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W R O C Ł A W S K A D R U K A R N I A N A U K O W A

Random theorems in topology

by

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Abstract. Let E and X be Polish spaces and A and B be two disjoint analytic subsets of $E\times X$ with closed vertical sections. We prove the following results.

(i) There is a Borel map $f\colon E\times X\to[0,1]$ such that $f\equiv 0$ on A, $f\equiv 1$ on B and for each $e\in E$, the map $x\to f(e,x)$ is continuous.

(ii) If Z is a retract of finite or countable product of intervals and if $f\colon A\to Z$ is a Borel map such that for every $e\in E$, the map $x\to f(e,x)$ is continuous then there is a Borel measurable extension $F\colon E\times X\to Z$ of f such that $x\to F(e,x)$ is continuous for each $e\in E$.

(iii) If A is Borel then (ii) holds for all convex subsets Z of a second countable affine space of type m.

1. Notation. For notation and basic results in Descriptive Set Theory we follow Moschovakis [11]. Throughout X is a Polish space with a bounded metric d. For $x\in X$ and positive real number r, $S_r(x)$ (resp. $\bar{S}_r(x)$) denotes the open (resp. closed) ball of X with centre x and radius r. Let E be an arbitrary set and $\mathscr{E}$ a family of subsets of E. A *multifunction* $F\colon E\to X$ is a map with domain E and values non-empty, closed subsets of X. We say that the multifunction $F\colon E\to X$ is *$\mathscr{E}$-measurable* if

$$F^{-1}(U)=\{e\in E\colon F(e)\cap U\neq\emptyset\}$$

belongs to $\mathscr{E}$ for every open set U in X. The set

$$\{(e,x)\in E\times X\colon x\in F(e)\}$$

will be called the *Graph* of F and will be denoted by $G(F)$. We consider a point map also as a multifunction.

Let Z be a topological space and $f\colon G(F)\to Z$ a point map. We call f a *G-Carathéodory* map if

(i) for each $e\in E$, $x\to f(e,x)$ is continuous and

(ii) for every $\mathscr{E}$-measurable selector $s\colon E\to X$ of E, the map $e\to f(e,s(e))$ is $\mathscr{E}$-measurable.

Let $\mathscr{E}$ be a σ-field and $G\subseteq E\times X$. Then a map $f\colon G\to Z$ will be called *$\mathscr{E}$-Carathéodory* or simply *Carathéodory* if for each $e\in E$, the map $x\to f(e,x)$ defined on the section $G(e)$ of G is continuous and f is $\mathscr{E}\times\mathscr{B}_X|G$-measurable, where $\mathscr{B}_X$ is the Borel σ-field of X and $\mathscr{E}\times\mathscr{B}_X$ is the product σ-field.

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Remark. If Z is a metrizable space and $(E, \mathscr{E})$ a measurable space, then for every $\mathscr{E}$-measurable multifunction $F\colon E \to X$, each G-Carathéodory map $f\colon G(F) \to Z$ is Carathéodory.

Proof. By [9], fix a sequence $\{s_i\}$ of $\mathscr{E}$-measurable selectors of F such that for every $e \in E$, $\{s_i(e)\}$ is dense in $F(e)$. Let C be a closed set in Z and for each positive integer n, let

$$C_n = \{z \in Z\colon \operatorname{dist}(z', z) < 1/n \text{ for some } z' \in C\}.$$

Then for every $(e, x) \in G(F)$,

$$f(e, x) \in C \Leftrightarrow \forall n \exists i \ (\operatorname{dist}(x, s_i(e)) < 1/n \text{ and } f(e, s_i(e)) \in C_n).$$

The rest of our notation is standard. If E is a metrizable space then unless otherwise mentioned, $\mathscr{E}$ will denote its Borel σ-field.

For concepts in General Topology we follow Dugundji [6].

2. Introduction. Motivated by results proved in [1, 4, 7] in [13] we proved, among others, the following two results.

THEOREM 1. *Let $(E, \mathscr{E})$ be a measurable space, $F\colon E \to X$ an $\mathscr{E}$-measurable multifunction and $f\colon G(F) \to \boldsymbol{R}$ a Carathéodory map. Then there is a Carathéodory map $g\colon E \times X \to \boldsymbol{R}$ which extends f and which satisfies*

$$g(e, X) \subseteq \operatorname{co}(f(\{e\} \times F(e))), \qquad e \in E,$$

where $\operatorname{co}(A)$ denotes the convex hull of A.

THEOREM 2. *Let E be a second countable metrizable space, Z a locally convex topological vector space, $F\colon E \to X$ a measurable multifunction and $f\colon G(F) \to Z$ a G-Carathéodory map. Then also the conclusions of Theorem 1 hold.*

In this paper we give generalizations of these two theorems when E is a Polish space. While proving Theorem 1 we needed some random analogues of the Urysohn Theorem. Here we study this in detail and also show that our random Urysohn theorems are sharp. At the end we prove a random analogue of Lusin's theorem and raise several open problems.

3. Random Urysohn theorems.

THEOREM 3. *If $(E, \mathscr{E})$ is a measurable space and $F_0, F_1\colon E \to X$ measurable multifunctions with $G(F_0) \cap G(F_1) = \emptyset$ then there is a Carathéodory map $f\colon E \times X \to [0, 1]$ such that $f \equiv 0$ on $G(F_0)$ and $f \equiv 1$ on $G(F_1)$.*

Proof. By [9], we get sequences $\{f_n^0\}$ and $\{f_n^1\}$ of measurable maps from E into X such that for every $e \in E$, $\{f_n^\varepsilon(e)\}$ is dense in $F_\varepsilon(e)$ where $\varepsilon = 0$ or 1. Now we define

$$f(e, x) = \frac{\operatorname{dist}(x, F_0(e))}{\operatorname{dist}(x, F_0(e)) + \operatorname{dist}(x, F_1(e))}, \qquad (e, x) \in E \times X$$

$$= \frac{\inf_n d(x, f_n^0(e))}{\inf_n d(x, f_n^0(e)) + \inf_n d(x, f_n^1(e))}.$$

The map f has the desired properties.

We give an example to show that Theorem 3 cannot be extended to the case $\mathscr{E} = \mathscr{L}_\sigma$ where $\mathscr{L}$ is a field.

EXAMPLE 1. Let $E = \omega^\omega$ and $\mathscr{E} = \Sigma_2^0$. Let A_0 and A_1 be two disjoint Σ_2^0-sets in the space of irrationals ω^ω such that there do not exist disjoint Π_2^0-sets C_0 and C_1 satisfying $A_0 \subseteq C_0$ and $A_1 \subseteq C_1$ [11, p. 205]. Define $F_0, F_1\colon \omega^\omega \to [0, 1]$ by

$$\begin{aligned} F_0(\alpha) &= [0, 3/4] && \text{if } \alpha \in A_0, \\ F_0(\alpha) &= \{0\} && \text{if } \alpha \in \omega^\omega \setminus A_0, \\ F_1(\alpha) &= [1/4, 1] && \text{if } \alpha \in A_1, \\ F_1(\alpha) &= \{1\} && \text{if } \alpha \in \omega^\omega \setminus A_1. \end{aligned}$$

Then F_0 and F_1 are two Σ_2^0-measurable, compact-valued multifunctions with $G(F_0) \cap G(F_1) = \emptyset$. If possible suppose there is a map $f\colon \omega^\omega \times [0, 1] \to [0, 1]$ such that $f \equiv 0$ on $G(F_0)$ and $f \equiv 1$ on $G(F_1)$ and such that for every $x \in [0, 1]$, $e \to f(e, x)$ is Σ_2^0-measurable. Now consider

$$C_0 = \{\alpha \in \omega^\omega\colon f(\alpha, 1/2) = 0\}, \qquad C_1 = \{\alpha \in \omega^\omega\colon f(\alpha, 1/2) = 1\}.$$

These are disjoint Π_2^0-sets such that $A_i \subseteq C_i$, $i = 0, 1$. This is a contradiction.

For semi-continuous multifunctions we have

LEMMA. *Let E be a metrizable space, X a Polish space and $F\colon E \to X$ a closed valued upper or lower semi-continuous multifunction. Then for each $x \in X$ the map*

$$e \to \operatorname{dist}(x, F(e))$$

is Σ_2^0-measurable.

Proof. Fix $x \in X$, $e \in E$ and reals $a < b$. We have

$$\operatorname{dist}(x, F(e)) < b$$

(i) $\quad \Leftrightarrow S_b(x) \cap F(e) \neq \emptyset$

(ii) $\quad \Leftrightarrow (\exists m)\ (\bar{S}_{b-1/m}(x) \cap F(e) \neq \emptyset)$,

$$\operatorname{dist}(x, F(e)) > a$$

(iii) $\quad \Leftrightarrow (\exists m)\ (S_{a+1/m}(x) \cap F(e) = \emptyset)$

(iv) $\quad \Leftrightarrow (\exists m)\ (\bar{S}_{a+1/m}(x) \cap F(e) = \emptyset)$.

Equivalences (i) and (iii) prove the lemma when F is lower semi-continuous. For upper semi-continuous F we use (ii) and (iv).

We now have

THEOREM 4. *Let E be a metrizable space and $F_0, F_1\colon E \to X$ be lower or upper semi-continuous multifunctions such that $F_0(e) \cap F_1(e) = \emptyset$ for every $e \in E$. Then there is a map $f\colon E \times X \to [0, 1]$ such that*

(i) *$f \equiv 0$ on F_0, and $f \equiv 1$ on F_1,*

(ii) *$x \to f(e, x)$ is continuous for every $e \in E$, and*

(iii) *$e \to f(e, x)$ is Σ_2^0-measurable, for each $x \in X$.*

Before we proceed to prove our next theorem we present an example.

EXAMPLE 2. Let A be a Σ_1^1 but non-Borel subset of $[0, 1]$. Fix a metric ϱ on ω^ω and let α, β be two distinct points of ω^ω. Let U be a non-empty clopen subset of $S_{(1/2)\varrho(\alpha,\beta)}(\alpha)$. Let B be a Borel subset of $[0, 1]\times U$ with closed sections such that $A = \operatorname{proj}(B)$. Let

$$F = B\cup([0, 1]\times\{\beta\}).$$

Then F is a Borel subset of $[0, 1]\times\omega^\omega$ with non-empty closed sections. If possible suppose the map $e\to\operatorname{dist}(\alpha, F(e))$ defined on $[0, 1]$ is Borel. Then, as

$$e\in A \Leftrightarrow \operatorname{dist}(\alpha, F(e))\leqslant \tfrac{1}{2}\varrho(\alpha, \beta),$$

A is Borel. Therefore, the map $e\mapsto\operatorname{dist}(\alpha, F(e))$ is not Borel.

The above example shows that the simple-minded arguments contained in the proofs of Theorems 3 and 4 do not work for our main random Urysohn theorem mentioned in the abstract. Instead we shall use the following three results.

THEOREM A (Saint-Raymond, [12]). *Let E and X be two Polish spaces and A and B be two Σ_1^1-subsets of $E\times X$ such that for every $e\in E$, $\overline{A(e)}\cap B(e) = \emptyset$. Then there is a Borel set C in $E\times X$ such that for every $e\in E$, $C(e)$ is closed and $A\subseteq C\subseteq (E\times X)\setminus B$.*

THEOREM B (Dellacherie, [5]). *If E and X are Polish spaces and $B\subseteq E\times X$ is a Borel set with $B(e)$ open for every $e\in E$ then*

$$B = \bigcup_{n\in\omega}(B_n\times U_n)$$

where B_n is Borel in E and U_n open in X.

THEOREM C (Miller, [10]). *Let E be a second countable metrizable space. Denote by $\mathcal{T}$ the topology on E. Then given any sequence $\{B_n\}$ of Borel sets in E there is a second countable metrizable topology $\mathcal{T}'$ on E such that*

(i) *each of $B_n\in\mathcal{T}'$, and*

(ii) *the σ-fields generated by $\mathcal{T}$ and $\mathcal{T}'$ are the same.*

Actually this is a simpler case of Miller's theorem and a proof of it is also presented in ([13, Theorem 5]).

From now on E will be a Polish space.

THEOREM 5. *Let F_0 and F_1 be two disjoint Σ_1^1 sets in $E\times X$ such that for each $e\in E$, the sections $F_0(e)$ and $F_1(e)$ are closed. Then there is a Carathéodory map f: $E\times X\to[0, 1]$ such that $f\equiv 0$ on F_0 and $f\equiv 1$ on F_1.*

Proof. By applying Theorem A twice, we get two disjoint Borel sets C_0 and C_1 in $E\times X$ with $C_0(e)$ and $C_1(e)$ closed, $F_0\subseteq C_0$ and $F_1\subseteq C_1$. By Theorem B, we write

$$(E\times X)\setminus C_i = \bigcup_{n\in\omega}(B_n^i\times U_n^i),\qquad i = 0 \text{ or } 1$$

with B_n^i Borel in E and U_n^i open in X. Denote the topology on E by $\mathcal{T}$. By Theorem C, let $\mathcal{T}'$ be a second countable metrizable topology on E such that

(i) $B_n^i\in\mathcal{T}'$, $n\in\omega$, $i = 0$ or 1, and

(ii) the σ-fields generated by $\mathcal{T}$ and $\mathcal{T}'$ are the same.

Now C_0 and C_1 are disjoint closed sets in $E\times X$ when E is equipped with $\mathcal{T}'$ and X has its own Polish topology, say $\mathcal{T}''$. By Urysohn's theorem there is a $\mathcal{T}'\times\mathcal{T}''$-continuous map f: $E\times X\to[0, 1]$ such that $f\equiv 0$ on C_0 and $f\equiv 1$ on C_1. Since the σ-fields generated by $\mathcal{T}$ and $\mathcal{T}'$ are the same, this f has all the desired properties.

Our next example shows that Theorem 5 does not hold if F_0, F_1 are Π_1^1.

EXAMPLE 3. In Example 1 take A_0 and A_1 to be two disjoint Π_1^1 sets such that there do not exist disjoint Borel sets C_0 and C_1 with $A_0\subseteq C_0$ and $A_1\subseteq C_1$. Define F_0 and F_1 exactly the same way. The same arguments show that there does not exist a Carathéodory map f: $E\times X\to[0, 1]$ such that $f\equiv 0$ on F_0 and $f\equiv 1$ on F_1.

Remark 1. It is worth noting that the following generalization of Theorem 5 also holds.

THEOREM 6. *Let E and X be Polish spaces, F_0, F_1 be two disjoint Σ_1^1 sets in $E\times X$ such that for all $e\in E$, $F_0(e)$ and $F_1(e)$ are Π_ξ^0. Then there is a Borel map f: $E\times X\to[0, 1]$ such that*

(i) *$f\equiv 0$ on F_0, $f\equiv 1$ on F_1 and*

(ii) *for every $e\in E$, $x\to f(e, x)$ is Σ_ξ^0-measurable.*

Proof. For $\xi = 1$ this is Theorem 5. Let $1<\xi<\omega_1$. Embed X in a recursively presented Polish space H, say the Hilbert cube. We now invoke a result of R. Barua ([2]) (which, in fact, is a simple extension of a result of A. Louveau [8]) and get a Borel set B in $E\times X$ such that

(i) $F_1\subseteq B\subseteq E\times X\setminus F_0$, and

(ii) $B(e)$ is Δ_ξ^0 for every $e\in E$.

We take $f = I_B$, the indicator function of B.

Remark 2. The argument above also works when $\xi = 1$ and X a zero-dimensional Polish space. In this case embed X in and as a closed subspace of ω^ω.

4. Random extension theorems. Using the ideas contained in the proof of Theorem 5 we prove

THEOREM 7. *Let A be a Borel set in $E\times X$ such that the sections $A(e)$ are closed for every $e\in E$. Suppose Z is a second countable convex subspace of an affine space of type m and f: $A\to Z$ a Carathéodory map. Then there is a Carathéodory map g: $E\times X\to Z$ which extends f.*

Proof. Fix a countable base $W_1, W_2, \ldots$ of Z. Let $A_0 = A$ and

$$A_n = A\setminus f^{-1}(W_n),\qquad n = 1, 2, \ldots$$

By the arguments contained in the proof of Theorem 5 we get a finer second countable metrizable topology $\mathcal{T}'$ such that each of A_i is closed when E is equipped with $\mathcal{T}'$ and the Borel σ-field of E remains the same. This makes A closed and f continuous when E has the new topology. By the extension theorem of Dugundji ([6], p. 188) there is a continuous extension g: $E\times X\to Z$ of f. This g is a Carathéodory map when E has the original topology.

THEOREM 8. *Let A be a Σ_1^1-set in $E\times X$ and Z a retract of a finite or countable product of intervals in $\mathbf{R}$. Let $f\colon A\to Z$ be a Borel measurable Carathéodory map. Then there is a Carathéodory map $g\colon E\times X\to Z$ which extends f.*

Proof. *Case* 1. $Z=[-1,1]$.

We define a sequence of Carathéodory maps $g_i\colon E\times X\to[-1,1]$, $i=0,1,\dots$ such that for every i

(i) $|g_i(e,x)|\leqslant\frac{1}{3}\cdot(\frac{2}{3})^i$, for all (e,x), and

(ii) $|f(e,x)-g_0(e,x)-\dots-g_i(e,x)|\leqslant(\frac{2}{3})^i$, for all $(e,x)\in A$.

To see that such a sequence can be defined we proceed inductively. Let

$$F_0^0=\{(e,x)\in A\colon f(e,x)\leqslant-\tfrac{1}{3}\}\quad\text{and}\quad F_1^0=\{(e,x)\in A\colon f(e,x)\geqslant\tfrac{1}{3}\}.$$

By Theorem 5 we get a Carathéodory $g_0\colon E\times X\to[-1/3,1/3]$ having the required properties. Having defined $g_0,g_1,\dots,g_i$ satisfying (i)–(iii), we let

$$F_0^{i+1}=\{(e,x)\in A\colon f(e,x)-g_0(e,x)-\dots-g_i(e,x)\leqslant-\tfrac{1}{3}\cdot(\tfrac{2}{3})^i\},$$

$$F_1^{i+1}=\{(e,x)\in A\colon f(e,x)-g_0(e,x)-\dots-g_i(e,x)\geqslant\tfrac{1}{3}\cdot(\tfrac{2}{3})^i\}.$$

By Theorem 5, we get a Carathéodory map $g_{i+1}\colon E\times X\to[-\frac{1}{3}\cdot(\frac{2}{3})^i,\frac{1}{3}\cdot(\frac{2}{3})^i]$ such that $g_{i+1}\equiv-\frac{1}{3}\cdot(\frac{2}{3})^i$ on F_0^{i+1} and is $\equiv\frac{1}{3}\cdot(\frac{2}{3})^i$ on F_1^{i+1}.

We define

$$g(e,x)=\lim_i g_i(e,x),\qquad(e,x)\in E\times X.$$

Case 2. $Z=(-1,1)$

Using case 1, we get a Carathéodory map $h\colon E\times X\to[-1,1]$ which extends f. Let

$$B=\{(e,x)\in E\times X\colon |h(e,x)|=1\}.$$

Then A and B are two disjoint Σ_1^1-sets with closed sections. By Theorem 5, we get a Carathéodory map

$$u\colon E\times X\to[0,1]$$

such that $u\equiv1$ on A and $\equiv0$ on B. Put $g=u\cdot h$.

Remaining cases. It is now clear that the result is true for all intervals. When Z is a finite or countable product of intervals we extend each of the coordinate functions. Finally, let Z' be a finite or countable product of intervals and Z a retract of Z'. Fix a retraction $r\colon Z'\to Z$. If $f\colon A\to Z$ is a given Carathéodory map, first get a Carathéodory map $h\colon E\times X\to Z'$ which extends f and then take $g=r\circ h$. This completes the proof.

In Theorems 1 and 2 we get extensions satisfying

$$g(e,X)\subseteq\operatorname{co}\big(f(\{e\}\times F(e))\big),\qquad e\in E.$$

our next example shows that we cannot have this in Theorems 7 and 8 even when $Z=\mathbf{R}$.

EXAMPLE 4. Let A_0 and A_1 be two Σ_1^1-sets in $[0,1]$ such that $A_0\cup A_1=[0,1]$ but there does not exist a Borel set B such that $B\subseteq A_0$ and $B^c\subseteq A_1$. Let I_0 be the space of all irrationals contained in $[0,1/3]$ whereas I_1 is the space of all those irrationals which are contained in $[2/3,1]$. Let C_i be a Borel set in $[0,1]\times I_i$ whose sections are closed in I_i and such that $\operatorname{Proj}(C_i)=A_i$, $i=0$ or 1. Let $C=C_0\cup C_1$. Let X be the set of all irrationals in $[0,1]$. Then we have a Borel set in $[0,1]\times X$ whose sections are closed in X. Define $f\colon C\to\mathbf{R}$ by

$$f(e,x)=x,\qquad(e,x)\in C.$$

If possible suppose there is a Carathéodory map $g\colon[0,1]\times X\to\mathbf{R}$ which extends f and which satisfies

$$g(e,X)\subseteq\operatorname{co}\big(f(\{e\}\times C(e))\big),\qquad e\in[0,1].$$

Let

$$B=\{e\in[0,1]\colon g(e,1/\sqrt{2})\leqslant1/2\}.$$

Then B is Borel, $B\subseteq A_0$ and $B^c\subseteq A_1$. Contradiction.

5. A random Lusin theorem.

THEOREM 9. *Let $f\colon E\times X\to[0,1]$ be a Borel map. Let $\mu(e,B)$ be a transition function on $E\times\mathscr{B}_X$. Then for every $\varepsilon>0$ there exists a Carathéodory map $g\colon E\times X\to[0,1]$ such that for every $e\in E$*

$$\mu(e,\{x\in X\colon g(e,x)\neq f(e,x)\})<\varepsilon.$$

Proof. Define a sequence $\{E_n\}$ of subsets of $E\times X$ as follows:

$$E_n=\Big\{(e,x)\in E\times X\colon\frac{2k-1}{2^n}\leqslant f(e,x)<\frac{2k}{2^n}\ \text{for some}\ k=1,2,\dots,2^{n-1}\ \text{or}\ f(e,x)=1\Big\}.$$

Then $f=\sum_{n=1}^{\infty}(1/2^n)I_{E_n}$.

By [3], get Borel sets F_n and U_n in $E\times X$ such that

(i) $F_n\subseteq E_n\subseteq U_n$, $n=1,2,\dots$;

(ii) $\mu(e,F_n(e)\setminus U_n(e))<\varepsilon/2^n$ for $n=1,2,\dots$ and $e\in E$; and

(iii) $F_n(e)$ and $X\setminus U_n(e)$ are compact for each n and e.

Now, $e\to F_n(e)$ and $e\to X\setminus U_n(e)$ are measurable, closed-valued multifunctions for each n. Hence by Theorem 3, there exist Carathéodory maps $g_n\colon E\times X\to[0,1]$ such that

$$g_n(e,x)=\begin{cases}0 & \text{if }(e,x)\in X\setminus U_n,\\ 1 & \text{if }(e,x)\in F_n.\end{cases}$$

Put $g=\sum_{n=1}^{\infty}(1/2^n)g_n(e,x)$.

6. Open problems.

PROBLEM 1. In Theorem 5 suppose we take F_0, F_1 to be Borel but E an arbitrary second countable metrizable or even a Π^1_1-set. Do the conclusions of Theorem 5 hold in this case?

PROBLEM 2. Does Theorem 7 hold for a Σ^1_1-set A? We do not know the answer even when Z is a convex subset of $\boldsymbol{R}^2$.

PROBLEM 3. Can Theorem 8 be extended for Π^1_1-sets A? We do not know the answer even when $Z = \boldsymbol{R}$.

A question related to Problem 3 is the following:

PROBLEM 4. Let C_0 and C_1 be two disjoint Π^1_1-sets in $E \times X$ such that for every $e \in E$, the sections $C_0(e)$ and $C_1(e)$ are closed. Further assume that there is a Borel set B containing C_0 but disjoint from C_1. Do there exist disjoint Borel sets B_0 and B_1 such that $C_0 \subseteq B_0$, $C_1 \subseteq B_1$ and for every $e \in E$, the sections $B_0(e)$ and $B_1(e)$ are closed in X?

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A splitting theorem for $\mathscr{F}$-products

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Abstract. Let $\mathscr{F}$ be a non-principal ultrafilter on an infinite index set A. Let $\{M_a\colon a \in A\}$ be a family of left R-modules. We define the $\mathscr{F}$-product of the M_a's to be $\prod^{\mathscr{F}}_{a\in A} M_a = \{(m_a)_{a\in A} \in \prod_{a\in A} M_a\colon \{a \in A\colon m_a = 0\} \in \mathscr{F}\}$. In the present paper, we determine a necessary condition for the $\mathscr{F}$-product of the M_a's to split in the corresponding direct product. This condition will be given in terms of the lattice of ideals which are annihilators of subsets of a certain factor ring of R, and will depend on $\mathscr{F}$.

R will always denote a ring with identity, all modules will be unital, A will always denote an infinite index set and $|X|$ will always denote the cardinality of a set X.

1. Introduction. The question of when the canonical embedding of a direct sum of modules splits in the corresponding direct product has been extensively studied. See, for example, [2], [9], [11], [18], and [19]. Recently modules that are in between direct sums and direct products, called $\varkappa$-products, have been introduced. See, for example, [5], [6], [13], [14] and [17]. In [14], a necessary condition was determined for the canonical embedding of the $\varkappa$-product of modules in the corresponding direct product to split. This result generalized the above-mentioned classical theorems on the sum-product splitting property.

The study of $\varkappa$-products can be done in a more natural setting. Indeed, the $\varkappa$-products are special cases of a larger class of submodules of the direct product, called $\mathscr{F}$-products, where $\mathscr{F}$ is a filter on the index set A. The main objective of this paper is to determine a necessary condition for the $\mathscr{F}$-product of modules to split in the corresponding direct product. This condition will be given in terms of the lattice of ideals which are annihilators of subsets of a certain factor ring of R, and will depend on $\mathscr{F}$.

Let $\operatorname{cpl}(\mathscr{F})$ be the largest cardinal number $\varkappa$ such that $\mathscr{F}$ is $\varkappa$-complete. If $\operatorname{cpl}(\mathscr{F}) = \aleph_0$ (e.g., if $|A| <$ first measurable cardinal number) and if the $\mathscr{F}$-product splits in the corresponding direct product, then a certain factor ring of R has the Ascending Chain Condition (ACC) on annihilators (Theorem 3.5). Under these hypotheses, if R is simple or if $M_a = R$ for every $a \in A$, then R itself has ACC on annihilators (Corollaries 3.6 and 3.7).

The proof of Theorem 3.5 extends a technique that was used in [2] and [14]. Following the classical definition (see, for example, [1], [3], [4], [8], [12] and [16]),