[DT] T. Dobrowolski and H. Toruńczyk, *Separable complete* ANR's *admitting a group structure are Hilbert manifolds*, Topology Appl. 12 (1981), 229–235.

[Du] J. Dugundji, *Locally equiconnected spaces and absolute neighborhood retracts*, Fund. Math. 57 (1965), 187–193.

[He] D. W. Henderson, *Infinite-dimensional manifolds are open subsets of Hilbert space*, Topology 9 (1970), 25–33.

[Hi] C. J. Himmelberg, *Some theorems on equiconnected and locally equiconnected spaces*, Trans. Amer. Math. 115 (1965), 43–53.

[Mo] E. E. Moise, *Grille decomposition and convexification theorems for compact locally connected continua*, Bull. Amer. Math. Soc. 55 (1949), 1111–1121.

[Na] S. B. Nadler Jr., *A characterization of locally connected continua by hyperspace retractions*, Proc. Amer. Math. Soc. 67 (1977), 167–176.

[Sa] K. Sakai, *The space of retractions of a compact Q-manifold is an l_2-manifold*, ibid. 83 (1981), 421–424.

[To_1] H. Toruńczyk, *Characterizing Hilbert space topology*, Fund. Math. 111 (1981), 247–262.

[To_2] — *A correction of two papers concerning Hilbert manifolds*, ibid. 125 (1985), 89–93.

INSTITUTE OF MATHEMATICS
P.O. Box 631
Bo Ho, Hanoi
Vietnam

INSTITUTE OF MATHEMATICS
UNIVERSITY OF TSUKUBA
Tsukuba-city
305 Japan

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
Santa Barbara, Cal. 93106
U.S.A.

Received 16 January 1989;
in revised form 26 June 1989

On decomposition of 3-polyhedra into a Cartesian product

by

Witold Rosicki (Gdańsk)

Abstract. It is well known that the uniqueness of decomposition of 3-polyhedra into a Cartesian product in general does not hold. In this paper we prove that if there is nonuniqueness then one of the factors is an arc. We also answer the question when $K\times I\approx L\times I$, where K and L are compact 2-polyhedra.

1. Introduction. In 1938 K. Borsuk [1] proved that decomposition of a compact polyhedron into a Cartesian product of 1-dimensional factors is unique. However, if one of the factors is a 2-polyhedron, or 2-manifold with boundary, the uniqueness of decomposition does not hold. We give some examples.

EXAMPLE 1.1 (R. H. Fox (1947) [2]). The sets K and L are unions of an annulus and two intervals as in Fig. 1.

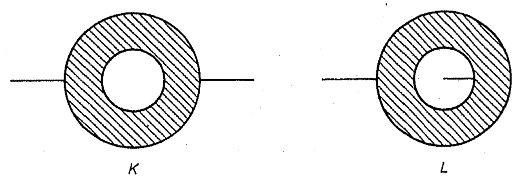

Fig. 1

Then $K\times I\approx L\times I$. ∎

EXAMPLE 1.2. The sets K and L are unions of a disc and six intervals as in Fig. 2.

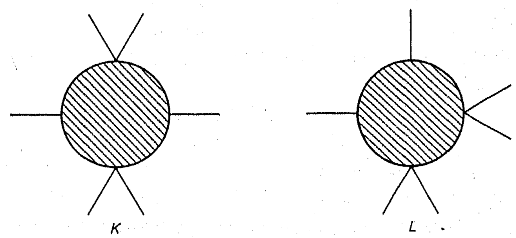

Fig. 2

Then $K\times I\approx L\times I$. ∎

EXAMPLE 1.3 (J. H. C. Whitehead (1940) [5]). Let K be a disc with two holes and L a torus with one hole. Then $K\times I\approx L\times I$. ■

Observe that in all these examples the 1-dimensional factor is homeomorphic to an interval. In this paper, we will prove (Theorem 3.1) that if a 3-polyhedron has two different decompositions into a Cartesian product then an arc is its topological factor. In particular (Lemma 3.5), if K, L are 2-polyhedra and $K\times S^1\approx L\times S^1$ then $K\approx L$.

In the proof we investigate the structure of a 3-polyhedron. We compare the non-Euclidean parts $n_i(K\times X)$ and $n_i(L\times Y)$ (see Definition 2.1) of two homeomorphic 3-polyhedra $K\times X$ and $L\times Y$.

In Section 4 we give some necessary conditions for the formula $K\times I\approx L\times I$ to be true (see Theorem 4.1). The non-Euclidean parts nK and nL of K and L must be homeomorphic. The "Euclidean parts" $M(A)$ and $M(A')$ (see Definition 2.2) may not be homeomorphic, but some conditions for them must hold. The third condition of Theorem 4.1 describes the way in which the Euclidean parts of K and L have to be connected for the formula $K\times I\approx L\times I$ to be true.

2. Preliminaries. We first present some definitions and list some properties of the notions defined.

DEFINITION 2.1. If P is a k-dimensional polyhedron then we define inductively the sets n_iP for $i=0,1,\ldots,k$ as follows:

(i) $n_0P=P$,

(ii) n_iP denotes the subset of $n_{i-1}P$ consisting of the points which have no neighborhood in $n_{i-1}P$ homeomorphic to $\mathbf{R}^{k-i+1}$ or $\mathbf{R}_+^{k-i+1}$.

We set $nP=n_1P$.

Remark. It is easy to see that every set n_iP is a polyhedron and $\dim n_iP\leqslant k-i$.

The above definition was presented in [4]. The following lemma, analogous to Lemma 3.1 of [4], is also true.

LEMMA 2.1. *If K is a 2-polyhedron and X is a 1-polyhedron then*

$$n_i(K\times X)=\bigcup\{n_pK\times n_qX\colon p+q=i\},\quad \textit{where } i=0,1,2,3. \ ■$$

If K is a 2-polyhedron and $A\in\square(K-nK)$ (the symbol $\square$ denotes the set of components) then we define a 2-manifold $M(A)$ as in [4] (Definition 4.1):

DEFINITION 2.2. (1) We denote by $N(A)$ the set of all sequences $\{x_n\}$ in A which converge in K and are such that for every neighborhood U of $\lim x_n$ in K there exist $U_0\in\square(U-nK)$ and a natural number n_0 such that for every $n>n_0$ we have $x_n\in U_0$.

(2) We define an equivalence relation "$\sim$" in $N(A)$ by writing $\{x_n\}\sim\{y_n\}$ iff

(i) $\lim x_n=\lim y_n=x_0$ in K,

(ii) for every neighborhood U of x_0 in K there exist $U_0\in\square(U-nK)$ and a natural number n_0 such that for every $n>n_0$ we have $x_n\in U_0$ and $y_n\in U_0$.

(3) We denote by $M(A)$ the set $N(A)/\sim$.

(4) We define a basis for a topology in $M(A)$. Let $[\{x_n^0\}]\in M(A)$ and $\lim x_n^0=x^0$. Let U be a neighborhood of x^0 in K and let U_0 denote the component of $U-nK$ such that for almost all n we have $x_n^0\in U_0$. We denote by $V(U,[\{x_n^0\}])$ the set of $[\{x_n\}]\in M(A)$ such that $\lim x_n\in U$ and for almost all n, $x_n\in U_0$. The collection of the sets $V(U,[\{x_n^0\}])$ is a basis for the topology of $M(A)$.

DEFINITION 2.3. We define a map $g_A\colon M(A)\to\bar{A}$ by the formula $g_A([\{x_n\}])=\lim x_n$.

Similarly to [4] we give some properties of $M(A)$ and g_A.

Property 2.1. If $\lim x_n=x\in A$ then $[\{x_n\}]=[\{x\}]$ (where $\{x\}$ is a constant sequence).

Property 2.2. The map $h_A\colon A\to M(A)$ given by $h_A(x)=[\{x\}]$ is a topological embedding.

Property 2.3. The space $M(A)$ is a compact 2-manifold with boundary.

Property 2.4. The set $g_A^{-1}(nK\cap\bar{A})$ is a subpolyhedron of $M(A)$ and there exists a subtriangulation of $g_A^{-1}(nK\cap\bar{A})$ such that for every 1-simplex I of this subtriangulation the map $g_A|_I$ is a homeomorphism. All such 1-simplexes are contained in $\partial M(A)$.

Property 2.5. If K, L are 2-polyhedra, X, Y are 1-manifolds (S^1 or I), $F\colon K\times X\to L\times Y$ is a homeomorphism and $F(A\times X)=A'\times Y$, where $A\in\square(K-nK)$ and $A'\in\square(L-nL)$, then there exists a homeomorphism $F_A\colon M(A)\times X\to M(A')\times Y$ such that

$$(g_{A'}\times \mathrm{id}_Y)\circ F_A=F|_{\bar{A}\times X}\circ(g_A\times\mathrm{id}_X).$$

Remark. Let $P_1\colon L\times Y\to L$ and $P_2\colon L\times Y\to Y$ denote the canonical projections. The homeomorphism F_A is given by

$$F_A([\{x_n\}],t)=\big([\{P_1F(x_n,t)\}],\,P_2(\lim F(x_n,t))\big).$$

3. 1-dimensional factor. In this section we prove that if a decomposition of a 3-polyhedron into a Cartesian product is not unique then the 1-dimensional factor has to be homeomorphic to an interval. We assume that all polyhedra are compact and connected.

THEOREM 3.1. *If K and L are 2-polyhedra, X and Y are 1-polyhedra and $K\times X\approx L\times Y$ then either*

(i) *$K\approx L$ and $X\approx Y$, or*

(ii) *there exists a 1-polyhedron P such that $K\approx P\times Y$ and $L\approx P\times X$, or*

(iii) *$X\approx Y\approx[0,1]$.*

In the following lemmas K, L, X and Y satisfy the assumptions of Theorem 3.1.

LEMMA 3.1. *If K has local cut points then $X\approx Y$.*

Proof. Let DK denote the set of local cut points of K, and DL the analogous set for L. It is obvious that if $F\colon K\times X\to L\times Y$ is a homeomorphism then $F(DK\times X)=DL\times Y$. If DK has isolated points then the lemma is proved. Otherwise $F((\overline{K-nK}\cap DK)\times X)=(\overline{L-nL}\cap DL)\times Y$. Since K is connected, $\overline{K-nK}\cap DK$ is not empty. This last set consists of a finite number of points, so the lemma is proved. ■

LEMMA 3.2. *If $K\times S^1\approx L\times I$ then there exists a 1-polyhedron P such that $K\approx P\times I$ and $L\approx P\times S^1$.*

Proof. If K and L are 2-manifolds the assertion is obvious and the polyhedron P is an arc or a simple closed curve.

The polyhedra K and L have no local cut points by Lemma 3.1.

Since $nK\times S^1 = n(K\times S^1)\approx n(L\times I) = nL\times I$, we have $nK\approx I\times\{1,\ldots,k\}$ and $nL\approx S^1\times\{1,\ldots,k\}$ by Borsuk's theorem [1] on the uniqueness of decomposition into a Cartesian product of 1-dimensional factors. So $nK = \bigcup_{i=1}^{k} I_i$ and $nL = \bigcup_{i=1}^{k} S_i$, where $\{I_i\}_{i=1}^{k}$ is a family of pairwise disjoint arcs and $\{S_i\}_{i=1}^{k}$ is a family of pairwise disjoint simple closed curves.

Let $F\colon K\times S^1\to L\times I$ be a homeomorphism. If $A\in\square(K-nK)$ then $F(A\times S^1) = A'\times I$, where $A'\in\square(L-nL)$. So $M(A)\times S^1\approx M(A')\times I$ by Property 2.5. Hence, there exists N (homeomorphic to I or S^1) such that $M(A)\approx N\times I$ and $M(A')\approx N\times S^1$.

Observe that if $I_i\cap\bar{A}\neq\emptyset$ then $I_i\subset\bar{A}$ and if $S_i\cap\bar{A}'\neq\emptyset$ then $S_i\subset\bar{A}'$. Indeed, if $I_i\cap\bar{A}\neq\emptyset$ and $I_i\not\subset\bar{A}$ then $F\big((I_i\cap\bar{A}\cap\overline{K-A})\times S^1\big) = (S_i\cap\bar{A}'\cap\overline{L-A'})\times I$. Hence, $S^1\approx I$ because the sets $I_i\cap\bar{A}\cap\overline{K-A}$ and $S_i\cap\bar{A}'\cap\overline{L-A'}$ are finite.

If $nL\neq\emptyset$ then $g_{A'}^{-1}(S_i)\subset\partial M(A')$ by Property 2.4. Hence, $\partial M(A')\neq\emptyset$. So $N = I$ and $M(A)\approx I\times I$ and $M(A')\approx I\times S^1$.

It is easy to see that $\bar{A}$ and $\bar{A}'$ are 2-manifolds. The set $\partial M(A')$ has only two components so $g_A^{-1}(nK)$ consists of no more than two arcs lying in $\partial M(A)$ (by Property 2.4). We obtain the set $\bar{A}$ from the manifold $M(A)\approx I\times I$ by identifying these arcs with one or two components of nK by homeomorphisms.

Now, let us assign to every $I_i\in\square nK$ a point a_i and to every $A\in\square(K-nK)$ an arc (or a simple closed curve) I_A. If $g_A^{-1}(I_i)\neq\emptyset$ then a_i is an end-point of I_A. If $g_A^{-1}(I_i)$ consists of two components then the end-points of I_A coincide and I_A is a simple closed curve. The sets I_A without end-points are pairwise disjoint. We define a 1-polyhedron P by $P = \bigcup\{I_A\colon A\in\square(K-nK)\}$. (The sets I_A are not subsets of K. The construction of P is abstract.) It is easy to see that $K\approx P\times I$ and $L\approx P\times S^1$. ∎

Lemma 3.3. *Let P, R be 1-polyhedra, L a 2-polyhedron and S a 1-manifold. If $P\times R\times S\approx L\times S$ then $P\times R\approx L$.*

Proof. We can assume $nP\neq\emptyset$ and $nR\neq\emptyset$. In the opposite case P (or R) is homeomorphic to S^1 or I and the proof is simple and we omit it.

If $F\colon P\times R\times S\to L\times S$ is a homeomorphism then $F\big(n(P\times R\times S)\big) = n(L\times S)$. Hence, $F\big(n(P\times R)\times S\big) = nL\times S$ by Lemma 2.1. We know that $n(P\times R)\approx nL$ by Borsuk's theorem [1] mentioned above.

Moreover, for every $A\in\square\big(P\times R-n(P\times R)\big)$ we have $n(P\times R)\cap\bar{A}\approx nL\cap\bar{A}'$, where $F(A\times S) = A'\times S$. We can construct a homeomorphism $f\colon n(P\times R)\to nL$ such that $f\big(n(P\times R)\cap\bar{A}\big) = nL\cap\bar{A}'$ for every $A\in\square(K-nK)$.

Observe that $F(\bar{A}\times S) = \bar{A}'\times S$. Every $\bar{A}$ is homeomorphic either to I^2, to $I\times S^1$ or to $S^1\times S^1$. So $\bar{A}\approx\bar{A}'$. It is easy to see that we can extend f to a homeomorphism $f_A\colon\bar{A}\cup n(P\times R)\to\bar{A}'\cup nL$. We define the required homeomorphism $\tilde{f}\colon P\times R\to L$ by $\tilde{f}(x) = f_A(x)$ for $x\in\bar{A}$. ∎

Recall that we have assumed that $K\times X\approx L\times Y$, K, L are 2-polyhedra and X, Y are 1-polyhedra.

Lemma 3.4. *If $nX\neq\emptyset$ then either*

(i) $K\approx L$ and $X\approx Y$, or

(ii) there exists a 1-polyhedron P such that $K\approx P\times Y$ and $L\approx P\times X$.

Proof. Let $F\colon K\times X\to L\times Y$ be a homeomorphism. Then $F\big(n(K\times X)-n_2(K\times X)\big) = n(L\times Y)-n_2(L\times Y)$. Since $n(K\times X) = K\times nX\cup nK\times X$ and $n_2(K\times X) = nK\times nX\cup n_2K\times X$ (by Lemma 2.1) we obtain $n(K\times X)-n_2(K\times X) = (K-nK)\times nX\cup(nK-n_2K)\times(X-nX)$, and analogously for $L\times Y$.

Therefore, if $A\in\square(K-nK)$ and $x\in nX$, two cases are possible:

1° $F(A\times\{x\}) = A'\times\{x'\}$, where $A'\in\square(L-nL)$ and $x'\in nY$, or

2° $F(A\times\{x\}) = V'\times V''$, where $V'\in\square(nL-n_2L)$ and $V''\in\square(Y-nY)$.

Let us consider case 1° first.

We prove that

(a) if $B\in\square(K-nK)$ and $\bar{A}\cap\bar{B}\neq\emptyset$ then $F(B\times\{x\}) = B'\times\{x'\}$, where $B'\in\square(L-nL)$.

Two cases are possible: either $F(B\times\{x\}) = B'\times\{x'\}$ or $F(B\times\{x\}) = U'\times U''$, where $U'\in\square(nL-n_2L)$ and $U''\in\square(Y-nY)$. Suppose that the second condition holds. Then $F(\bar{A}\times\{x\})\cap F(\bar{B}\times\{x\})\neq\emptyset$ because $\bar{A}\cap\bar{B}\neq\emptyset$. So there exist $u'\in\bar{A}'\cap\bar{U}'$ and $x'\in\bar{U}''$. Choose $a'\in A'$ and $u''\in U''$. There exist arcs $a'u'\subset A'\cup\{u'\}$ and $x'u''\subset U''\cup\{x'\}$. Therefore, there exists an arc $(a',x')(u',u'')$ whose interior lies in $A'\times U''$ and therefore is disjoint from $n(L\times Y)$. Then $F^{-1}(a',x')\in A\times\{x\}$ and $F^{-1}(u',u'')\in\bar{B}\times\{x\}-\bar{A}\times\{x\}$ because $(a',x')\in A'\times\{x'\}$, $(u',u'')\in\bar{U}'\times\bar{U}''$ and $(u',u'')\notin\bar{A}'\times\{x'\}$. Therefore, the interior of any arc which joins $F^{-1}(a',x')$ to $F^{-1}(u',u'')$ in $K\times X$ meets $n(K\times X)$. However, the interior of the arc $F^{-1}\big((a',x')(u',u'')\big)$ lies in $F^{-1}(A'\times U'')$ and so is disjoint from $n(K\times X)$. Hence, $F(B\times\{x\}) = B'\times\{x'\}$.

Observe that

(b) if $Z\in\square nK$ and $Z\cap\bar{A}\neq\emptyset$ then $F(Z\times\{x\}) = Z'\times\{x'\}$, where $Z'\in\square nL$.

Indeed, we know that $Z\times X\approx Z'\times Y$ and Z, X, Z', Y are 1-polyhedra. So as in [1], either $F(Z\times\{x\}) = Z'\times\{x'\}$ or $F(Z\times\{x\}) = \{z'\}\times Y$. It is enough to show that if $C\in\square(nK-n_2K)$ and $C\cap\bar{A}\neq\emptyset$ then $F(C\times\{x\}) = C'\times\{x'\}$, where $C'\in\square(nL-n_2L)$.

Observe that we have either $F(C\times\{x\}) = C'\times\{x'\}$ or $F(C\times\{x\}) = \{z'\}\times C'$. If $C\cap\bar{A}$ consists of more than one point then so does $F(C\times\{x\})\cap F(\bar{A}\times\{x\})$. But $F(\bar{A}\times\{x\}) = \bar{A}'\times\{x'\}$ so $F(C\times\{x\}) = C'\times\{x'\}$. Suppose $C\cap\bar{A} = \{v\}$ and $F(C\times\{x\}) = \{z'\}\times C'$. Then $\bar{A}'\times\{x'\}\cap\{z'\}\times C' = \{(z',x')\}$. We have $\{z'\}\times C'\subset\overline{A'\times C'}$ because $z'\in\bar{A}'$. But $C\times\{x\}\subset\overline{A\times V}$ for any $V\in\square(X-nX)$.

Also, observe that

(c) if K has local cut points then $F(B\times\{x\}) = U'\times U''$ for no $B\in\square(K-nK)$.

To see this, let as before DK, DL denote the sets of local cut points of K and L, respectively. Then $F(DK\times X) = DL\times Y$. Suppose $\bar{B}\cap DK\neq\emptyset$ and $F(B\times\{x\}) = U'\times U''$. Then either $\dim(\bar{U}'\times\bar{U}'')\cap(DL\times Y) = 1$ or $(\bar{U}'\times\bar{U}'')\cap(DL\times Y) = \emptyset$, which is impossible. The polyhedron K is connected and the condition (a) holds, so (c) holds too.

The conditions (a), (b), (c) yield

(d) if $F(A\times\{x\}) = A'\times\{x'\}$ for some $A\in\square(K-nK)$ then $F(K\times\{x\}) = L\times\{x'\}$.

So $K \approx L$.

We need to prove that in case 1° we have $X \approx Y$.

The sets nX and nY have the same number of points because $F(K \times nX) = L \times nY$. Let $F(K \times \{x_i\}) = L \times \{y_i\}$. The components of $X - nX$ and $Y - nY$ are open arcs. If $I \in \square(X - nX)$ then $F(K \times I) = L \times I'$, where $I' \in \square(Y - nY)$. Observe that $x_i \in \bar{I}$ iff $y_i \in \bar{I}'$. So $X \approx Y$.

Hence, in case 1° condition (i) holds.

Now we consider case 2°: $F(A \times \{x\}) = V' \times V''$. Then for every $B \in \square(K - nK)$ we have $F(B \times \{x\}) = U' \times U''$, where $U' \in \square(nL - n_2 L)$, $U'' \in \square(Y - nY)$. We know that K has no local cut points by (c).

For components of $nL - n_2 L$ we define a relation: $W' \sim U'$ iff there exists a sequence $W' = W_0', W_1', \ldots, W_n' = U'$ of components of $nL - n_2 L$ such that for all $i = 0, 1, \ldots, k-1$, $\bar{W}_i' \cap \bar{W}_{i+1}' \neq \emptyset$ and if $W_i' \neq W_{i+1}'$ then no component C' of $L - nL$ satisfies $W_i' \cap \bar{C}' \neq \emptyset$ and $W_{i+1}' \cap \bar{C}' \neq \emptyset$.

We define a 1-polyhedron P by $P = \bigcup \{\bar{W}' \colon W' \in \square(nL - n_2 L) \wedge W' \sim V'\}$. We will prove that $F(K \times \{x\}) = P \times Y$.

First we note that $F(K \times \{x\}) \subset P \times Y$. Indeed, it is enough to observe that if $\dim \bar{A} \cap \bar{B} = 1$ then $F(B \times \{x\}) = U' \times U'' \subset P \times Y$ because K is connected and has no local cut points.

If $U' = V'$ then $U' \times U'' \subset P \times Y$ because $V' \times V'' \subset P \times Y$.

Suppose $U' \neq V'$. Then $V'' = U''$ because $\dim ((\bar{V}' \times \bar{V}'') \cap (\bar{U}' \times U'')) = 1$. Suppose that there exists a component C' of $L - nL$ such that $V' \cap \bar{C}' \neq \emptyset$ and $U' \cap \bar{C}' \neq \emptyset$. Let $a' \in V' \cap \bar{C}'$, $b' \in U' \cap \bar{C}'$ and $x'' \in U'' = V''$. There exists an arc $a'b'$ whose interior lies in C'. (The interior of an arc ab will be denoted by (ab).)

Observe that $(a'b') \times \{x''\} \subset C' \times U'' \subset L \times Y - n(L \times Y)$. Hence, $F^{-1}((a'b') \times \{x''\}) \subset K \times X - n(K \times X)$. The ends of the arc $F^{-1}(a'b' \times \{x''\})$ lie in $A \times \{x\}$ and $B \times \{x\}$, respectively. This is impossible because the interior of any arc in $K \times X$ with end-points $F^{-1}(a', x'')$, $F^{-1}(b', x'')$ meets $n(K \times X)$. So $U' \sim V'$. Therefore, $F(B \times \{x\}) \subset P \times Y$ and $F(K \times \{x\}) \subset P \times Y$.

Now, we prove that $P \times Y \subset F(K \times \{x\})$.

We know that $F^{-1}(V' \times V'') = A \times \{x\}$. Let us investigate where the set $F^{-1}(U' \times U'')$ lies if $\bar{V}' \cap \bar{U}' \neq \emptyset$, $V' \sim U'$ and $V'' = U''$. Suppose $F^{-1}(U' \times U'') = U_1 \times U_2$, where $U_1 \in \square(nK - n_2 K)$, $U_2 \in \square(X - nX)$. The set $U_1 \cap \bar{A}$ is not empty and $x \in \bar{U}_2$ because $\dim ((\bar{A} \times \{x\}) \cap (\bar{U}'_1 \times \bar{U}_2)) = 1$. Therefore, there exist $(a, x) \in A \times \{x\}$, $(u_1, u_2) \in U_1 \times U_2$ and an arc $(a, x)(u_1, u_2)$ such that $((a, x)(u_1, u_2)) \subset A \times U_2 \subset K \times X - n(K \times X)$. The points $F(a, x)$ and $F(u_1, u_2)$ lie in $V' \times V''$ and $U' \times U''$, respectively. The interior of any arc in $L \times Y$ with end-points $F(a, x)$ and $F(u_1, u_2)$ meets $n(L \times Y)$ because $V' \sim U'$ and the projection of this arc on the first factor meets nL. But $F(((a, x)(u_1, u_2))) \subset L \times Y - n(L \times Y)$. Therefore, $F^{-1}(U' \times U'') = B \times \{x\}$.

Similarly we prove $F^{-1}(U' \times U'') = B \times \{x\}$ in the case $U' = V'$ and $\bar{V}'' \cap \bar{U}'' \neq \emptyset$.

By a finite number of steps as above we can show that $F^{-1}(W' \times W'') = B \times \{x\}$ for any $W' \in \square(nL - n_2 L)$ such that $W' \sim V'$ and any $W'' \in \square(Y - nY)$. Hence, $F^{-1}(P \times Y) \subset K \times \{x\}$ because $P = \bigcup \{\bar{W}' \colon W' \sim V'\}$.

We have proved $F(K \times \{x\}) = P \times Y$, so $K \approx P \times Y$.

If $nY = \emptyset$ then $L \approx P \times X$ by Lemma 3.3.

Now, we assume $nY \neq \emptyset$ and consider the homeomorphism $F^{-1} \colon L \times Y \to K \times X$. Let $A' \in \square(L - nL)$ and $x' \in nY$. Then two cases are possible. The first: $F^{-1}(A' \times \{x'\}) = A \times \{x\}$ for some $A \in \square(K - nK)$ and $x \in nX$. Hence, $K \approx L$ by 1°. The second: $F^{-1}(A' \times \{x'\}) = V_1 \times V_2$, where $V_1 \in \square(nK - n_2 K)$, $V_2 \in \square(X - nX)$. Hence, $L \approx P_1 \times X$. We have $(P \times Y) \times X \approx K \times X \approx L \times Y \approx (P_1 \times X) \times Y$. The 1-polyhedra P and P_1 are homeomorphic because decomposition into Cartesian product of 1-polyhedra is unique.

So, in case 2° condition (ii) holds. ■

Corollary. *If $nX = \emptyset$ then either*

(i) *$K \approx L$ and $X \approx Y$, or*

(ii) *there exists a 1-polyhedron P such that $K \approx P \times Y$ and $L \approx P \times X$, or*

(iii) *$X \approx Y$.*

Proof. If $nY \neq \emptyset$, we consider the homeomorphism $F^{-1} \colon L \times Y \to K \times X$ and (ii) holds by Lemma 3.4. If $nY = \emptyset$ then either $X \approx Y$ or (ii) holds by Lemma 3.2. ■

Now we prove

Lemma 3.5. *If K, L are 2-polyhedra and $K \times S^1 \approx L \times S^1$ then $K \approx L$.*

Proof. We have assumed K, L to be compact and connected. The lemma is an insignificant generalization of Proposition 4.2 from [4]. Notice that the assumptions (*) and (**) of that proposition are not essential.

If $F \colon K \times S^1 \to L \times S^1$ is a homeomorphism then $F(nK \times S^1) = nL \times S^1$. Hence, $nK \approx nL$. Moreover, $F(n_2 K \times S^1) = n_2 L \times S^1$. If $D_1 K$ and $D_1 L$ are the sets of local cut points of K and L, respectively, then $F(D_1 K \times S^1) = D_1 L \times S^1$. Let $D_2 K$ and $D_2 L$ denote the sets of those points of nK and nL, respectively, whose regular neighborhoods are not homeomorphic to $(\mathrm{cone}\{1, \ldots, n\}) \times I$, for any $n \in N$. Then also $F(D_2 K \times S^1) = D_2 L \times S^1$. Observe that if $D_3 K$ is the set of points of nK which have an open neighborhood in nK homeomorphic to $[0, 1)$ then $F(D_3 K \times S^1) = D_3 L \times S^1$, where $D_3 L$ is the analogous subset of nL.

We can construct a homeomorphism $f \colon nK \to nL$ such that $f(n_2 K) = n_2 L$ and $f(D_i K) = D_i L$ for $i = 1, 2, 3$. Moreover, if for a component U of $n_2 K$ or of $D_i K$ $(i = 1, 2, 3)$ we have $F(U \times S^1) = U' \times S^1$, where U' is component of $n_2 L$ or of $D_i L$, then $f(U) = U'$.

This means that we can divide nK into arcs ab, simple closed curves S and isolated points c and divide nL into arcs $a'b'$, simple closed curves S' and isolated points c' such that $f(ab) = a'b'$, $f(S) = S'$ and $f(c) = c'$ iff $F(ab \times S^1) = a'b' \times S^1$, $F(S \times S^1) = S' \times S^1$ and $F(\{c\} \times S^1) = \{c'\} \times S^1$. Moreover, if $F(ab \times S^1) = a'b' \times S^1$ and $F(\{a\} \times S^1) = \{a'\} \times S^1$ then $f(a) = a'$.

We want to extend the homeomorphism f to a homeomorphism $\bar{f} \colon K \to L$.

By Property 2.5, if $F(A \times S^1) = A' \times S^1$, where $A \in \square(K - nK)$, $A' \in \square(L - nL)$, then there exists a homeomorphism $F_A \colon M(A) \times S^1 \to M(A') \times S^1$ such that the diagram

$$\begin{array}{ccc} M(A) \times S^1 & \xrightarrow{F_A} & M(A') \times S^1 \\ {\scriptstyle g_A \times \mathrm{id}} \downarrow & & \downarrow {\scriptstyle g_{A'} \times \mathrm{id}} \\ \bar{A} \times S^1 & \xrightarrow{F|_{\bar{A} \times S^1}} & \bar{A} \times S^1 \end{array}$$

commutes. ($M(A)$, $M(A')$ are given by Definition 2.2 and g_A, $g_{A'}$ are given by Definition 2.3.)

Of course $M(A) \approx M(A')$.

We can subdivide the polyhedron $g_A^{-1}(nK \cap \bar{A})$ so that for every 1-simplex I the map $g_A|_I$ is a homeomorphism (Property 2.4). The components of $g_A^{-1}(ab)$, where ab is an arc as above, are such 1-simplexes. So, we can assume $F_A(I \times S^1) = I' \times S^1$.

We define a homeomorphism $f_A\colon g_A^{-1}(nK \cap \bar{A}) \to g_{A'}^{-1}(nL \cap \bar{A}')$ by

$f_A(x) = (g_{A'}|_{I'})^{-1} \circ f \circ g_A(x)$ for $x \in I$ and

$f_A(x) = x'$ iff $F_A(\{x\} \times S^1) = \{x'\} \times S^1$ and x is an isolated point.

As in the proof of Proposition 4.1 of [4], we first extend f_A to $\partial M(A)$ and next to $M(A)$ to obtain a homeomorphism $\tilde{f}_A\colon M(A) \to M(A')$.

The required homeomorphism $\tilde{f}\colon K \to L$ is defined by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in nK, \\ g_{A'} \circ \tilde{f}_A(g_A^{-1}(x)) & \text{for } x \in A. \ \blacksquare \end{cases}$$

Lemmas 3.2, 3.4, 3.5 immediately imply Theorem 3.1.

Corollary. *If X, Y, Z are 1-polyhedra, K is a 2-polyhedron and $X \times Y \times Z \approx K \times Z$ then $X \times Y \approx K$.*

Remark. By Kosiński's theorem [3], every 2-dimensional topological divisor of a polyhedron is a polyhedron. So, we have considered all possible decompositions of a 3-polyhedron and Theorem 3.1 can be formulated a bit more generally: *A 3-polyhedron can be represented as a Cartesian product in two different ways only if the interval is its topological divisor.*

4. Conditions for $K \times I \approx L \times I$.

Theorem 4.1. *If K and L are compact 2-polyhedra then $K \times I \approx L \times I$ iff the following conditions hold:*

(1) *There exists a homeomorphism $h\colon nK \to nL$.*

(2) *There exists a one-to-one correspondence $A \leftrightarrow A'$ between components of $K - nK$ and of $L - nL$ such that: the manifolds $M(A)$ and $M(A')$ are either both orientable or both nonorientable, their first Betti numbers ($\beta(M(A)) = \operatorname{rank} H_1(M(A))$) are equal; if $g_A(\partial M(A)) \subset nK$ then $M(A) \approx M(A')$; and*

(3) *There exist homeomorphisms $h_A\colon g_A^{-1}(nK \cap \bar{A}) \to g_{A'}^{-1}(nL \cap \bar{A}')$, for every $A \in \square(K - nK)$, such that:*

(a) *The diagrams*

$$(A) \qquad \begin{array}{ccc} g_A^{-1}(nK \cap \bar{A}) & \xrightarrow{h_A} & g_{A'}^{-1}(nL \cap \bar{A}') \\ \downarrow{\scriptstyle g_A|_{g_A^{-1}(nK\cap\bar{A})}} & & \downarrow{\scriptstyle g_{A'}|_{g_{A'}^{-1}(nL\cap\bar{A}')}} \\ nK & \xrightarrow{h} & nL \end{array}$$

commute for every A.

(b) *If we choose suitable orientations of the orientable manifolds $M(A)$ and $M(A')$ and orientations of the boundaries of the nonorientable ones then all h_A preserve the induced orientations.*

(c) $h_A(g_A^{-1}(nK \cap \bar{A}) \cap \operatorname{int} M(A)) = g_{A'}^{-1}(nL \cap \bar{A}') \cap \operatorname{int} M(A')$.

Proof. First we show that if $K \times I \approx L \times I$ then the above conditions hold.

The first condition is obvious: From Lemma 2.1, $n(K \times I) = nK \times I$ and $n(L \times I) = nL \times I$. So, if $F\colon K \times I \to L \times I$ is a homeomorphism then $F(nK \times I) = nL \times I$. Hence, $nK \approx nL$ by Borsuk's theorem [1].

Before we prove the second condition, we observe that for every $A \in \square(K - nK)$ there exists exactly one $A' \in \square(L - nL)$ such that $F(A \times I) = A' \times I$. We recall that there exists a homeomorphism $F_A\colon M(A) \times I \to M(A') \times I$. So, either both $M(A)$ and $M(A')$ are orientable or both are nonorientable. The first Betti numbers of $M(A)$ and $M(A')$ are equal because $H_1(M(A)) \approx H_1(M(A) \times I) \approx H_1(M(A') \times I) \approx H_1(M(A'))$.

If $g_A(\partial M(A)) \subset nK$ then $F_A(\partial M(A) \times I) = \partial M(A') \times I$ because the diagram

$$(*) \qquad \begin{array}{ccc} M(A) \times I & \xrightarrow{F_A} & M(A') \times I \\ \downarrow{\scriptstyle g_A \times \mathrm{id}} & & \downarrow{\scriptstyle g_{A'} \times \mathrm{id}} \\ \bar{A} \times I & \xrightarrow{F|_{\bar{A} \times I}} & \bar{A}' \times I \end{array}$$

commutes. Indeed, $\partial M(A) \times I$ is the union of those components of $(g_A \times \mathrm{id})^{-1}((nK \cap \bar{A}) \times I)$ which are not intervals, and similarly for $\partial M(A') \times I$. Hence, $H_1(M(A), \partial M(A)) \approx H_1(M(A) \times I, \partial M(A) \times I) \approx H_1(M(A') \times I, \partial M(A') \times I) \approx H_1(M(A'), \partial M(A'))$. Also, since $H_1(M(A)) \approx H_1(M(A'))$ and both manifolds are orientable or both nonorientable we have $M(A) \approx M(A')$.

Now, we prove the third condition. Since $F((nK \cap \bar{A}) \times I) = (nL \cap \bar{A}') \times I$ and $nK \cap \bar{A}$, $nL \cap \bar{A}'$ are 1-polyhedra, it follows that $nK \cap \bar{A} \approx nL \cap \bar{A}'$. Set $A_1 = g_A^{-1}(nK \cap \bar{A})$ and $A_1' = g_{A'}^{-1}(nL \cap \bar{A}')$. Since the diagram $(*)$ commutes, so does the diagram

$$(**) \qquad \begin{array}{ccc} A_1 \times I & \xrightarrow{F_A|_{A_1 \times I}} & A_1' \times I \\ \downarrow{\scriptstyle g_A|_{A_1} \times \mathrm{id}} & & \downarrow{\scriptstyle g_{A'}|_{A_1'} \times \mathrm{id}} \\ (nK \cap \bar{A}) \times I & \xrightarrow{F|_{(nK \cap \bar{A}) \times I}} & (nL \cap \bar{A}') \times I \end{array}$$

Therefore $A_1 \approx A_1'$.

By Property 2.4 we can assume that g_A and $g_{A'}$ are homeomorphisms on every 1-simplex of A_1 and A_1'. So, we can choose homeomorphisms $h\colon nK \to nL$ and $h_A\colon A_1 \to A_1'$ in such a way that the diagram (A) commutes.

Part (b) of condition (3) is also easy to see: If a homeomorphism h_A preserves orientation on some 1-simplex and reverses it on another one then identifying these simplexes we obtain 2-manifolds whose Cartesian products with an interval are not homeomorphic.

The last part of condition (3) follows from the facts that $F_A(\operatorname{int}(M(A) \times I)) = \operatorname{int}(M(A') \times I)$ and the diagrams $(*)$ and (A) commute.

Of course, conditions (1)–(3) do not imply $K \approx L$. However, we will prove that they imply $K \times I \approx L \times I$.

We have to extend the homeomorphism $h \times \mathrm{id}\colon nK \times I \to nL \times I$ to a homeomorphism $F\colon K \times I \to L \times I$.

By (2), $M(A)\times I \approx M(A')\times I$. Indeed, if $\beta(M(A)) = \beta(M(A')) = n$ then the manifolds $M(A)$ and $M(A')$ are constructed from a disc D by identifying n pairs of disjoint arcs lying in ∂D. If $M(A)$, $M(A')$ are orientable we identify all arcs with the inverse orientation, otherwise with the same orientation. The manifolds $M(A)\times I$ and $M(A')\times I$ are obtained from the ball $D\times I$ by identifying n disjoint pairs of discs lying in $\partial(D\times I)$. The result of the first identification depends on the order of the arcs on the simple closed curve ∂D. On the contrary, the result of the second identification does not depend on the position of the discs on $\partial(D\times I)$.

The set $A_1 = g_A^{-1}(nK \cap \bar{A})$ consists of disjoint points, arcs and simple closed curves lying in $\partial M(A)$ and isolated points lying in $\operatorname{int} M(A)$, and similarly for A_1'. Since $h_A(A_1 \cap \operatorname{int} M(A)) = A_1' \cap \operatorname{int} M(A')$ and the interior of a manifold is homogeneous, we do not have to consider the interior points.

If $g_A(\partial M(A)) \subset nK$ and $g_{A'}(\partial M(A')) \subset nL$ then we can extend $h\colon nK \to nL$ to a homeomorphism $\tilde{h}_A\colon nK \cup A \to nL \cup A'$ because $M(A) \approx M(A')$ and the diagram (A) commutes. So, we can extend $h\times \mathrm{id}$ to a homeomorphism $\tilde{h}_A \times \mathrm{id}\colon (nK \cup A)\times I \to (nL \cup A')\times I$.

Now, we can assume that not all components of $\partial M(A)$ are contained in A_1. The sets A_1 and A_1' are homeomorphic. However, we cannot extend h to A because the manifolds $M(A)$ and $M(A')$ need not be homeomorphic and the components of A_1 and A_1' can have different positions on $\partial M(A)$ and $\partial M(A')$.

Let $S_1, \ldots, S_k$ be all components of $\partial M(A)$ contained in A_1. We can represent $M(A)$ as follows: Take a disc D with k holes. Let $T_1, \ldots, T_k, T_{k+1}$ denote the components of the boundary. If $\beta(M(A)) = n$ we choose $n-k$ disjoint pairs of arcs I_i, J_i on T_{k+1} and $n-k$ homeomorphisms $\varphi_i\colon I_i \to J_i$. If $M(A)$ is orientable the homeomorphisms φ_i are assumed to reverse orientation, otherwise they should preserve orientation. Then the quotient space $D/\sim$, where

$$x \sim y \text{ iff } x = y \text{ or } \varphi_i(x) = y \text{ or } x = \varphi_i(y) \text{ for some } i = 1, \ldots, n-k,$$

is homeomorphic to $M(A)$.

Let $p_A\colon D \to M(A)$ be the quotient map. Then $p_A(T_i) = S_i$ for $i = 1, \ldots, k$. The intersection of T_{k+1} with $A_2 = p_A^{-1}(A_1) = p_A^{-1} g_A^{-1}(nK \cap \bar{A})$ consists of disjoint arcs and points. We can assume that A_2 is disjoint from the union of the arcs I_i and J_i, $i = 1, \ldots, n-k$.

Analogously we represent the manifold $M(A')$ as a disc D' with k holes where we identify $n-k$ pairs of disjoint arcs lying all in one component of the boundary. Let $p_{A'}\colon D' \to M(A')$ denote the quotient map.

Since $A_2 \cap \bigcup_{i=1}^{n-k} (I_i \cup J_i) = \emptyset$ the map $p_A|_{A_2}$ is a homeomorphism.

Similarly $p_{A'}|_{A_2'}$ is a homeomorphism, where $A_2' = p_{A'}^{-1}(A_1')$. Since the diagram (A) commutes, so does the diagram

$$\begin{array}{ccc} A_2 & \xrightarrow{\;h_A' = (p_{A'}|_{A_2'})^{-1} \circ h_A \circ p_A|_{A_2}\;} & A_2' \\ {\scriptstyle g_A \circ p_A|_{A_2}} \downarrow & & \downarrow {\scriptstyle g_{A'} \circ p_{A'}|_{A_2'}} \\ nK & \xrightarrow{\quad h \quad} & nL \end{array}$$

We have $h_A'(T_i) = T_i'$, where the T_i' are components of $\partial D'$, $i = 1, \ldots, k$, and $h_A'(A_2 \cap T_{k+1}) = A_2' \cap T_{k+1}'$.

The homeomorphism h_A' need not extend to T_{k+1} to a homeomorphism $\tilde{h}_A'$ such that $\tilde{h}_A'(I_i) = I_i'$ and $\tilde{h}_A'(J_i) = J_i'$ because $(A_2 \cap T_{k+1}) \cup \bigcup_{i=1}^{n-k} (I_i \cup J_i)$ and $(A_2' \cap T_{k+1}') \cup \bigcup_{i=1}^{n-k} (I_i' \cup J_i')$ can have different positions on T_{k+1} and T_{k+1}'. However, $\partial(D\times I) \approx \partial(D'\times I)$ is a closed 2-manifold, so $[A_2 \cup \bigcup_{i=1}^{n-k} (I_i \cup J_i)]\times I$ and $[A_2' \cup \bigcup_{i=1}^{n-k} I_i' \cup J_i')]\times I$ are in the same position on it. The homeomorphism $h_A' \times \mathrm{id}$ can be extended to a homeomorphism F_A'; $D\times I \to D'\times I$. We can, additionally, require that $F_A'(I_i \times I) = I_i' \times I$ and $F_A'(J_i\times I) = J_i' \times I$ for $i = 1, \ldots, n-k$. Moreover, we can require:

$\varphi_i(x) = y$ iff $(\varphi_i' \times \mathrm{id}) \circ F_A'(x, t) = F_A'(y, t)$ for $x \in I_i$ and

$g_A p_A(x) = g_A p_A(y)$ iff $(g_{A'} p_{A'} \times \mathrm{id})(F'(x, t)) = (g_{A'} p_{A'} \times \mathrm{id})(F'(y, t))$ for $x, y \in A_2$.

Now we can define a homeomorphism $F_A\colon M(A)\times I \to M(A')\times I$ by $F_A(x, t) = (p_{A'}, \times \mathrm{id})(F_A'(p_A^{-1}(x), t))$.

The desired homeomorphism $F\colon K\times I \to L\times I$ is defined by

$$F(x, t) = \begin{cases} (h(x), t) & \text{for } x \in nK, \\ (g_{A'} \times \mathrm{id})(F_A(g_A^{-1}(x), t) & \text{for } x \in \bar{A}, \text{ when } g_A(\partial M(A)) \not\subset nK, \\ (\tilde{h}_A(x), t) & \text{for } x \in \bar{A}, \text{ when } g_A(\partial M(A)) \subset nK. \end{cases}$$

This completes the proof. ■

References

[1] K. Borsuk, *Sur la décomposition des polyèdres en produits cartésiens*, Fund. Math. 31 (1983), 137–148.

[2] R. H. Fox, *On a problem of S. Ulam concerning Cartesian products*, Fund. Math. 34 (1947), 278–287.

[3] A. Kosiński, *On 2-dimensional topological divisors of polytopes*, Bull. Acad. Polon. Sci. 2 (1957), 325–328.

[4] W. Rosicki, *On a problem of S. Ulam concerning Cartesian squares of 2-dimensional polyhedra*, Fund. Math. 127 (1986), 101–125.

[5] J. H. C. Whitehead, *On the homotopy type of manifolds*, Ann. of Math. 41 (1940), 825–832.

INSTYTUT MATEMATYCZNY
UNIWERSYTET GDAŃSKI
INSTITUTE OF MATHEMATICS
GDAŃSK UNIVERSITY
Wita Stwosza 57
80-952 Gdańsk
Poland

Received 19 January 1989;
in revised form 24 April 1989