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On the additivity of the fixed point property for 1-dimensional continua

by

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Abstract. Two rational arcwise connected continua X_0 and Y_0 with the fixed point property are constructed such that $X_0 \cap Y_0$ is contractible and $X_0 \cup Y_0$ does not have the fixed point property. The problem of the additivity of the fixed point property for 1-dimensional continua is summarized in the remark at the end of the paper.

1. Introduction. It is known that if X and Y are 1-dimensional continua with the fixed point property and $X \cap Y$ is an AR, then $X \cup Y$ has the fixed point property ([6], p. 1292). The aim of the present paper is to show that, roughly speaking, nothing more can be proved on the additivity of the fixed point property, for non-planar 1-dimensional continua, answering simultaneously a problem raised in [5] (p. 237).

To formulate the main result of the paper precisely, recall that a continuum X is said to be *rational* if X has a base of neighbourhoods with countable boundaries, and *arcwise connected* if any two points of X can be joined by an arc in X. Any homeomorphic image of a cone over a convergent sequence of points together with its limit is called a *harmonic brush*; such a brush is contractible and has the fixed point property (see, for instance, [1], p. 20). The main result will be the following

THEOREM. *There exist two rational arcwise connected continua* X_0 *and* Y_0 *with the fixed point property such that* $X_0 \cap Y_0$ *is a straightline harmonic brush and* $X_0 \cup Y_0$ *does not have the fixed point property.*

The continua X_0 and Y_0 will be *uniquely arcwise connected*, i.e. for any two points of X_0 or Y_0 there will be exactly one arc between them, so that the results from [5] can be applied. All other topological notions used, but not defined in the present paper, can be found in [4].

The continua X_0 and Y_0 will be constructed almost wholly on the plane E^2 (except an arc of Y_0 lying in E^3 outside E^2). A basic role in their description will be played by certain geometrical functions on E^2, which we shall define now.

Namely, for the points

(1.1) $\quad a = (1, 1), \quad c = (0, 2) \quad$ and $\quad d_{2n-1} = (0, 2+2^{-n}), \qquad n = 1, 2, \ldots,$

we take into account the following functions, defined for all $p \in E^2$, $k = -1, 0, 1, 2, \ldots$ and $n = 1, 2, \ldots$:

(1.2) $$\psi(p) = a-(p-a),$$

(1.3) $$\varphi_k(p) = c+2^{-k}(p-c) \quad \text{and} \quad \varphi_k^{(n)}(p) = d_{2n-1}+2^{-k}(p-d_{2n-1}),$$

i.e. the rotation ψ about the point a through the angle π, and the homotheties φ_k and $\varphi_k^{(n)}$ with the ratio 2^{-k} and the fixed point c and d_{2n-1}, respectively. Note that

(1.4) $\varphi_k^{(n)}$ uniformly converges to φ_k for all $k = -1, 0, 1, 2, \ldots$

Throughout the paper, we will use (for the reader's convenience) the primed letters for the images of points or sets under the rotation (1.2).

2. The auxiliary cycle $C(A)$ and the spiral $S_{C(A)}$. The cycle $C(A)$ will be contained in the square with vertices

$$s = (2, 2), \quad c = (0, 2), \quad s' = (0, 0) \quad \text{and} \quad c' = (2, 0)$$

in such a way that the perpendicular sides $\overline{[c, s']}$ and $\overline{[c', s]}$ of this square will be included in $C(A)$, and its remaining two sides will be replaced by other continua A and A'. To describe these continua, we start with a continuum A_0 homeomorphic to the condensed sinusoid

$$\overline{[(0, 1), (0, -1)]} \cup \{(x, y) \in E^2\colon\ y = \sin(\pi/x) \text{ and } 0 < x \leqslant 1\}.$$

Namely, setting for $m = 1, 2, \ldots$

$$u_m = (1+2^{-m+1}, 2), \quad v_m = \left(1+2^{-m}, 2-\frac{1-2^{-m}}{4}\right) \quad \text{and} \quad V_m = \overline{[u_m, v_m]} \cup \overline{[v_m, u_{m+1}]}$$

let

$$A_0 = \overline{[(1, 2), (1, \tfrac{7}{4})]} \cup \bigcup_{m=1}^{\infty} V_m.$$

In general, set

(2.1) $$A_k = \varphi_k(A_0) \quad \text{for } k = 0, 1, 2, \ldots$$

and define

(2.2) $$A = \{c\} \cup \bigcup_{k=0}^{\infty} A_k, \qquad A' = \psi(A),$$

(2.3) $$C(A) = A \cup A' \cup \overline{[c, s']} \cup \overline{[c', s]}.$$

Now we will deal with continuous fixed point free functions in $C(A)$. First, denoting for $k = 1, 2, \ldots$

(2.4) $$a_k = (2^{-k+1}, 2), \quad b_k = \left(2^{-k+1}, 2-\frac{2^{-k+1}}{4}\right), \qquad a'_k = \psi(a_k), \quad b'_k = \psi(b_k),$$

(so that $\overline{[a_k, b_k]}$ is the segment of condensation, and a_{k-1} is the end point of the condensed sinusoid A_{k-1} for $k = 1, 2, \ldots$ ($a_0 = s$)), take the projection π_{-1} from the point c onto the straightline $\overline{a_1 b_1}$ which contains the segment $\overline{[a_1, b_1]}$. Define

(2.5) $$\gamma(p) = \begin{cases} \psi(p) & \text{for } p \in \overline{[c, s']} \cup A_0, \\ \psi(\pi_{-1}(p)) & \text{for } p \in A_1, \\ \psi(\varphi_{-1}(p)) & \text{for } p \in \bigcup_{k=2}^{\infty} A_k, \\ s & \text{for } p \in A'_0, \\ \varphi_{-1}(\psi(p)) & \text{for } p \in \bigcup_{k=1}^{\infty} A'_k, \\ \psi(p) & \text{for } p \in \overline{[c', s]}. \end{cases}$$

Lemma 1. *The function $\gamma\colon C(A) \to C(A)$ is continuous, has no fixed point and its values at the end points of the arc components of $C(A)$ are as follows:*

(2.6) $$\gamma(c) = c', \quad \gamma(c') = c,$$

(2.7) $$\gamma(b_1) = b'_1, \quad \gamma(b'_1) = s,$$
$$\gamma(b_k) = b'_{k-1} \quad \textit{and} \quad \gamma(b'_k) = b_{k-1} \quad \textit{for } k = 2, 3, \ldots$$

Proof. The function γ is continuous on each of the six parts of $C(A)$ indicated in (2.5) and its definitions on any two of these parts agree on their intersection. As each of the above parts is disjoint from its image under γ, the function γ has no fixed point. The equalities (2.6) and (2.7) follow directly from the definition (2.5) of the function γ.

Lemma 2. *If $f(C(A)) = C(A)$ is a fixed point-free continuous function, then there is a k_1 such that*

$$f(\{b_k\colon k = 1, 2, \ldots\}) \cap A(b'_k) \neq \emptyset \quad \textit{for all } k > k_1,$$

where $A(b'_k)$ is the arc component of b'_k in $C(A)$.

Proof. The union $\bigcup_{k=1}^{\infty} \overline{[a_k, b_k]} \cup \overline{[a'_k, b'_k]}$ is contained in its image $f(\bigcup_{k=1}^{\infty} \overline{[a_k, b_k]}) \cup f(\bigcup_{k=1}^{\infty} \overline{[a'_k, b'_k]})$ (by [3], (3), p. 28). If $f(\bigcup_{k=1}^{\infty} \overline{[a'_k, b'_k]})$ contained infinitely many of the segments $\overline{[a'_k, b'_k]}$, which converges to $\{c'\}$ by construction, then we would have $f(c') = c'$ by the continuity of f — and this contradicts the assumption that f has no fixed point. Thus there is a k_1 such that $f(\bigcup_{k=1}^{\infty} \overline{[a_k, b_k]})$ contains $\bigcup_{k>k_1} \overline{[a'_k, b'_k]}$, from which Lemma 2 follows.

To define the spiral $S_{C(A)}$, we will describe now a line S such that

(2.8) $$S \underset{\text{top}}{=} [0, \infty), \quad \bar{S} = S \cup C(A) \quad \text{and} \quad S \cap C(A) = \emptyset,$$

which lies in the square determined by the vertices

$$s_1 = (\tfrac{5}{2}, \tfrac{5}{2}), \quad c_1 = (-\tfrac{1}{2}, \tfrac{5}{2}), \quad s_2 = (-\tfrac{1}{2}, -\tfrac{1}{2}), \quad c_2 = (\tfrac{5}{2}, -\tfrac{1}{2}).$$

For $n = 1, 2, \ldots$, denote

$$s_{2n-1} = (2+2^{-n}, 2+2^{-n}), \quad c_{2n-1} = (-2^{-n}, 2+2^{-n}) \quad \text{and}$$

$$s_{2n} = \psi(s_{2n-1}), \quad c_{2n} = \psi(c_{2n-1}).$$

Then we have

$$\text{(2.9)} \qquad \operatorname*{Lt}_n \overline{[c_{2n-1}, d_{2n-1}]} = \{c\} \quad \text{and} \quad \operatorname*{Lt}_n \overline{[c_{2n}, d_{2n}]} = \{c'\},$$

where d_{2n-1} are defined as in (1.1) and

$$d_{2n} = \psi(d_{2n-1})$$

for $n = 1, 2, \ldots$ Setting

$$u_1^{(n)} = s_{2n-1},$$

$$u_m^{(n)} = (1+2^{-m+1}+2^{-n}, 2+2^{-n}), \quad v_m^{(n)} = \left(1+2^{-m}+2^{-n-1}, 2-\frac{1-2^{-m}}{4}+2^{-n-1}\right),$$

$$V_m^{(n)} = \overline{[u_m^{(n)}, v_m^{(n)}]} \cup \overline{[v_m^{(n)}, u_{m+1}^{(n)}]},$$

for $n, m = 1, 2, \ldots$, let

$$\text{(2.10)} \qquad W_k^{(n)} = \varphi_k^{(n)}\Big(\bigcup_{m=1}^{n+k} V_m^{(n)}\Big)$$

for $k = 0, 1, 2, \ldots$ If we denote

$$e_k^{(n)} = \varphi_k^{(n)}(u_{n+1}^{(n)}), \quad a_{k+1}^{(n)} = \varphi_{k+1}^{(n)}(u_1^{(n)})$$

for $k = 0, 1, 2, \ldots$, $n = 1, 2, \ldots$, then each $W_k^{(n)}$ is the arc

$$W_k^{(n)} = [a_k^{(n)}, e_k^{(n)}]^\frown \quad \text{with } a_0^{(n)} = s_{2n-1} = u_1^{(n)}$$

and

$$\text{(2.11)} \qquad \operatorname*{Lt}_n W_k^{(n)} = A_k, \quad \operatorname*{Lt}_n \overline{[e_k^{(n)}, a_{k+1}^{(n)}]} = \{a_{k+1}\},$$

as well as $\operatorname*{Lt}_n \overline{[e_k^{(n)}, \varphi_k^{(n)}(v_n^{(n)})]} = \overline{[a_{k+1}, b_{k+1}]}$, which implies that

$$\text{(2.12)} \qquad \operatorname*{Lt}_n \overline{[e_0^{(n)}, v_n^{(n)}]} = [a_1, b_1].$$

Denoting

$$\text{(2.13)} \qquad A_k^{(n)} = W_k^{(n)} \cup \overline{[e_k^{(n)}, a_{k+1}^{(n)}]} \quad \text{for } n = 1, 2, \ldots, \ k = 0, 1, 2, \ldots$$

we define the arcs

$$\text{(2.14)} \qquad [s_{2n-1}, c_{2n-1}]^\frown = \bigcup_{k=0}^{\infty} A_k^{(n)} \cup \overline{[d_{2n-1}, c_{2n-1}]},$$

$$\text{(2.15)} \qquad [s_{2n}, c_{2n}]^\frown = \psi([s_{2n-1}, c_{2n-1}]^\frown).$$

Finally, we take

$$\text{(2.16)} \qquad S = \bigcup_{l=1}^{\infty} [s_l, c_l]^\frown \cup \overline{[c_l, s_{l+1}]},$$

$$\text{(2.17)} \qquad S_{C(A)} = S \cup C(A).$$

Now we shall describe a fixed point-free continuous function in $S_{C(A)}$. First, observe that if for $n = 1, 2, \ldots$ we take the translation τ_n determined by the vector from d_{2n-1} to d_{2n+1}, then the similarity $\chi_{-1}^{(n)} = \varphi_{-1}^{(n)} \circ \tau_n$ transforms $\bigcup_{k=1}^{\infty} A_k^{(n)}$ onto $\bigcup_{k=0}^{\infty} A_k^{(n+1)}$. Of course, each $\chi_{-1}^{(n)}$ is continuous and, in view of (1.3) and (2.9),

$$\text{(2.18)} \qquad \chi_{-1}^{(n)} \text{ converges uniformly to } \varphi_{-1} \text{ on } S_{C(A)}.$$

Now take the projection $\pi_{-1}^{(n)}$ from d_{2n-1} onto the straightline $\overline{e_0^{(n)} v_n^{(n)}}$. Then, in view of (2.5) and (2.12),

$$\text{(2.19)} \qquad \pi_{-1}^{(n)} \text{ converges to } \pi_{-1} \text{ uniformly on } S_{C(A)}.$$

Further, consider the projection $\pi^{(n)}$ from c onto the straightline $\overline{d_{2n+1}, c_{2n+1}}$. Then by (2.9) we have

$$\operatorname*{Lt}_n \pi^{(n)}(\overline{[d_{2n-1}, c_{2n-1}]}) = \{c\}.$$

Define now

$$\sigma^{(2n-1)}(p) = \begin{cases} \psi(p) & \text{for } p \in \overline{[c_{2n-1}, d_{2n-1}]} \cup W_0^{(n)}, \\ \psi(\pi_{-1}^{(n)}(p)) & \text{for } p \in \overline{[e_0^{(n)}, a_1^{(n)}]} \cup A_1^{(n)}, \\ \psi(\varphi_{-1}^{(n)}(p)) & \text{for } p \in \bigcup_{k=2}^{\infty} A_k^{(n)}, \end{cases}$$

$$\sigma^{(2n)}(p) = \begin{cases} s_{2n+1} & \text{for } p \in \psi(A_0^{(n)}), \\ \chi_{-1}^{(n)}(\psi(p)) & \text{for } p \in \psi\Big(\bigcup_{k=1}^{\infty} A_k^{(n)}\Big), \\ \pi^{(n)}(\psi(p)) & \text{for } p \in \overline{[d_{2n}, c_{2n}]} \end{cases}$$

for $n = 1, 2, \ldots$ Then each $\sigma^{(l)}$, $l = 1, 2, \ldots$, is continuous, fixed point-free and maps $[s_l, c_l]^\frown$ onto $[s_{l+1}, c_{l+1}]^\frown$ so that

$$\text{(2.20)} \qquad \sigma^{(l)}(s_l) = s_{l+1} \quad \text{and} \quad \sigma^{(l)}(c_l) = c_{l+1} \quad \text{for } l = 1, 2, \ldots$$

Comparing the above formulas defining the functions $\sigma^{(l)}$ with the formulas defining γ in (2,5), in view of (1.5), (2.11), (2.12), (2.18) and (2.19) we obtain the following (cf. [4], vol. II, p. 89, Remark 1).

LEMMA 3. *The function* $\sigma\colon S_{C(A)}\to S_{C(A)}$ *defined by the formula*

$$\sigma(p)=\begin{cases}\gamma(p) & \text{for } p\in C(A),\\ \sigma^{(l)}(p) & \text{for } p\in[s_l, c_l]^\frown,\ l=1,2,\ldots,\\ \psi^{(l)}(p) & \text{for } p\in\overline{[c_l, s_{l+1}]},\ l=1,2,\ldots,\end{cases}$$

where each $\psi^{(l)}$ *is a linear function from* $\overline{[c_l, s_{l+1}]}$ *onto* $\overline{[c_{l+1}, s_{l+2}]}$ *with* $\psi^{(l)}(c_l)=c_{l+1}$, *is continuous and has no fixed point. By* (2.6), (2.7) *and* (2.20), *the values of* σ *at the end points of the arc components of* $S_{C(A)}$ *are as follows*:

(2.21) $$\sigma(s_1)=s_2,$$

(2.22) $$\sigma(c)=c' \text{ and } \sigma(c')=c,$$

(2.23) $$\sigma(b_1)=b_1' \text{ and } \sigma(b_1')=s,$$

(2.24) $$\sigma(b_k)=b_{k-1}' \text{ and } \sigma(b_k')=b_{k-1} \quad \text{for } k=2,3,\ldots$$

3. The main continua X_0 **and** Y_0. Consider the straightline brush

(3.1) $$B=\overline{[a, c]}\cup\bigcup_{k=1}^{\infty}\overline{[a, b_k]}$$

and the following sets L_+ and L_-:

(3.2) $$L_+=\left\{(x, y)\in E^2\colon y>-\frac{x}{4}+\frac{3}{2}\right\},\qquad L_-=\left\{(x, y)\in E^2\colon y\leqslant -\frac{x}{4}+\frac{3}{2}\right\}.$$

Then

(3.3) $$B\cap L_+=\overline{[c, r)}\cup\bigcup_{k=1}^{\infty}\overline{[b_k, r_k)},\qquad B\cap L_-=\overline{[r, a]}\cup\bigcup_{k=1}^{\infty}\overline{[r_k, a]},$$

where r and r_k are determined in B by the straightline $L\colon y=-x/4+\frac{3}{2}$; i.e.

(3.4) $$\{r\}\cup\{r_k\in\overline{[a, b_k]}\colon k=1,2,\ldots\}=B\cap L.$$

LEMMA 4. *There exists in the triangle determined by* $\overline{[a, b_1]}$ *and* $\overline{[a, c]}$ *a rational arcwise connected continuum* $\tilde{B}$ *containing* B *such that* $\tilde{B}$ *has the fixed point property and*:

1° *for each convergent sequence* $p_i\in B$ *with* $p=\lim p_i\in\overline{[a, c]}$, $i=1,\ldots$ *there exist continua* $K_1, K_2,\ldots$ *such that*

(3.5) $$p,\ p_i\in K_i\subset\tilde{B}\quad \text{and}\quad \lim_i \operatorname{diam} K_i=0;$$

2° *there is a homeomorphism* h_0 *from* $\tilde{B}\cap L_+-\overline{[b_1, r_1)}$ *onto* $\tilde{B}-\{a\}$ *with*

(3.6) $$h_0(c)=c\quad \text{and}\quad h_0(b_k)=b_{k-1}\quad \text{for } k=2,3,\ldots$$

Proof. For $k=1,2,\ldots$, let $d_{k,j}\in\overline{[a, b_k]}$ be the end points of pairwise disjoint condensed sinusoids $S_{k,j}$ lying in the triangle determined by $\overline{[a, b_k]}$ and $\overline{[a, b_{k+1}]}$ and having segments $I_{k,j}$ of $\overline{[a, b_{k+1}]}$ as the segment of condensation for $j=1,2,\ldots,2^k-1$. Also, we can assume that $S_{k,j}\subset L_+$ for $j<(2^k-1)/2$ and $S_{k,j}\subset L_-$ for $j\geqslant(2^k-1)/2$ (cf. [4], vol. II, p. 247, Remark, where a continuum similar to $\overline{[a, c]}\cup\bigcup_{k=1}^{\infty}\bigcup_{j=1}^{2^k-1}S_{k,j}$ is presented).

We can assert that $d_{k,j}$ is the middle point of $I_{k-1,2j-1}$, $k=2,3,\ldots$, all the segments $I_{k,j}$ have the same length and

$$\operatorname{Ls}_k\Big(\bigcup_{j=1}^{2^k-1}S_{k,j}\Big)=\overline{[a, c]}=\operatorname{Ls}_{k,j}\{d_{k,j}\colon j=1,2,\ldots,2^k-1,\ k=1,2,\ldots\}.$$

Then the set

(3.7) $$\tilde{B}=B\cup\bigcup_{k=1}^{\infty}\bigcup_{j=1}^{2^k-1}S_{k,j}$$

is obviously a rational uniquely arcwise connected continuum satisfying conditions 1° and 2° of Lemma 4, and posessing the fixed point property (cf. [5], p. 233, Corollary 2(b)).

Define now the *main continua* as

(3.8) $$X_0=\psi(\tilde{B})$$

and

(3.9) $$Y_0=\tilde{B}\cup\psi(B)\cup[a, s_1]^\frown\cup S_{C(A)},$$

where $[a, s_1]^\frown$ is an arbitrary arc in E^3 having only its end points a and s_1 in common with the plane E^2.

Then $X_0\cap Y_0$ is the brush $\psi(B)=\overline{[a, c']}\cup\bigcup_{k=1}^{\infty}\overline{[a, b_k']}$, and X_0 has the fixed point property as a homeomorphic image of $\tilde{B}$.

Proof of the fixed point property for Y_0. The continuum Y_0 is uniquely arcwise connected, because $\tilde{B}\cup\psi(B)\cup[a, s_1]^\frown$ joins end points of all arc components of $S_{C(A)}$, which are indicated in (2.21)–(2.24), without producing any simple closed curve. Hence, supposing on the contrary that there is a fixed point-free continuous function $f\colon Y_0\to Y_0$, we have

(3.10) $$f(a)\in(a, s_1]^\frown\cup S$$

and $f(C(A))=C(A)$ (by [5], p. 231, Theorem 2). Hence from Lemma 2, in view of (3.10) and the unique arcwise connectedness of Y_0, it follows that $f(B)$ contains the brush $\underline{B}'=\psi(B)-\bigcup_{k=1}^{k_1}\overline{(a, b_k']}$:

(3.11) $$\underline{B}'\subset f(B).$$

Now let $\{r'_k\}$, $k \geq k_1+1$, be a sequence of points convergent to r':

(3.12) $$r'_k = \psi(r_k) \in B', \quad r' = \psi(r) = \lim_k r'_k \in B',$$

where r, r_k are taken from (3.4). Then for every $k \geq k_1+1$

(3.13) $$\operatorname{diam} L_k \geq \frac{1}{3}$$

for each continuum L_k such that $r', r'_k \in L_k \subset Y_0$. Hence from (3.11) and (3.12) we obtain a convergent sequence of points $p_i \in B$ such that $f(p_i) = r'_{k_i}$ for $i = 1, 2, \ldots$ Let $p = \lim_i p_i$.

Suppose that $p \notin \overline{[a, c]}$. Then there would be a locally connected subcontinuum M of B (namely a finite union of the segments defining B in (3.11)) containing almost all points p and p_i, $i = 1, 2, \ldots$, whose image $f(M)$ would not be locally connected, e.g. at the point r'.

Therefore $p \in \overline{[a, c]}$. By (3.5) and (3.13), this contradicts the continuity of f (cf. [4], vol. I, p. 207).

Proof of the fact that $X_0 \cup Y_0$ does not have the fixed point property. By (3.8) and (3.9), we have

(3.14) $$X_0 \cup Y_0 = T \cup S_{C(A)},$$

where

(3.15) $$T = \tilde{B} \cup \psi(\tilde{B}) \cup [a, s_1]^\frown.$$

Hence $T \cap S_{C(A)}$ is just the set of all end points of the arc components of $S_{C(A)}$:

(3.16) $$T \cap S_{C(A)} = \{s_1, c, c', b_k, b'_k\colon k = 1, 2, \ldots\}.$$

We now define a continuous fixed point-free function

$$\tau\colon T \to T \cup [s_1, s_2]^\frown$$

in such a way that τ will coincide with σ (cf. Lemma 3) on $T \cap S_{C(A)}$.

Let π_0 be the projection in the direction of the straight line L (cf. (3.2)–(3.4)) of the set $\tilde{B} \cap L_- \cup \psi(\tilde{B} \cap L_-)$ onto the segment $\overline{[r, r']} = \overline{[r, a]} \cup \overline{[a, r']}$. Further, let ϱ_0 be any homeomorphism of $\overline{[r, a]}$ and $\overline{[r', a]}$ onto the arc $[a, s_1]^\frown$ such that $\varrho_0(r) = a = \varrho_0(r')$, and $\varrho_0 \circ \pi_0$ agrees with the homeomorphism from Lemma 4 (2°), i.e. $\varrho_0(\pi_0(r)) = a = \varrho_0(\pi_0(r'))$, $\varrho_0(\pi_0(r_k)) = a = \varrho_0(\pi_0(r'_k))$ for $k = 1, 2, \ldots$, and

$$\varrho_0(\pi_0(a)) = s_1.$$

Define

(3.17) $$\tau(p) = \begin{cases} \psi(h_0(p)) & \text{for } p \in \tilde{B} \cap L_+ - \overline{[b_1, r_1)}, \\ h_0(\psi(p)) & \text{for } p \in \psi(\tilde{B} \cap L_+) - \overline{[b'_1, r'_1)}, \\ \varrho_0(\pi_0(p)) & \text{for } p \in \tilde{B} \cap L_- \cup \psi(\tilde{B} \cap L_-), \\ h_1(p) & \text{for } p \in \overline{[b_1, r_1)} \cup \overline{[b'_1, r'_1)} \cup [a, s_1]^\frown, \end{cases}$$

where h_1 is any homeomorphism of the union of pairwise disjoint arcs $\overline{[b_1, r_1)}$, $\overline{[b'_1, r'_1)}$ and $[a, s_1]^\frown$ onto the union of pairwise disjoint arcs $\overline{[b'_1, a)}$, $[s, a)^\frown$ and $[s_1, s_2]^\frown$ such that

$$h_1(b_1) = b'_1, \quad h_1(b'_1) = s \quad \text{and} \quad h_1(a) = s_1.$$

Then we have

(3.18) $$\tau(s_1) = s_2,$$

(3.19) $$\tau(b_1) = b'_1 \quad \text{and} \quad \tau(b'_1) = s$$

and, in view of (3.6) and (3.17),

(3.20) $$\tau(c) = c' \quad \text{and} \quad \tau(c') = c,$$

(3.21) $$\tau(b_k) = b'_{k-1} \quad \text{and} \quad \tau(b'_k) = b_{k-1} \quad \text{for } k = 2, 3, \ldots$$

Hence the function $\tau\colon T \to T \cup [s_1, s_2]^\frown$ has no fixed point in each of the four parts of T which are indicated in (3.17). Moreover, τ is continuous, because its definitions on any of the two parts of T agree on the intersection of the closures of these parts.

Now the function

$$f_0(p) = \begin{cases} \sigma(p) & \text{for } p \in S_{C(A)}, \\ \tau(p) & \text{for } p \in T, \end{cases}$$

transforms $X_0 \cup Y_0$ onto itself without having a fixed point (by Lemma 3 and (3.17)). Further, f_0 is continuous by (2.21)–(2.24) and (3.18)–(3.21) (cf. (3.16)).

Remark. Let us quote here the known examples of rational arcwise connected continua X_0 and Y_0 with the fixed point property such that $X_0 \cup Y_0$ does not have the fixed point property, although $X_0 \cap Y_0$ is a maximally simple non-locally connected continuum with the fixed point property:

(1) $X_0 \cap Y_0$ is a harmonic brush (as in the Theorem stated in the introduction),

(2) $X_0 \cap Y_0$ is a "Warsaw circle" (as in [5], p. 236),

(3) $X_0 \cap Y_0$ is a condensed sinusoid (as in [7], p. 156, Ex. 3).

The third example is planar, and the first two are in E^3; let us mention that, by a theorem of [2], it is not possible to exhibit such examples in the plane. However, the following problem remains open on the plane E^2: does the union $X \cup Y$ of two 1-dimensional continua $X, Y \subset E^2$ with the fixed point property such that $X \cap Y$ is arcwise connected have the fixed point property?

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The nonexistence of expansive homeomorphisms of dendroids

by

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Abstract. In this paper, we prove that no dendroid admits an expansive homeomorphism. Also, we show that no uniformly arcwise connected continuum admits an expansive homeomorphism.

0. Introduction. Let X be a compact metric space with metric d. A homeomorphism f of X is *expansive* if there exists $c > 0$ (called an expansive constant for f) such that $d(f^n(x), f^n(y)) \leqslant c$ for all integers n implies $x = y$. This property is important in the topological theory of dynamical systems.

It is well known that the Cantor set, the 2-adic solenoid and the 2-torus admit expansive homeomorphisms ([11]).

Also, Bryant, Jacobson and Utz proved that there exists no expansive homeomorphism on an arc and a circle (see [1] and [6]). By using those results, Kawamura showed that if X is a Peano continuum and X contains a free arc, then X admits no expansive homeomorphism ([7]). In [8], we showed that if X is a Peano continuum which contains a 1-dimensional open ANR then X does not admit an expansive homeomorphism. In particular, 1-dimensional compact ANRs admit no expansive homeomorphism. Also, Jacobson and Utz [6] asserted that the shift homeomorphism of the inverse limit of any continuous surjection of an arc is not an expansive homeomorphism (see [5] for a simple proof). The limit is a special type of arc-like continua and arc-like continua are tree-like. Naturally, the following problem arises: Is it true that no tree-like continuum admits an expansive homeomorphism?

The purpose of this paper is to prove that no dendroid (= arcwise connected tree-like continuum) admits an expansive homeomorphism, and no uniformly arcwise connected continuum admits an expansive homeomorphism.

1. Preliminaries. All spaces under consideration are assumed to be metric. A *continuum* is a compact connected nondegenerate space.

A continuum X is said to be *unicoherent* provided that if $X = A \cup B$ and A, B are subcontinua of X, the intersection $A \cap B$ is connected. A continuum X is *hereditarily*

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