6. Open problems.

**Problem 1.** In Theorem 5 suppose we take $F_0$, $F_1$, to be Borel but $E$ an arbitrary second countable metrizable or even a $\mathcal{P}$-set. Do the conclusions of Theorem 5 hold in this case?

**Problem 2.** Does Theorem 7 hold for a $\mathcal{I}$-set $A$? We do not know the answer even when $Z$ is a convex subset of $R^2$.

**Problem 3.** Can Theorem 8 be extended for $\mathcal{I}$-sets $A$? We do not know the answer even when $Z = R$.

A question related to Problem 3 is the following:

**Problem 4.** Let $C_0$ and $C_1$ be two disjoint $\mathcal{I}$-sets in $E \times X$ such that for every $e \in E$, the sections $C_0(e)$ and $C_1(e)$ are closed. Further assume that there is a Borel set $B$ containing $C_0$ but disjoint from $C_1$. Do there exist disjoint Borel sets $B_0$ and $B_1$ such that $C_0 \subseteq B_0$, $C_1 \subseteq B_1$, and for every $e \in E$, the sections $B_0(e)$ and $B_1(e)$ are closed in $X$?

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**References**


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**A splitting theorem for $\mathcal{F}$-products**

by

Philippe Loustau (Fairfax, Va.)

**Abstract.** Let $\mathcal{F}$ be a non-principal ultrafilter on an infinite index set $A$. Let $(M_a; a \in A)$ be a family of left $R$-modules. We define the $\mathcal{F}$-product of the $M_a$'s to be $\prod_{a \in A} M_a = \{ (m_a)_{a \in A} | \prod_{a \in A} m_a = 0 \}$. In the present paper, we determine a necessary condition for the $\mathcal{F}$-product of the $M_a$'s to split in the corresponding direct product. This condition will be given in terms of the lattice of ideals which are annihilators of subsets of a certain factor ring of $R$, and will depend on $\mathcal{F}$.

$R$ will always denote a ring with identity, all modules will be unital, $A$ will always denote an infinite index set and $|X|$ will always denote the cardinality of a set $X$.

**1. Introduction.** The question of when the canonical embedding of a direct sum of modules splits in the corresponding direct product has been extensively studied. See, for example, [2], [9], [11], [18], and [19]. Recently modules that are in between direct sums and direct products, called $x$-products, have been introduced. See, for example, [5], [6], [13], [14] and [17]. In [14], a necessary condition was determined for the canonical embedding of the $x$-product of modules in the corresponding direct product to split. This result generalized the above-mentioned classical theorems on the sum-product splitting property.

The study of $x$-products can be done in a more natural setting. Indeed, the $x$-products are special cases of a larger class of submodules of the direct product, called $\mathcal{F}$-products, where $\mathcal{F}$ is a filter on the index set $A$. The main objective of this paper is to determine a necessary condition for the $\mathcal{F}$-product of modules to split in the corresponding direct product. This condition will be given in terms of the lattice of ideals which are annihilators of subsets of a certain factor ring of $R$, and will depend on $\mathcal{F}$.

Let $\text{cpl}(\mathcal{F})$ be the largest cardinal number $\kappa$ such that $\mathcal{F}$ is $\kappa$-complete. If $\text{cpl}(\mathcal{F}) = \aleph_0$ (e.g., if $|A| < \text{first measurable cardinal}$) and if the $\mathcal{F}$-product splits in the corresponding direct product, then a certain factor ring of $R$ has the Ascending Chain Condition (ACC) on annihilators (Theorem 3.5). Under these hypotheses, if $R$ is simple or if $M_a = R$ for every $a \in A$, then $R$ itself has ACC on annihilators (Corollaries 3.6 and 3.7).

The proof of Theorem 3.5 extends a technique that was used in [2] and [14]. Following the classical definition (see, for example, [1], [3], [4], [8], [12] and [16]),
we say that $\mathcal{F}$ is a filter on $A$ if it is a subset of the power set of $A$ that satisfies the following conditions:

(1) $\emptyset \notin \mathcal{F}$ and $A \in \mathcal{F}$;

(2) If $B_1, B_2 \in \mathcal{F}$, then $B_1 \cap B_2 \in \mathcal{F}$;

(3) If $B \in \mathcal{F}$ and $B \subseteq C \subseteq A$, then $C \in \mathcal{F}$.

Let $\mathcal{F}$ be a filter on $A$ and let $\{M_a; a \in A\}$ be a family of left $R$-modules. We define an equivalence relation $\sim$ on $\prod_a M_a$ as follows. For $(m)_{a \in A}$ and $(m')_{a \in A}$ in $\prod_a M_a$, we have $(m)_{a \in A} \sim (m')_{a \in A}$ if and only if $\{a \in A; m_a = m'_a\} \in \mathcal{F}$. The equivalence class of $(0, 0, \ldots)$ is called the $\mathcal{F}$-product of the $M_a$'s, and is denoted by $\prod_{a \in A}^\mathcal{F} M_a$. Clearly, $\prod_{a \in A}^\mathcal{F} M_a$ is a submodule of $\prod_{a \in A} M_a$. Also, $(m)_{a \in A} \in \prod_{a \in A}^\mathcal{F} M_a$ if and only if $\{a \in A; m_a = 0\} \in \mathcal{F}$.

Example. Let $x$ be an infinite cardinal number such that $x \leq |A|$. Then $\mathcal{F} = \{B \subseteq A; |A \setminus B| < x\}$ is a filter on $A$ and the $\mathcal{F}$-product is the $x$-product. In particular, if $x = \aleph_0$, the $\mathcal{F}$-product is the direct sum.

A filter $\mathcal{F}$ on $A$ is an ultrafilter if $\mathcal{F}$ is maximal, or equivalently, if whenever $B \in \mathcal{F}$, then either $B \in \mathcal{F}$ or $A \setminus B \in \mathcal{F}$. Let $\mathcal{F}$ be an ultrafilter on $A$ and let $\{M_a; a \in A\}$ be a family of left $R$-modules. The set of equivalence classes under $\sim$, denoted $\prod_{a \in A} M_a/\mathcal{F}$, is a left $R$-module and is called the ultra product of the $M_a$'s. Hence, in case $\mathcal{F}$ is an ultra filter on $A$, the $\mathcal{F}$-product can be viewed as the kernel of the natural epimorphism $\prod_a M_a \twoheadrightarrow \prod_{a \in A} M_a/\mathcal{F}$. The ultra product is a very useful construction in Algebra, Model Theory, Topology and Set Theory. See, for example, [1], [3], [4], [7], [8], [12], [15] and [16].

2. Preliminaries. In this section, we first look at the case where $\mathcal{F}$ is a principal filter, we then establish some properties of non-principal filters that will be referred to in proving the main theorem in Section 3. Some of these results have straightforward proofs that will be omitted.

Definition 2.1. Let $\mathcal{F}$ be a filter on $A$ and let $B \subseteq \mathcal{F}$. We define

$$\mathcal{F}_B = \{C; C \subseteq B \land C \in \mathcal{F}\} = \{B \setminus C; C \in \mathcal{F}\}.$$

Note that $\mathcal{F}_B$ is a filter on $B$. Moreover, if $\mathcal{F}$ is an ultrafilter on $A$, then $\mathcal{F}_B$ is an ultrafilter on $B$.

Lemma 2.2. Let $\mathcal{F}$ be a filter on $A$ and let $B \subseteq \mathcal{F}$. Then, for any family $\{M_a; a \in A\}$ of left $R$-modules, we have:

$$\prod_{a \in A} M_a \cong \prod_{a \in B \cup \mathcal{F}} M_a \otimes \prod_{a \in B} M_a.$$

Therefore if $\prod_{a \in A}^\mathcal{F} M_a$ is a direct summand of $\prod_{a \in A} M_a$, then $\prod_{a \in B \cup \mathcal{F}}^\mathcal{F} M_a$ is a direct summand of $\prod_{a \in B} M_a$.

3. Main theorems. Using the Axiom of Choice, we can extend any filter $\mathcal{F}$ on $A$ to an ultrafilter $\mathcal{F}$. This extension induces the inclusion $\prod_{a \in A}^\mathcal{F} M_a \subseteq \prod_{a \in A} M_a$ for any family $\{M_a; a \in A\}$ of left $R$-modules. If is therefore natural to only consider ultrafilters $\mathcal{F}$ in the study of the embedding of the $\mathcal{F}$-product in the corresponding direct product.

Definition 2.3. A filter $\mathcal{F}$ is a principal filter on $A$ if there exists $B \subseteq \mathcal{F}$ such that $\mathcal{F} = \{C; C \subseteq A \land B \subseteq C\}$. $\mathcal{F}$ is said to be generated by $B$.

Theorem 2.4. Let $\mathcal{F}$ be the principal filter on $A$ generated by $B$. Then, for any family $\{M_a; a \in A\}$ of left $R$-modules, we have $\prod_{a \in A}^\mathcal{F} M_a \cong \prod_{a \in B \cup \mathcal{F}} M_a$. Hence $\prod_{a \in B \cup \mathcal{F}}^\mathcal{F} M_a$ is a direct summand of $\prod_{a \in A} M_a$.

We now turn our attention to the case where $\mathcal{F}$ is a non-principal filter. There are $2^{2^{\aleph_0}}$ non-principal filters on $A$, and in fact $2^{2^{\aleph_0}}$ non-principal ultrafilters on $A$ (see, for example, Theorem 1.5, p. 108, in [1]). The rest of this section will be devoted to establishing some important properties of non-principal filters on $A$.

Definition 2.5. Let $\mathcal{F}$ be a filter on $A$, let $x$ be an infinite cardinal number. $\mathcal{F}$ is said to be $x$-complete if for every $S \subseteq \mathcal{F}$, with $|S| < x$, we have $\bigcap S \subseteq \mathcal{F}$. $\mathcal{F}$ is said to be $x$-complete if $\mathcal{F}$ is not $x$-complete. See, for example, [1], [4] and [15].

Note that a filter $\mathcal{F}$ is either $\aleph_0$-principal or $\aleph_0$-incomplete for some $\aleph_0$.

Lemma 2.6. Let $\mathcal{F}$ be a non-principal filter on $A$. Let $x$ be the least cardinal number such that $\mathcal{F}$ is $x$-incomplete. Then $x$ cannot be a limit cardinal, for otherwise, $\mathcal{F}$ would be $\gamma^+$-complete for all $\gamma < x$ and hence $\mathcal{F}$ would be closed under all intersections of fewer than $x$ sets, i.e., $\mathcal{F}$ would be $x$-complete. Therefore $x$ is a successor cardinal number. We define $\text{cpl}(\mathcal{F})$ to be the predecessor of $x$. Note that $\text{cpl}(\mathcal{F})$ is the largest cardinal number $\beta$ such that $\mathcal{F}$ is $\beta$-complete.

Recall that if a cardinal number $x$ is measurable, if there exists a nonprincipal $x$-complete ultrafilter on $x$. By [3, Proposition 4.2.7], $\text{cpl}(\mathcal{F})$ is a measurable cardinal number.

The next few results will give some information on $\text{cpl}(\mathcal{F})$ and will be referred to in Section 3.

Lemma 2.7 (Lemma 1.12, p. 113 in [1]). Let $\mathcal{F}$ be a non-principal ultrafilter on $A$. Then, whenever $\beta < \text{cpl}(\mathcal{F})$ and $\{B_\alpha; \alpha < \beta\}$ is a partition of $A$, then there exists a unique $\xi < \beta$ such that $B_\xi \subseteq \mathcal{F}$.

Lemma 2.8 (Lemma 1.13, p. 114, in [1]). Let $\mathcal{F}$ be a non-principal filter on $A$. Then there exists a sequence $\{B_\alpha; \alpha < \text{cpl}(\mathcal{F})\}$ of elements of $\mathcal{F}$ such that $B_\xi \notin \mathcal{F}$ for all $\alpha < \beta < \text{cpl}(\mathcal{F})$, and $\bigcap_{\alpha < \text{cpl}(\mathcal{F})} B_\alpha \notin \mathcal{F}$. Moreover, if $\mathcal{F}$ is an ultrafilter, then we may choose $\{B_\alpha; \alpha < \text{cpl}(\mathcal{F})\}$ such that $\bigcap_{\alpha < \text{cpl}(\mathcal{F})} B_\alpha = \emptyset$.

Corollary 2.9. Let $\mathcal{F}$ be a filter on $A$ and let $B \subseteq \mathcal{F}$. Then $\text{cpl}(\mathcal{F}) = \text{cpl}(\mathcal{F}_B)$.

Remark. Let $x$ be a regular cardinal number such that $\aleph_0 < x < |A|$, and let $\mathcal{F} = \{B \subseteq A; |A \setminus B| < x\}$. Then $\text{cpl}(\mathcal{F}) = x$.
The proof of the main theorem (Theorem 3.5) extends a technique that was used in [2] and [14]. But first we need some definitions and few lemmas.

A $p$-functor is a subfunctor of the forgetful functor $F: R\to Mod\to Z\to Mod$ that commutes with direct products. Let $(U_i;_x < \delta)$ be a well-ordered set of $p$-functors. Then $(U_i;_x < \delta)$ is a descending chain of $p$-functors if, for all $M \in R\to Mod$, $U_i(M) \supseteq U_j(M)$ for all $_x \beta < \delta$.

Example. For a subset $S$ of $R$, define $U(M) = \text{ann}_R(S)$, for $M \in R\to Mod$. Then $U(n)$ is a $p$-functor.

**Lemma 3.1.** Let $\mathcal{F}$ be a non-principal ultrafilter on $A$. Let $(M_a; a \in A)$ be a family of left $R$-modules and let $(U_i;_x < \sigma)$ be a descending chain of $p$-functors with $\text{cpl}(\mathcal{F}) \subseteq \sigma$.

Suppose that $A = \bigcup_{a \in A} A_a$ for some $A_a \subseteq A$ satisfying:

1. $A_a \neq \emptyset$ for all $a < \sigma$;
2. $A_a \cap A_b = \emptyset$ for all $a \neq b, a, b < \sigma$;
3. $\bigcup_{a < \sigma} A_a \in \mathcal{F}$ for all $\beta < \sigma$;
4. $\bigcap_{a < \sigma} (M_a \supseteq U_{x+1}(M_a))$ for all $a \in A_a$ and for all $a < \sigma$.

Then $\prod_{a < \sigma} M_a$ is not a direct summand of $\prod_{a < \sigma} M_a$.

Proof. Let $\varphi$ be the natural isomorphism $\prod_{a < \sigma} M_a \cong \prod_{a < \sigma} (\prod_{a < \sigma} M_a)$. Let $S = \varphi(\prod_{a < \sigma} M_a) = \bigcap_{a < \sigma} (\prod_{a < \sigma} M_a)$. Note that $(m_{a_{x_{a < \sigma}}}; x_{a < \sigma} \in S)$ if and only if

$\bigcup_{a < \sigma} \{a \in A_a; m_{a_{x_{a < \sigma}}} = 0\} \in \mathcal{F}$.

Now, suppose to the contrary that $\prod_{a < \sigma} M_a$ is a direct summand of $\prod_{a < \sigma} M_a$. Then $S$ is a direct summand of $P$. Let $P = S \oplus Q$ and let $p: P \to S$ be the natural projection. By (4), there exists $0 \neq x_{a_{x_{a < \sigma}}} \in U_{x+1}(M_a)$ for all $a \in A_a$ and all $a < \sigma$. Let $z = \{x_{a_{x_{a < \sigma}}}; x_{a < \sigma} \in S\}$ and for $\beta < \sigma$, let $z_{\beta} = \{x_{a_{x_{a < \sigma}}}; x_{a < \sigma} \in S\}$.

Claim. $z_{\beta} \in S$ for all $\beta < \sigma$.

Let $\beta < \sigma$. Since $x_{a_{x_{a < \sigma}}} \neq 0$ for all $a \in A_a$ and for all $a < \sigma$, the set of indices where the coordinates of $x_{a_{x_{a < \sigma}}}$ are zero is exactly $\bigcup_{a < \sigma} A_a$, which is in $\mathcal{F}$ by (3). Hence $z_{\beta} \in S$.

Now let $u_{z_{\beta}} = \{z_{\beta} + z_{\beta} = z_{\beta} + (x_{a_{x_{a < \sigma}}} + y_{a_{x_{a < \sigma}}})$ and hence, $s = \{x_{a_{x_{a < \sigma}}} + y_{a_{x_{a < \sigma}}} \}$. Then $a < \beta < \sigma$. Then

$\{x_{a_{x_{a < \sigma}}} \in \bigcap_{a < \sigma} (M_a \supseteq U_{x+1}(M_a)) = U_{x+1}(\prod_{a < \sigma} M_a)$.}

Hence, $u_{z_{\beta}} = \{0, \ldots, 0, (x_{a_{x_{a < \sigma}}} + y_{a_{x_{a < \sigma}}} \} \in \bigcap_{a < \sigma} (U_{x+1}(\prod_{a < \sigma} M_a) = U_{x+1}(P))$.

Now, $s = p(u_{z_{\beta}}) \in \bigcap_{a < \sigma} (U_{x+1}(P)) \subseteq U_{x+1}(S) = U_{x+1}(\prod_{a < \sigma} (\prod_{a < \sigma} M_a)) \subseteq \prod_{a < \sigma} (\prod_{a < \sigma} U_{x+1}(M_a))$.

Let $a \in A$, then $a \in A_a$ for some unique $\alpha < \sigma$. Let $x_{a_{x_{a < \sigma}}} \in U_{x+1}(P)$ be the natural projection. Since $\alpha \in S$, there exist $\alpha < \sigma$ and $b \in A_a$ such that $\varphi_{a_{x_{a < \sigma}}} = 0$. Now, $x_{a_{x_{a < \sigma}}} = \varphi_{a_{x_{a < \sigma}}} - (s_{\alpha_{x_{a < \sigma}}} = -\varphi_{a_{x_{a < \sigma}}} \in U_{x+1}(M_a)$ by (4), which is a contradiction to the choice of $x_{a_{x_{a < \sigma}}}$.

**Lemma 3.2.** Let $\mathcal{F}$ be a non-principal ultrafilter on $A$. Let $(M_a; a \in A)$ be a family of left $R$-modules. If $\prod_{a < \sigma} M_a$ is a direct summand of $\prod_{a < \sigma} M_a$, then $\text{ann}_R(P_a) \in \mathcal{F}$ for all $a < \sigma$.

Proof. Suppose to the contrary that $\text{ann}_R(P_a) \in \mathcal{F}$ has no maximal element.

**Claim 1.** There exists an ordinal number $\alpha$ and a sequence $(B_\xi)_{\xi < \alpha}$ such that:

1. $B_\xi \in \mathcal{F}$ for all $\xi < \sigma$;
2. $B_{\alpha} \supseteq B_{\xi + 1}$ for all $\xi < \sigma$;
3. $\text{ann}_R(P_a) \subseteq \text{ann}_R(P_{a_{x_{a < \sigma}}})$ for all $\alpha < \sigma$;
4. $(\text{ann}_R(P_{a_{x_{a < \sigma}}}) \subseteq \text{ann}_R(P_{a_{x_{a < \sigma}}}))$ if and only if $a \in B_\alpha$ for all $\alpha < \sigma$;
5. $\bigcap_{\alpha < \sigma} B_\alpha \in \mathcal{F}$.

Proof of Claim 1. Let $B_0 = A$. We proceed by induction. Let $\alpha > 1$. Suppose that $B_{\alpha - 1}$ has been defined for all $\xi < \alpha$, satisfying 1-4.

Case 1. $\alpha$ is not a limit ordinal. Then $B_{\alpha - 1}$ has been defined. By hypothesis, $\text{ann}_R(P_{a_{x_{a < \sigma}}}) \not\subseteq \text{ann}_R(P_{a_{x_{a < \sigma}}})$. Thus, there exists $B_\alpha \in \mathcal{F}$ such that $\text{ann}_R(P_{a_{x_{a < \sigma}}}) \not\subseteq \text{ann}_R(P_{a_{x_{a < \sigma}}})$. Then $B_\alpha \subseteq B_{\alpha - 1}$ and hence $B_\alpha \in \mathcal{F}$. Also, if $a \in B_\alpha$, then $\text{ann}_R(P_{a_{x_{a < \sigma}}}) \subseteq \text{ann}_R(P_{a_{x_{a < \sigma}}})$ and hence by induction hypothesis, $a \in B_{\alpha - 1}$. Therefore $B_\alpha \subseteq B_{\alpha - 1}$. Now, $\text{ann}_R(P_{a_{x_{a < \sigma}}}) \not\subseteq \text{ann}_R(P_{a_{x_{a < \sigma}}}) \not\subseteq \bigcap_{\alpha < \sigma} \text{ann}_R(P_{a_{x_{a < \sigma}}})$.

Case 2. $\alpha$ is a limit ordinal. If $\bigcap_{\xi < \alpha} B_\xi \in \mathcal{F}$, we let $\sigma = \alpha$ and we stop the induction. Otherwise, let $B_\alpha = \bigcap_{\xi < \alpha} B_\xi \in \mathcal{F}$. If $a \in B_\alpha$, then $\text{ann}_R(P_{a_{x_{a < \sigma}}}) \subseteq \text{ann}_R(P_{a_{x_{a < \sigma}}})$. Conversely, if $\text{ann}_R(P_{a_{x_{a < \sigma}}}) \subseteq \text{ann}_R(P_{a_{x_{a < \sigma}}})$, then $\text{ann}_R(P_{a_{x_{a < \sigma}}}) \subseteq \text{ann}_R(P_{a_{x_{a < \sigma}}})$. Hence $a \in B_\alpha$ for all
\( \xi < \alpha \) by induction hypothesis; i.e., \( a \in B_\alpha \). Clearly, there exists a limit ordinal \( \sigma \) such that
\[
B = \bigcap \{ B_\xi : \xi < \alpha \}.
\]
Note that \( \sigma \geq \text{cpl}(\mathcal{F}) \). This completes the proof of Claim 1.

**Claim 2.** \( A \cup \mathcal{B} = \bigcup_{\xi < \alpha} (B_\xi \setminus B_{\xi+1}) \).

**Proof of Claim 2.** Let \( a \in A \cup \mathcal{B} \). Then \( a \notin B_\xi \) for some \( \xi < \alpha \). Let \( a \in B \) be the least such ordinal number. Then \( a \) is not a limit ordinal. For, if \( \alpha \) is a limit ordinal, then \( a \notin B_\alpha \) by the construction of \( B_\alpha \) in Claim 1, and therefore \( a \notin B_\xi \) for some \( \xi < \alpha \), which is a contradiction to the choice of \( a \). Hence \( a \notin B_\xi \) for all \( \xi < \alpha \). This proves Claim 2.

Now, for \( a < \sigma \) and for \( M \in \mathcal{R} \), define \( U_\alpha(M) = \text{ann}_a \text{ann}_b (P_\alpha) \). Then \( U_\alpha : a < \sigma \) is a descending chain of \( p \)-functors.

**Claim 3.** \( U_\alpha(M) \subseteq U_{\alpha+1}(M) \) for all \( a \in B_\alpha \setminus B_{\alpha+1} \).

**Proof of Claim 3.** Suppose to the contrary that \( a \in B_\alpha \) but \( a \notin U_{\alpha+1}(M) \) for some \( \alpha < \sigma \). Since \( a \in B_\alpha \), we have \( M_\alpha = U_\alpha(M) \), and hence \( M_\alpha = U_{\alpha+1}(M) = \text{ann}_a \text{ann}_b (P_{\alpha+1}) \). Therefore \( \text{ann}_a (P_{\alpha+1}) \subseteq \text{ann}_a (M_\alpha) \), so that \( a \notin B_\alpha \) by Claim 1, which is a contradiction to the choice of \( a \).

Now, let \( \mathcal{F} = \mathcal{F}_{< \sigma} \) (See Definition 2.1.). Then \( \mathcal{F} \) is a non-principal ultrafilter on \( A \cup \mathcal{B} \). By Lemma 2.2 and by hypothesis, \( \prod_{\alpha < \sigma} M_\alpha \) is a direct summand of \( \prod_{\alpha < \sigma} M_\alpha \). Also, by Corollary 2.9, \( \text{cpl}(\mathcal{F}) = \text{cpl}(\mathcal{F}) \). Now, for \( \beta < \sigma \),
\[
\bigcup_{\alpha < \beta} B_\alpha (B_\alpha \setminus B_{\alpha+1}) \in \mathcal{F}.
\]
We thus obtain a contradiction to Lemma 3.1.

**Lemma 3.3.** Let \( \mathcal{F} \) be a non-principal ultrafilter on \( A \). Let \( \{ M_\alpha : a \in A \} \) be a family of left \( R \)-modules. Suppose that \( \prod_{\alpha < \sigma} M_\alpha \) is a direct summand of \( \prod_{\alpha < \sigma} M_\alpha \). Then \( \text{cpl}(\mathcal{F}) = \text{cpl}(\mathcal{F}) \).

**Proof.** Suppose to the contrary that \( B \notin \mathcal{F} \). Then, by Lemma 2.2, \( \prod_{\alpha < \sigma} M_\alpha \) is a direct summand of \( \prod_{\alpha < \sigma} M_\alpha \). Also, by Corollary 2.9, \( \text{cpl}(\mathcal{F}) = \text{cpl}(\mathcal{F}) \). Now, by Lemma 2.8, there exists a sequence \( (B_\alpha : \alpha < \text{cpl}(\mathcal{F})) \) of elements of \( \mathcal{F} \) such that \( B_\alpha \supsetneq B_{\alpha+1} \) for all \( \alpha < \beta \in \text{cpl}(\mathcal{F}) \) and \( \bigcap_{\alpha < \text{cpl}(\mathcal{F})} B_\alpha = \emptyset \). We may choose clearly \( B_\alpha \). Then
\[
B = \bigcup_{\alpha < \text{cpl}(\mathcal{F})} E_\alpha,
\]
where \( E_\alpha = (\bigcap_{\xi < \alpha} B_\xi) \setminus B_{\alpha} \). For \( \alpha < \text{cpl}(\mathcal{F}) \), \( E_\alpha \neq \emptyset \) and \( E_\alpha \notin \mathcal{F} \), since \( B_\alpha \notin \mathcal{F} \). Moreover, \( E_\alpha \cap E_\beta = \emptyset \) for all \( \alpha \neq \beta \). However, if \( \beta \in \text{cpl}(\mathcal{F}) \), then
\[
E_\beta = B_{\beta} \notin \mathcal{F}.
\]
Finally, if \( \alpha < \text{cpl}(\mathcal{F}) \), then \( \bigcup_{\alpha < \text{cpl}(\mathcal{F})} U_\alpha (M) \subseteq U_{\alpha+1}(M) \). By assumption, therefore the hypotheses of Lemma 3.1 are satisfied and we obtain a contradiction. Hence \( B \notin \mathcal{F} \).

At this point, we need to specialize \( \text{cpl}(\mathcal{F}) \). As was shown in the remark after Corollary 2.9, given a regular cardinal number \( \kappa \) such that \( \text{cf}(\kappa) < \kappa \), there exists a filter \( \mathcal{F} \) on \( A \) such that \( \text{cpl}(\mathcal{F}) = \kappa \). The question arises as to whether there exists an ultrafilter \( \mathcal{F} \) on \( A \) such that \( \text{cpl}(\mathcal{F}) = \kappa \). The answer is trivial for \( \kappa \) countable: every filter \( \mathcal{F} \) on a countable set \( A_0 \) satisfies \( \text{cpl}(\mathcal{F}) = \aleph_0 \). The existence of a non-principal ultrafilter \( \mathcal{F} \) on an uncountable set such that \( \text{cpl}(\mathcal{F}) > \aleph_0 \) is an axiom which is often used in modern set theory. In the next few paragraphs, we will state some results dealing with this problem that will be used later; a more detailed exposition of these results and further information on the properties and uses of the definitions that will be given can be found in [1], [4], [7], [10] and [15].

A cardinal number \( \kappa > \aleph_0 \) is said to be \( \kappa \)-measurable (or \( \kappa \)-ulam-measurable) if there exists a non-principal ultrafilter \( \mathcal{F} \) on \( A \) of cardinality \( \kappa \) such that \( \text{cpl}(\mathcal{F}) = \kappa \). By Theorem 8.31 in [4], a cardinal number \( \kappa > \aleph_0 \) is \( \kappa \)-measurable if and only if it is greater or equal to the first uncountable measurable cardinal. Hence, the existence of a non-principal ultrafilter \( \mathcal{F} \) on a set \( A \) such that \( \text{cpl}(\mathcal{F}) > \aleph_0 \) is equivalent to the existence of uncountable measurable cardinal numbers (3MC). It cannot be proved in standard set theory ZF that such cardinals exist. Moreover, the consistency of the theory ZF + 3MC cannot be proved in ZF. However, ZF + 3MC is often used in descriptive set theory and even stronger theories, so-called natural extensions of ZF, are used and believed to be consistent.

If uncountable measurable cardinals exist, they are very large. In fact, if \( \kappa \) is the first measurable cardinal number, then it is inaccessible (i.e., it is a regular limit cardinal number) and there are \( \kappa \) inaccessible cardinal numbers less than \( \kappa \) (Theorem 3, p. 26 in [15]). The existence of such cardinal numbers is independent of ZF. Also, the axiom of constructibility, that "V = L," in the sense of Gödel, implies that measurable cardinal numbers do not exist (Theorem 6.9, p. 305 in [11]).

In view of the above, the case \( \text{cpl}(\mathcal{F}) = \aleph_0 \) is a very important case. The following few results deal with this case.

**Lemma 3.4.** Let \( \mathcal{F} \) be a non-principal ultrafilter on \( A \) such that \( \text{cpl}(\mathcal{F}) = \aleph_0 \). Let \( \{ U_n : n = 1, 2, \ldots \} \) be a descending chain of \( p \)-functors. Let \( \{ M_\alpha : a \in A \} \) be a family of left \( R \)-modules. If \( \prod_{\alpha < \sigma} M_\alpha \) is a direct summand of \( \prod_{\alpha < \sigma} M_\alpha \), then there exist \( B \notin \mathcal{F} \) and \( n \) such that for all \( a \in B \) and for all \( p \geq n \), \( U_p(M_\alpha) = U_p(M_\alpha) \).

**Proof.** For \( p \geq 1 \), define
\[
C_p = \{ a \in A : U_p(M_\alpha) \supsetneq U_{p+1}(M_\alpha) \text{ for some } \alpha \geq p \}.
\]
If \( C_p \notin \mathcal{F} \) for some \( n \geq 1 \), then \( B = A \setminus C_0 \notin \mathcal{F} \). Moreover, for \( a \in B \) and \( p \geq n \), we have \( U_p(M_\alpha) = U_p(M_\alpha) \). Therefore, we may assume that \( C_p \notin \mathcal{F} \) for all \( p \geq 1 \). We will obtain a contradiction. Note that we have \( C_p \subseteq C_{p+1} \) for all \( p \geq 1 \). Let \( C = \bigcap_{p \geq 1} C_p \). Then
\[
C = C \cup \bigcup_{p \geq 1} C_{p+1}.
\]
Step 1. \( F = \bigcup_{p \geq 1} C_p \setminus C_{p+1} \notin \mathcal{F} \).

Proof. Suppose to the contrary that \( F \in \mathcal{F} \). Since \( C_p \notin \mathcal{F} \) for all \( p \geq 1 \), we have \( C_p \setminus C_{p+1} \notin \mathcal{F} \) for all \( p \geq 1 \). Moreover, \( (C_p \setminus C_{p+1}) \cap (C_{p+1} \setminus C_{p+2}) = \emptyset \) for all \( p \neq q \). Therefore, by Lemma 27, there are \( N_0 \) non-empty \( C_p \setminus C_{p+1} \)'s. Hence, we may assume that \( C_p \setminus C_{p+1} \notin \mathcal{F} \) for all \( p \geq 1 \). For \( n > 1 \),

\[
\bigcup_{p \geq 1} \bigcup_{1 \leq k \leq n-1} C_p \setminus C_{p+1} = \bigcup_{1 \leq k \leq n-1} \bigcup_{p \geq 1} (C_p \setminus C_{p+1}) \notin \mathcal{F},
\]

since \( C_p \setminus C_{p+1} \notin \mathcal{F} \).

Now, for \( p > 1 \) and \( a \in C_p \setminus C_{p+1} \), \( U_p(M) \sim U_{p+1}(M) \). Also, \( \mathcal{F} \) is a non-principal ultrafilter on \( F \). (See Definition 2.1.) By Corollary 2.9, \( \text{cpl}(\mathcal{F}) = \text{cpl}(\mathcal{F}) = N_0 \). Therefore the hypothesis of Lemma 3.1 is satisfied and hence \( \prod_{p \geq 1} M_p \) is not a direct summand of \( \prod_{p \geq 1} M_p \). By Lemma 2.2 and by hypothesis, \( \prod_{p \geq 1} M_p \) is a direct summand of \( \prod_{p \geq 1} M_p \). We thus obtain a contradiction to our assumption.

Step 2. \( C = \bigcap_{p \geq 1} C_p \notin \mathcal{F} \).

Proof. Suppose to the contrary that \( C \in \mathcal{F} \). For \( a \in C \) and for \( p > 1 \), let \( \eta(a, p) = \inf \{ n : a \in C \cap U_{p,n}(M) \} \). Then \( \eta(a, p) \) is well defined for all \( a \in C \) and all \( p > 1 \). For \( p < q \), let \( B_q = \{ a \in C : \eta(a, p) = q \} \). We construct inductively a sequence \( (p_i) \) of natural numbers such that \( p_i < p_{i+1} \) and \( B_{p_i} \notin \mathcal{F} \) for all \( i \). Let \( p_0 = 0 \). Now, \( C = \bigcup_{q \geq 1} B_{p_0} \).

CLAIM 1. \( B_{p_0} \notin \mathcal{F} \) for some \( q \geq 1 \).

Proof. Suppose to the contrary that \( B_{p_0} \notin \mathcal{F} \) for all \( q \geq 1 \). Since \( B_{p_0} \setminus B_{p_0} \notin \mathcal{F} \) for all \( q \geq 1 \), there are \( N_0 \) non-empty \( B_{p_0} \)'s. We may therefore assume that \( B_{p_0} \notin \mathcal{F} \) for all \( q \geq 1 \). For \( n > 1 \),

\[
\bigcup_{q \geq 1} B_{p_0} = \bigcup_{1 \leq q \leq n-1} B_{p_0} = \bigcap_{1 \leq q \leq n-1} (C \setminus B_{p_0}) \notin \mathcal{F},
\]

since \( B_{p_0} \notin \mathcal{F} \). For \( a \in B_{p_0} \), \( U_p(M) \sim U_{p+1}(M) \). Finally, since \( C \in \mathcal{F} \), \( C \) is a non-principal ultrafilter on \( C \). By Lemma 2.2 and by hypothesis, \( \prod_{p \geq 1} M_p \) is a direct summand of \( \prod_{p \geq 1} M_p \). Also, by Corollary 2.9, \( \text{cpl}(\mathcal{F}) = \text{cpl}(\mathcal{F}) = N_0 \). Therefore, we obtain a contradiction to Lemma 3.1. Let \( p_1 \) be the natural number such that \( B_{p_1} \notin \mathcal{F} \). Then \( p_1 > p_0 \). Now, suppose that \( p_0, p_1, \ldots, p_n \) have been defined such that \( p_i < p_{i+1} \) and \( B_{p_i} \notin \mathcal{F} \) for all \( i < n \). Since \( B_{p_i} \notin \mathcal{F} \), \( B_{p_i} \setminus B_{p_{i+1}} \notin \mathcal{F} \) for all \( i < n \). A similar argument to the one in the proof of Claim 1 shows that \( B_{p_i} \setminus B_{p_{i+1}} \notin \mathcal{F} \) for some \( q > p_{i+1} \). Let \( p_{i+1} \) be that natural number. Then \( p_{i+1} < p_{i+1} \).
Since $C \in \mathcal{F}$, $P_C$ is a faithful $R$-module, and since $\text{ann}_R(\mathcal{F}) = \mathcal{V}_R(P_C) = \mathcal{V}_R(P_C) = \text{ann}_R(\mathcal{F})$ for all $p \geq n$, we have $\text{ann}_R(\mathcal{F}) = \text{ann}_R(\mathcal{F})$ for all $p \geq n$, by Claim 2. Hence $R$ has ACC on annihilators.

**Corollary 3.6.** Let $R$ be a simple ring. Let $\mathcal{F}$ be a non-principal ultrafilter on $A$ such that $\text{clp}(\mathcal{F}) = \mathcal{N}_R$. Let $\{M_a : a \in A\}$ be a family of left $R$-modules. If $\prod_{a \in A} M_a$ is a direct summand of $\prod_{a \in A} M_a$, then $R$ has ACC on annihilators.

**Corollary 3.7.** Let $\mathcal{F}$ be a non-principal ultrafilter on $A$ such that $\text{clp}(\mathcal{F}) = \mathcal{N}_R$. If $\prod_{a \in A} R$ is a direct summand of $\prod_{a \in A} R$, then $R$ has ACC on annihilators.

Note that Theorems 3.5 and Corollaries 3.6 and 3.7 can be applied to any non-principal ultrafilter on any set $A$ with $|A| < \text{first measurable cardinal number}$. In particular, these results can be applied to any non-principal ultrafilter on any set $A$ if there are no measurable cardinals.

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**References**


