect products of injectives*, preprint.
[6] — *Uniform dimensions and subdirect products*, Pacific J. Math. 126 (1987), 1–19.
[7] F. R. Drake, *Set Theory: an Introduction to Large Cardinals*, North-Holland, New York 1974.
[8] P. C. Eklof, *Ultraproducts for algebraists*, in: *Handbook of Mathematical Logic*, edited by J. Barwise, North-Holland, Amsterdam 1977, pp. 105–137.
[9] C. F. Faith, *Rings with ascending chain conditions on annihilators*, Nagoya Math. J. 27 (1966), 179–191.
[10] T. Jech, *Set Theory*, Monographs in Pure Appl. Math., Academic Press, 1978.
[11] H. Lenzing, *Direct sums of modules as direct summands of their direct products*, Comm. Algebra 4 (1976), 681–691.
[12] H. Lenzing and C. U. Jensen, *Model Theoretic Algebra*, Gordon and Breach, to appear.
[13] P. Loustaunau, *Large subdirect products of projective modules*, Comm. Algebra 17 (1) (1989), 197–215.
[14] — *Large subdirect products of modules as direct summand of their direct products*, ibid. 17 (2) (1989), 393–412.
[15] J. R. Schoenfield, *Measurable cardinals*, in: *Logic Colloquium* 69, edited by R. D. Gaudy and C. M. E. Yates, North-Holland, Amsterdam 1969, pp. 19–49.
[16] A. W. Schurle, *Topics in Topology*, North-Holand, New York 1979.
[17] M. L. Teply, *Large subdirect products*, preprint.
[18] B. Zimmermann-Huisgen, *The sum-product splitting property and injective direct sums of modules over von Neumann regular rings*, Proc. Amer. Math. Soc. 83 (1981), 251–254.

[19] W. Zimmermann, *Rein injektiv direkte Summen von Moduln*, Comm. Algebra 5 (1977), 1083–1117.

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