§ 3. Families satisfying weak conditions. We now examine the question for the cases when the parameter $\eta$ in the condition $C(\eta, \lambda)$ is greater than three. The proofs are left to the reader.

**Theorem 3.1.** (i) $[2^\kappa, \kappa, 0, 4, 1] \rightarrow [\kappa]^2$ if $3 \leq \theta \leq \kappa$;
(ii) $[2^\kappa, \kappa, 0, 4, 1] \rightarrow [\kappa]^2$ if $2 \leq \theta \leq \kappa$.

Theorem 3.1 shows that under these weaker conditions all questions are solved in the negative. This appears to correspond to the case $C(3, \lambda)$ in [2].

References


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More on locally atomic models

by

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Abstract.

**Theorem.** Assume $T$ is a stable theory with $x(T) = \aleph_1$. If $|T| < \text{cov} K + b + \min(\text{cov} L, b)$, then any $A \subseteq \mathcal{C}$ can be extended to a model of $T$ locally atomic over $A$.

This improves Theorem 2.2 from [N2], in which we have a stronger assumption that $x(T) = \aleph_0$ (i.e. $T$ is superstable). The coefficients bounding $|T|$ above are defined in terms of measure and category on the real line, and can vary between $\aleph_1$ and $2^{\aleph_0}$.

§ 1. Introduction. Throughout, we use the same standard terminology as in [N2]. In particular, $T$ is a fixed first-order theory in language $L$, $\mathcal{C}$ is the monster model of $T$, i.e. a very saturated model of $T$ of high cardinality, such that all models of $T$ under consideration are elementary submodels of $\mathcal{C}$. For a formula $\varphi \in L(\mathfrak{C})$, $[\varphi]$ is the class of types containing $\varphi$. $A$ is a set of parameters from $\mathcal{C}$, $L(A)$ is the set of formulas with parameters from $A$, $S(A)$ is the set of complete $1$-types over $A$, $p \in S(A)$ is locally isolated if for every $\psi(p, \omega) \in L$ there is a $\psi(p, \omega)$ such that $\varphi(p, \omega) \in L(\mathfrak{A})$, i.e. for every $\varphi \in L(A)$, $\psi(p, \omega)$ implies either $\varphi(p, \omega)$ or $\neg \varphi(p, \omega)$. A model $M$ of $T$ containing $A$ is locally atomic over $A$ if for each $\varphi \in L(M)$, $\text{tp}(\psi/A)$ is locally isolated. The notion of local isolation, invented by Shelah, is fundamental in stability theory. It is one of the main tools to construct models of stable theories in the non-totally transcendental case.

To understand the paper, no deep understanding of stability theory is necessary. In particular, the reader does not have to know what $x(T)$ is, provided he is willing to accept Lemma 1, (2)-(3) without proof.

Now we explain what the real line coefficients $b, b$, cov $K$ and cov $L$ are.

$$b = \min \{|A| : A \subseteq \omega \land \forall \varphi \exists \exists \varphi \in A \exists \alpha \in \mathbb{R} \land \varphi(n) < \varphi(n)\},$$

$$\text{cov} L = \min \{|A| : A \subseteq I \land \bigcup A = X\}.$$
Now, \( K, L \) denote the family of meager subsets of the real line and subsets of the real line of Lebesgue measure 0, respectively.

\( b, b, \text{cov} K, \text{cov} L \) and other 6 coefficients regarding \( K \) and \( L \) are arranged in the so-called Cichoń's diagram, which is an extended version of the Kunen–Miller chart (cf. \([F],[M2]\)). Encapsulated in this diagram are the known inequalities between the coefficients of the real line. One finds out from Cichoń's diagram that \( \text{cov} K, b, b, \text{min} \{\text{cov} L, b\} \) are all \( b < 2^{\aleph_0} \) and \( \geq N_\omega \), and can vary quite independently between \( N_\omega \) and \( 2^{\aleph_0} \), which means that for every assignment of \( N_\omega \) and, say, \( \text{min} \{\text{cov} K, b, b, \text{min} \{\text{cov} L, b\} \), there is a generic model of ZFC realizing this assignment.

From the point of view of stability hierarchy, the theorem cannot be improved anymore. In \([N2]\) we have found an example of a stable \( T \) of power \( N_\omega \) and with \( \chi(T) = N_\omega \) such that for some \( A \) there is no \( M \geq A \) locally atomic over \( A \). Also, we have shown in \([N2]\) an example of a superstable \( T \) (i.e. \( \chi(T) = N_\omega \)) of power \( N_\omega \) such that for some \( A \) there is no \( M \geq A \) locally atomic over \( A \), \( \chi(T) = N_\omega \) is the minimal power of a partition of the real line into compact sets. Now, \( b \leq \chi(T) \leq 2^{\aleph_0} \) and there are models of ZFC + \( \forall \chi \exists \theta \exists \chi \in L(A) \) in which \( x = N_\omega \) holds. This shows that in ZFC only, we cannot improve the coefficient \( \text{cov} K + b + \text{min} \{\text{cov} L, b\} \) bounding \( \{T\} \) in the theorem 1, say, \( 2^{\aleph_0} \). The theorem shows that some basic statements of stability theory are independent of ZFC and that ZFC does not determine a statement of stability theory regarding cardinals between \( N_\omega \) and \( 2^{\aleph_0} \).

The theorem was proved for countable stable \( T \) by Lachlan \([Ls]\). Our proof is a technical complication of the proof of Theorem 2.2 from \([N2]\). In the proof, depending on which of \( \{T\} < \text{cov} K \) and \( \{T\} < \text{cov} L \) holds, we use the compactness theorem in three different ways. As \( \text{cov} K, b, b, \text{min} \{\text{cov} L, b\} \) can vary independently between \( N_\omega \) and \( 2^{\aleph_0} \), none of these ways is stronger than the others.

**§ 2. Proof of the theorem.** As in \([N2]\), to prove the theorem it suffices to include each consistent formula \( \theta(x) \in L(A) \) into a locally isolated type \( \phi \in S(A) \). So let us fix \( A \in S \) and a consistent formula \( \phi \in L(A) \). First let us clarify what \( \chi(T) \leq N_\omega \) means.

Let \( \phi(x) \in L(A) \) and \( \mu = N_\omega \) or \( N_\omega \). We say that a sequence of formulas \( \phi_n(x, \bar{y}), a \preceq b, \mu \)-split below \( \phi \) for any finite subsequence \( \phi_{a0}, \ldots, \phi_{an-1} \), \( \mu(0) < \ldots < \mu(n-1) \), and any \( r \preceq \mu \), there are parameters \( \bar{m} \in A, \sigma \preceq \alpha^+, \zeta \preceq \nu^+, \eta \preceq \zeta \), called splitting parameters, such that for each \( \sigma \preceq \alpha^+ \), the formulas \( \phi_{a0}(x, \bar{m}, \sigma, \zeta, \eta) \), \( \zeta < \eta \), are

(a) pairwise contradictory,
(b) consistent with \( \phi(x) \in L(A) \) and \( \bar{m}_{a0} \neq \bar{m}_{a0+1} \).

**Lemma 1.** Let \( \phi(x) \in L(A) \) be a consistent formula and \( \phi_n(x, \bar{y}), a < \alpha, \) a sequence of formulas. Then \((1) \rightarrow (2) \rightarrow (3), \) where \( (1), (2), (3) \) are the conditions given below.

(1) \( \{\phi_n(x, \bar{y}), a < \alpha_1\} \) \( \text{\( N_\omega \)-splits below} \ \phi \),
(2) \( \{\phi_n(x, \bar{y}), a < \alpha_1\} \) \( \text{\( N_\omega \)-splits below} \ \phi \),
(3) \( \chi(T) > N_\omega \).

**Proof.** (1) \( \rightarrow (2) \) is trivial, \( (2) \rightarrow (3) \) follows easily by \([Sh, III]\).

The following characterization of local isolation, which is of some interest in its own right, is the only place in the proof of the theorem where we really use some basic stability theory.

**Lemma 2.** Assume \( T \) is stable. Then \( \phi \in S(A) \) is locally isolated if for every \( \psi(x, y) \in L \) there is \( \psi(x) \in L \) such that \( \phi \) does not \( N_\omega \)-split below \( \psi \).

**Proof.** This is an immediate conclusion from \([N2, \text{Lemma } 2.5]\) and the definition of locally isolated type.

Let \( \ast \) (respectively \( ** \ast \)) stand for:

there is a consistent formula \( \theta(x) \rightarrow \theta(x) \), \( \theta(x) \in L(A) \), such that for any consistent formulas \( \phi_n(x) \in L(A), \mu < \alpha_1 \), with \( \phi_n(x) \rightarrow \theta(x) \), there is a formula \( \chi(x, y) \in L \) which \( \text{\( N_\omega \)-splits below} \) infinitely many [respectively all] of \( \phi_n(x) \).

**Lemma 3.** (1) If \( \chi(T) \leq N_\sigma \) and \( \{T\} < \text{cov} K \) then there are formulas \( \theta(x) \in L(A) \) for \( n < \omega \) such that \( \theta(x), \theta(x) : n < \omega \) is consistent and for each \( \chi(x, y) \in L \) there is \( n < \sigma \) such that \( \chi \) does not \( N_\sigma \)-split below \( \theta \).

(2) If \( \mu < \sigma \), satisfy the conclusion of (1) then every type \( \phi \in S(A) \cap \{T\} \cap \{ \theta \} \) \( n < \omega \)

is locally isolated. In particular, there is a locally isolated \( \phi \in S(A) \) containing \( \theta \).

**Proof.** (1) We find a tree of consistent formulas \( \{\phi_n(x) : n < \omega \} \subseteq L(A) \) below \( \theta(x) \) such that

(a) \( \phi_n(x) \rightarrow \phi_n(x) \) for \( n < \sigma \) and \( \phi_n(x) \in \phi \),
(b) for each \( n < \omega \), \( \phi_n(x) \in L \), either \( \chi \) does not \( N_\sigma \)-split below \( \phi_n(x) \) for all but finitely many \( n \) (when \( \ast \) holds) or \( \chi \) does not \( N_\sigma \)-split below \( \phi_n(x) \) for some \( n \) (when \( ** \ast \) holds).

Let \( N(x) = \bigcup_{n < \omega} \chi \) \( N_\sigma \)-splits below \( \phi_n(x) \) for every \( n < \omega \). We prove that

(c) \( \chi < \phi_n(x) \).

When \( \ast \) holds then by (b) each \( x(x) \) is nowhere dense, hence if \( \{T\} < \text{cov} K \) then \( \chi \) is a clear definition of \( \text{cov} K \). When \( \ast \) holds then by (b) for each \( x \in L \) there is \( \phi_n(x) \) such that for every \( \sigma \in N(x) \), \( \phi_n(x) \) and (c) follows.

Let \( x(x) = \bigcup_{n < \omega} \chi \) \( \text{\( N_\sigma \)-splits below} \) \( N(x) \). Clearly the formulas \( \theta(x) \in \phi_n(x) < \omega_0 \), satisfy our demands. (2) follows by Lemma 2.

The following lemma concludes the proof of the theorem. In \([N2]\) we used superstability here.

**Lemma 4.** If \( \chi(T) < N_\sigma \) then \( \chi(\ast) \). If additionally \( \{T\} < \text{cov} L + b \), then \( \chi(\ast) \).

**Proof.** Suppose not. We are heading towards a contradiction with \( \chi(T) < N_\omega \) via Lemma 1.

(1) \( \{T\} < \text{cov} L + b \).

By induction we find a sequence \( \psi_n, a < \omega_1 \), which \( \text{\( N_\omega \)-splits below} \) \( \theta \). To begin with, by \( \ast \) there is a \( \psi_n \) which \( \text{\( N_\omega \)-splits below} \) \( \theta \).

Suppose \( b \leq \omega, \) and we have a sequence \( \psi_n, a < \beta, \) which \( \text{\( N_\omega \)-splits below} \) \( \theta \). Let \( b = \bigcup_{n < \sigma} \psi_n \), where \( \{B_n : n < \omega \} \) is a non-decreasing sequence of non-empty subsets of \( \beta \) such that \( |B_n| = \beta(n) < \eta \) for \( n > 0 \). Let \( x_i, i < b(n), \) be the increasing enumeration of
Let $\mu_0 = n^2$ and let $\bar{m}_i(\delta), \sigma \in \mathbb{N}^{\omega}$ be splitting parameters for $\psi_{n}(0), \ldots, \psi_{n}(b(n) - 1)$. Set $S_n = \mu_0^{\mu_0}$ and for $\sigma \in S_n$, let $\phi_\sigma^\chi$ be

$$
\bigwedge_{i<n} \psi_{n}(i, \bar{m}_i(\delta + 1)) \wedge \theta(y).
$$

Finally, for $\chi \in L$ let

$$
A_\chi = \{ \sigma \in S_n : \chi \text{ does not } N_\sigma \text{-split below } \phi_\sigma^\chi \}, \ A_\chi = \bigcup_{n < \omega} \prod \{ n \in \mathbb{N}^{\omega} \} \times \prod A_\chi.
$$

Suppose there is no suitable $\psi_\sigma$ such that $\langle \psi_{n, \sigma} \alpha \beta \chi \rangle$ below $\theta$, Consider the statement

$$(1) \quad (\exists \alpha < \beta < \gamma \in \mathbb{N}^{\omega} \exists \bar{m}_i(\delta) \exists \psi_{n, \sigma} \alpha \beta \chi \langle \psi_{n, \sigma} \alpha \beta \chi \rangle \text{ does not } N_\sigma \text{-split below } \theta' \rangle.
$$

If for some $\chi \in L$, each $\sigma \in S_n$ satisfies $\langle \psi_{n, \sigma} \alpha \beta \chi \rangle$ below $\theta$, the proof consists first in choosing suitable subtrees of the trees of parameters $\{ \bar{m}_i(\delta) : \sigma \in \mathbb{N}^{\omega} \}$, $n < \omega$, and then applying the definition of $N_\sigma$-splitting.

Hence for each $\chi \in L$ there is $k < S_n$ such that $\langle (1) \rangle$ is true. (true) implies that there is $m < \omega$ such that for $n > m$, $|A_\chi| \geq (n^2 - k)^\omega$. Let $\mu$ be the product measure on $\prod S_n$ arising from the measures $\mu_\sigma(n)$ at each point in $S_n$. Then

$$
\mu(A_\chi) = \lim_{n \to \omega} \left( \prod_{i<n} \frac{n^2 - k}{n^2} \right)\omega = 1
$$

As $\co(L) > \omega$, there is $\langle \sigma, n < \omega \rangle \in \bigcap A_\chi$. Consider the formulas $\varphi_\sigma^\chi = \psi_{n, \sigma} \alpha \beta \chi$, $n < \omega$. For any $\chi$, for all but finitely many $n$, $\sigma_\chi \in A_\chi$, hence $\chi$ $N_\sigma$-splits below $\theta$. This contradicts $(\ast)$.

**Case 2.** $|T| < b$ and $(\ast)$.

We proceed as in Case 1 but replace $N_\sigma$-splitting by $N_\sigma$-splitting, and therefore $\mu_\sigma = n^2$ by $\mu_\sigma = N_\sigma$. In $\langle (1) \rangle$ we now have $k < S_n$. For some $\chi \in L$ and $k < S_n$, $|A_\chi|$ is true, then again we can prove that $\langle \psi_{n, \sigma} \alpha \beta \chi \rangle$ below $\theta'$, a contradiction. As in Case 1, the proof consists first in choosing suitable subtrees of the trees of parameters $\{ \bar{m}_i(\delta) : \sigma \in \mathbb{N}^{\omega} \}$, $n < \omega$, and then applying the definition of $N_\sigma$-splitting.

Hence for each $\chi \in L$, $\langle (1) \rangle$ is true for $k = S_n$. It follows that there is $n^2 < \omega$ and a function $g_j : \bigcup_{n < \omega} \{ n \times (\tau^{n+1} \tau^{n+1}) \} \to \omega$ such that for $n > n^2$ and $\sigma \in \mathbb{N}^{\omega}$ satisfies $\sigma(n) \geq g_j(n, \sigma(n))$ for each $i < j(b)$, then $\sigma \in A_\chi$. Now the idea of the proof consists in choosing in virtue of $|T| < b$ a single function eventually dominating all the $g_j$. More precisely, we proceed as follows. We define by induction functions

$$
g_j : \bigcup_{n < \omega} \{ n \times (\tau^{n+1} \tau^{n+1}) \} \to \omega
$$

for $j < |T|$, and sequences $\sigma_\chi \in \tau^{n+1} \tau^{n+1}$ for $n, t < \omega$ and $j < b(n) - 1$ as follows.

1. Let $\sigma_\chi(n) = \chi$. For each $n, \chi$, $g_j(n, \sigma_\chi)$ is the function mapping $i$ to $g_j(n, \sigma_\chi)$. As $|T| < b$, there is $g_j$ such that for each $n < \omega$, $\chi \in L$, for all $t$ large enough we have $g_j(n, \sigma_\chi(n)) \geq g_j(n, \sigma_\chi)$.

References


On the additivity of the fixed point property for 1-dimensional continua

by

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Abstract. Two rational arcwise connected continua $X_0$ and $Y_0$ with the fixed point property are constructed such that $X_0 \cap Y_0$ is contractible and $X_0 \cup Y_0$ does not have the fixed point property. The problem of the additivity of the fixed point property for 1-dimensional continua is summarized in the remark at the end of the paper.

1. Introduction. It is known that if $X$ and $Y$ are 1-dimensional continua with the fixed point property and $X \cap Y$ is an AR, then $X \cup Y$ has the fixed point property ([6], p. 1292). The aim of the present paper is to show that, roughly speaking, nothing more can be proved on the additivity of the fixed point property, for non-planar 1-dimensional continua, answering simultaneously a problem raised in [5] (p. 237).

To formulate the main result of the paper precisely, recall that a continuum $X$ is said to be rational if $X$ has a base of neighbourhoods with countable boundaries, and arcwise connected if any two points of $X$ can be joined by an arc in $X$. Any homeomorphic image of a cone over a convergent sequence of points together with its limit is called a harmonic brush; such a brush is contractible and has the fixed point property (see, for instance, [1], p. 20). The main result will be the following

Theorem. There exist two rational arcwise connected continua $X_0$ and $Y_0$ with the fixed point property such that $X_0 \cap Y_0$ is a straightline harmonic brush and $X_0 \cup Y_0$ does not have the fixed point property.

The continua $X_0$ and $Y_0$ will be uniquely arcwise connected, i.e., for any two points of $X_0$ or $Y_0$ there will be exactly one arc between them, so that the results from [5] can be applied. All other topological notions used, but not defined in the present paper, can be found in [4].

The continua $X_0$ and $Y_0$ will be constructed almost wholly on the plane $E^2$ (except an arc of $Y_0$ lying in $E^2$ outside $E^3$). A basic role in their description will be played by certain geometrical functions on $E^3$, which we shall define now.

Namely, for the points

(1.1) \[ a = (1, \, 1), \quad c = (0, \, 2), \quad \text{and} \quad d_{2n-1} = (0, \, 2+2^{-n}), \quad n = 1, \, 2, \, \ldots, \]

we take into account the following functions, defined for all $p \in E^2$, $k = -1, \, 0, \, 1, \, 2, \, \ldots$, and $n = 1, \, 2, \, \ldots$: