§ 3. Families satisfying weak conditions. We now examine the question for the cases when the parameter η in the condition $C[\eta, \lambda]$ is greater than three. The proofs are left to the reader.

THEOREM 3.1. (i) $[2^\varkappa, \varkappa, \theta, 4, 1] \nleftrightarrow R$ if $3 \leqslant \theta \leqslant \varkappa$;

(ii) $[2^\varkappa, \varkappa, \theta, 4, 1] \nleftrightarrow P'$ if $2 \leqslant \theta \leqslant \varkappa$. ∎

Theorem 3.1 shows that under these weaker conditions *all* questions are solved in the negative. This appears to correspond to the case $C(3, \lambda)$ in [2].

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF QUEENSLAND
St Lucia QLD 4067
Australia

Current address:
ELECTRONICS RESEARCH LABORATORY
DEPT. OF DEFENCE
P.O. Box 1600
Salisbury, SA 5108
Australia

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More on locally atomic models

by

Ludomir Newelski (Wrocław)

Abstract.

THEOREM. *Assume* T *is a stable theory with* $\varkappa(T) \leqslant \aleph_1$. *If* $|T| < \operatorname{cov} K + \mathfrak{b} + \min(\operatorname{cov} L, \mathfrak{d})$, *then any* $A \subseteq \mathfrak{C}$ *can be extended to a model of* T *locally atomic over* A.

This improves Theorem 2.2 from [N2], in which we have a stronger assumption that $\varkappa(T) = \aleph_0$ (i.e. T is superstable). The coefficients bounding $|T|$ above are defined in terms of measure and category on the real line, and can vary between $\aleph_1$ and $2^{\aleph_0}$.

§ 1. Introduction. Throughout, we use the same standard terminology as in [N2]. In particular, T is a fixed first-order theory in language L, $\mathfrak{C}$ is the monster model of T, i.e. a very saturated model of T of high cardinality, such that all models of T under consideration are elementary submodels of $\mathfrak{C}$. For a formula $\theta \in L(\mathfrak{C})$, $[\theta]$ is the class of types containing θ. A is a set of parameters from $\mathfrak{C}$. $L(A)$ is the set of formulas with parameters from A, $S(A)$ is the set of complete 1-types over A. $p \in S(A)$ is locally isolated if for every $\varphi(x, \bar{y}) \in L$ there is a $\psi \in p$ such that $\psi \vdash p|\varphi$, i.e. for every $\bar{a} \in A$, $\psi(x)$ implies either $\varphi(x, \bar{a})$ or $\neg\varphi(x, \bar{a})$. A model M of T containing A is locally atomic over A if for each $\bar{a} \in M$, $\operatorname{tp}(\bar{a}/A)$ is locally isolated. The notion of local isolation, invented by Shelah, is fundamental in stability theory. It is one of the main tools to construct models of stable theories in the non-totally transcendental case.

To understand the paper, no deep understanding of stability theory is necessary. In particular, the reader does not have to know what $\varkappa(T)$ is, provided he is willing to accept Lemma 1, (2)→(3) without proof.

Now we explain what the real line coefficients $\mathfrak{b}$, $\mathfrak{d}$, $\operatorname{cov} K$ and $\operatorname{cov} L$ are.

$$\mathfrak{b} = \min\{|A|:\ A \subseteq {}^\omega\omega \ \&\ \forall f \in {}^\omega\omega\ \exists g \in A\ \exists^\infty n\ f(n) < g(n)\},$$

$$\mathfrak{d} = \min\{|A|:\ A \subseteq {}^\omega\omega \ \&\ \forall f \in {}^\omega\omega\ \exists g \in A\ \forall n\ f(n) < g(n)\}.$$

Thus $\mathfrak{b}$ is the minimal power of an unbounded family of reals, and $\mathfrak{d}$ is the minimal power of a dominating family of reals. If $I \subseteq \mathscr{P}(X)$ then we define

$$\operatorname{cov} I = \min\{|A|:\ A \subseteq I \ \&\ \bigcup A = X\}.$$

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Now, $\boldsymbol{K}$, $\boldsymbol{L}$ denote the family of meager subsets of the real line and subsets of the real line of Lebesgue measure 0, respectively.

$\mathfrak{b}$, $\mathfrak{d}$, cov $\boldsymbol{K}$, cov $\boldsymbol{L}$ and other 6 coefficients regarding $\boldsymbol{K}$ and $\boldsymbol{L}$ are arranged in the so-called Cichoń's diagram, which is an extended version of the Kunen–Miller chart (cf. [F], [M2]). Encapsulated in this diagram are the known inequalities between the coefficients of the real line. One finds out from Cichoń's diagram that cov $\boldsymbol{K}$, $\mathfrak{b}$, and $\min\{\mathrm{cov}\,\boldsymbol{L}, \mathfrak{d}\}$ are all $\leqslant \mathfrak{d} \leqslant 2^{\aleph_0}$ and $\geqslant \aleph_1$, and can vary quite independently between $\aleph_1$ and $2^{\aleph_0}$, which means that for every assignment of $\aleph_1$ and, say, $\aleph_2$ to the symbols cov $\boldsymbol{K}$, $\mathfrak{b}$, and $\min\{\mathrm{cov}\,\boldsymbol{L}, \mathfrak{d}\}$, there is a generic model of ZFC realizing this assignment.

From the point of view of stability hierarchy, the theorem cannot be improved anymore. In [N2] we have found an example of a stable T of power $\aleph_1$ and with $\varkappa(T) = \aleph_2$ such that for some A there is no $M \supseteq A$ locally atomic over A. Also, we have shown in [N2] an example of a superstable T (i.e. $\varkappa(T) = \aleph_0$) of power $\varkappa_1$ such that for some A there is no $M \supseteq A$ locally atomic over A, $\varkappa_1$ is the minimal power of a partition of the real line into compact sets. Now, $\mathfrak{d} \leqslant \varkappa_1 \leqslant 2^{\aleph_0}$ and there are models of ZFC$+\neg$CH in which $\varkappa_1 = \aleph_1$ holds. This shows that in ZFC only, we cannot improve the coefficient $\mathrm{cov}\,\boldsymbol{K} + \mathfrak{b} + \min\{\mathrm{cov}\,\boldsymbol{L}, \mathfrak{d}\}$ bounding $|T|$ in the theorem to, say, $2^{\aleph_0}$. The theorem shows that some basic statements of stability theory are independent of ZFC and that ZFC does not determine a statement of stability theory regarding cardinals between $\aleph_1$ and $2^{\aleph_0}$.

The theorem was proved for countable stable T by Lachlan [La]. Our proof is a technical complication of the proof of Theorem 2.2 from [N2]. In the proof, depending on which of $|T| < \mathrm{cov}\,\boldsymbol{K}$, $|T| < \mathfrak{b}$, $|T| < \min\{\mathrm{cov}\,\boldsymbol{L}, \mathfrak{d}\}$ holds, we use the compactness theorem in three different ways. As cov $\boldsymbol{K}$, $\mathfrak{b}$, and $\min\{\mathrm{cov}\,\boldsymbol{L}, \mathfrak{d}\}$ can vary independently between $\aleph_1$ and $2^{\aleph_0}$, none of these ways is stronger than the others.

§ 2. Proof of the theorem. As in [N2], to prove the theorem it suffices to include each consistent formula $\theta(x) \in L(A)$ into a locally isolated type $p \in S(A)$. So let us fix $A \subseteq \mathfrak{C}$ and a consistent formula $\theta(x) \in L(A)$. First let us clarify what $\varkappa(T) \leqslant \aleph_1$ means.

Let $\varphi(x) \in L(A)$ and $\mu = \aleph_0$ or $\aleph_1$. We say that a sequence of formulas $\varphi_\alpha(x, \bar{y}_\alpha)$, $\alpha < \beta$, *μ-splits below φ* if for any finite subsequence $\varphi_{\alpha(0)}, \ldots, \varphi_{\alpha(n-1)}$, $\alpha(0) < \ldots < \alpha(n-1)$, and any $\nu < \mu$, there are parameters $\bar{m}_\sigma \in A$, $\sigma \in {}^{n\geqslant}\nu$, called *splitting parameters*, such that for each $\sigma \in {}^{n>}\nu$, the formulas $\varphi_{\alpha(|\sigma|)}(x, \bar{m}_{\sigma^\wedge\langle\xi\rangle})$, $\xi < \nu$, are

(a) pairwise contradictory,

(b) consistent with $\varphi(x) \& \bigwedge_{j<|\sigma|} \varphi_{\alpha(j)}(x, \bar{m}_{\sigma|(j+1)})$.

LEMMA 1. *Let $\varphi(x) \in L(A)$ be a consistent formula and $\varphi_\alpha(x, \bar{y}_\alpha)$, $\alpha < \omega_1$, a sequence of formulas. Then* (1)$\to$(2)$\to$(3), *where* (1), (2), (3) *are the conditions given below.*

(1) $\{\varphi_\alpha(x, \bar{y}_\alpha),\ \alpha < \omega_1\}$ *$\aleph_1$-splits below φ,*

(2) $\{\varphi_\alpha(x, \bar{y}_\alpha),\ \alpha < \omega_1\}$ *$\aleph_0$-splits below φ,*

(3) $\varkappa(T) > \aleph_1$.

Proof. (1)$\to$(2) is trivial, (2)$\to$(3) follows easily by [Sh, III].

The following characterization of local isolation, which is of some interest in its own right, is the only place in the proof of the theorem where we really use some basic stability theory.

LEMMA 2. *Assume T is stable. Then $p \in S(A)$ is locally isolated iff for every $\varphi(x, \bar{y}) \in L$ there is $\psi(x) \in p$ such that φ does not $\aleph_1$-split below ψ.*

Proof. This is an immediate conclusion from [N2, Lemma 2.5] and the definition of locally isolated type.

Let (∗) [respectively (∗∗)] stand for:

there is a consistent formula $\theta'(x) \vdash \theta(x)$, $\theta'(x) \in L(A)$, such that for any consistent formulas $\varphi_n(x) \in L(A)$, $n < \omega$, with $\varphi_n(x) \vdash \theta'(x)$, there is a formula $\chi(x, \bar{y}) \in L$ which $\aleph_1$-splits below infinitely many [respectively all] of φ_n's.

LEMMA 3. (1) *If* ($\neg$(∗) *and* $|T| < \mathfrak{d}$) *or* ($\neg$(∗∗) *and* $|T| < \mathrm{cov}\,\boldsymbol{K}$) *then there are formulas $\theta_n(x) \in L(A)$ for $n < \omega$ such that $\{\theta(x), \theta_n(x):\ n < \omega\}$ is consistent and for each $\chi(x, y) \in L$ there is $n < \omega$ such that χ does not $\aleph_1$-split below θ_n.*

(2) *If θ_n, $n < \omega$, satisfy the conclusion of* (1) *then every type $p \in S(A) \cap [\theta] \cap \bigcap_{n<\omega} [\theta_n]$ is locally isolated. In particular, there is a locally isolated $p \in S(A)$ containing θ.*

Proof. (1) We find a tree of consistent formulas $\{\varphi_\eta(x):\ \eta \in {}^{\omega>}\omega\} \subseteq L(A)$ below $\theta(x)$ such that

(a) $\varphi_\eta(x) \vdash \varphi_\nu(x)$ for $\nu < \eta \in {}^{\omega>}\omega$ and

(b) for each $\eta \in {}^{\omega>}\omega$ and $\chi \in L$, either χ does not $\aleph_1$-split below $\varphi_{\eta^\wedge\langle n\rangle}$ for all but finitely many n (when $\neg$(∗) holds) or χ does not $\aleph_1$-split below $\varphi_{\eta^\wedge\langle n\rangle}$ for some n (when $\neg$(∗∗) holds).

Let $N(\chi) = \{f \in {}^\omega\omega:\ \chi\ \aleph_1\text{-splits below } \varphi_{f|n} \text{ for every } n < \omega\}$. We prove that

(c) ${}^\omega\omega \neq \bigcup_{\chi \in L} N(\chi)$.

When $\neg$(∗∗) holds then by (b) each $N(\chi)$ is nowhere dense, hence if $|T| < \mathrm{cov}\,\boldsymbol{K}$ then (c) is clear by the definition of cov $\boldsymbol{K}$. When $\neg$(∗) holds then by (b) for each $\chi \in L$ there is $f_\chi \in {}^\omega\omega$ such that for every $f \in N(\chi)$, $f <^* f_\chi$. Now if $|T| < \mathfrak{d}$ then there is $g \in {}^\omega\omega$ such that for each χ, $\neg\, g <^* f_\chi$, hence $g \notin N(\chi)$ and (c) follows.

Let $f \in {}^\omega\omega - \bigcup_{\chi \in L} N(\chi)$. Clearly the formulas $\theta_n = \varphi_{f|n}$, $n < \omega$, satisfy our demands. (2) follows by Lemma 2.

The following lemma concludes the proof of the theorem. In [N2] we used superstability here.

LEMMA 4. *If $\varkappa(T) \leqslant \aleph_1$ then $\neg$(∗∗). If additionally $|T| < \mathrm{cov}\,\boldsymbol{L} + \mathfrak{b}$, then $\neg$(∗).*

Proof. Suppose not. We are heading towards a contradiction with $\varkappa(T) \leqslant \aleph_1$ via Lemma 1.

Case 1. $|T| < \mathrm{cov}\,\boldsymbol{L}$ and (∗).

By induction we find a sequence ψ_α, $\alpha < \omega_1$, which $\aleph_0$-splits below θ'. To begin with, by (∗) there is ψ_0 which $\aleph_1$-splits below θ'.

Suppose $\beta < \omega_1$, and we have a sequence ψ_α, $\alpha < \beta$, which $\aleph_0$-splits below θ'. Let $\beta = \bigcup_{n<\omega} B_n$, where $\{B_n:\ n < \omega\}$ is a non-decreasing sequence of non-empty subsets of β such that $|B_n| = b(n) \leqslant n$ for $n > 0$. Let $\alpha_n(i)$, $i < b(n)$, be the increasing enumeration of

B_n. Let $\mu_n = n^3$ and let $\bar{m}_n(\sigma)$, $\sigma \in {}^{b(n)\geq}\mu_n$, be splitting parameters for $\psi_{\alpha_n}(0), \ldots, \psi_{\alpha_n(b(n)-1)}$. Set $S_n = {}^{b(n)}\mu_n$ and for $\sigma \in S_n$ let Φ^n_σ be

$$\bigwedge_{i<b(n)} \psi_{\alpha_n(i)}(x, \bar{m}_n(\sigma|(i+1)) \,\&\, \theta'(x).$$

Finally, for $\chi \in L$ let

$$A^n_\chi = \{\sigma \in S_n\colon \chi \text{ does not } \aleph_1\text{-split below } \Phi^n_\sigma\}, \qquad A_\chi = \bigcup_{m<\omega} \Big(\prod_{n\leq m} S_n \times \prod_{n>m} A^n_\chi\Big).$$

Suppose there is no suitable ψ_β such that $\langle \psi_\alpha, \alpha \leq \beta \rangle$ $\aleph_0$-splits below θ'. Consider the statement

(†) $\qquad (\exists m < \omega)(\forall n > m)(\neg \exists^k n_0 \ldots \exists^k n_{b(n)-1} \langle n_0, \ldots, n_{b(n)-1} \rangle \in S_n - A^n_\chi)$.

If for some $\chi \in L$, for each $k < \aleph_0$, (†) is false, then we can prove that if $\psi_\beta = \chi$ then $\langle \psi_\alpha, \alpha \leq \beta \rangle$ $\aleph_0$-splits below θ', a contradiction. The proof consists first in choosing suitable subtrees of trees of parameters $\{\bar{m}_n(\sigma)\colon \sigma \in {}^{b(n)\geq}\mu_n\}$, $n < \omega$, and then applying the definition of $\aleph_1$-splitting.

Hence for each $\chi \in L$ there is $k < \aleph_0$ such that (†) is true. (†) implies that there is $m < \omega$ such that for $n > m$, $|A^n_\chi| \geq (n^3 - k)^{b(n)}$. Let μ be the product measure on $\prod_n S_n$ arising from measures which assing weight $n^{-3b(n)}$ to each point in S_n. Then

$$\mu(A_\chi) = \lim_m \Big(\prod_{n>m} \Big(\frac{n^3-k}{n^3}\Big)^{b(n)}\Big) = 1$$

As $\operatorname{cov} \boldsymbol{L} > |T|$, there is $\langle \sigma_n, n < \omega \rangle \in \bigcap_{\chi \in L} A_\chi$. Consider the formulas $\varphi_n = \Phi^n_{\sigma_n}$, $n < \omega$. For any χ, for all but finitely many n, $\sigma_n \in A^n_\chi$, hence χ $\aleph_1$-splits below φ_n. This contradicts (∗).

Case 2. $|T| < \mathfrak{b}$ and (∗).

We proceed as in case 1 but replace $\aleph_0$-splitting by $\aleph_1$-splitting, and therefore $\mu_n = n^3$ by $\mu_n = \aleph_0$. In (†) we have now $k = \aleph_0$. If for some $\chi \in L$ and $k = \aleph_0$, (†) is false, then again we can prove that if $\psi_\beta = \chi$ then $\langle \psi_\alpha, \alpha \leq \beta \rangle$ $\aleph_1$-splits below θ', a contradiction. As in case 1, the proof consists first in choosing suitable subtrees of trees of parameters $\{\bar{m}_n(\sigma)\colon \sigma \in {}^{b(n)\geq}\mu_n\}$, $n < \omega$, and then applying the definition of $\aleph_1$-splitting.

Hence for each $\chi \in L$, (†) is true for $k = \aleph_0$. It follows that there is $n_\chi < \omega$ and a function $g_\chi\colon (\bigcup_{n<\omega} \{n\} \times ({}^{b(n)>}\omega)) \to \omega$ such that if $n > n_\chi$ and $\sigma \in {}^{b(n)}\omega$ satisfies $\sigma(i) \geq g_\chi(n, \sigma|i)$ for each $i < b(n)$, then $\sigma \in A^n_\chi$. Now the idea of the proof consists in choosing in virtue of $|T| < \mathfrak{b}$ a single function eventually dominating all the g_χ's. More precisely, we proceed as follows. We define by induction functions

$$g_j\colon \Big(\bigcup_{n<\omega} \{n\} \times ({}^{j+1}\omega \cap {}^{b(n)>}\omega)\Big) \to \omega$$

for $j < |\beta|$, and sequences $\sigma^j_{n,t} \in {}^{j+1}\omega$ for $n, t < \omega$ and $j < b(n) - 1$ as follows.

(1) Let $\sigma^0_{n,t}(0) = t$. For each n, χ, $g_\chi(n, \sigma^0_{n,\cdot})$ is the function mapping t to $g_\chi(n, \sigma^0_{n,t})$. As $|T| < \mathfrak{b}$, there is g_0 such that for each $n < \omega$, $\chi \in L$, for all t large enough we have $g_0(n, \sigma^0_{n,t}) \geq g_\chi(n, \sigma^0_{n,t})$.

(2) Suppose we have defined $g_0, \ldots, g_{j-1}$ and $\sigma^{j-1}_{n,t}$ for $n, t < \omega$ and $j - 1 < b(n) - 1$. If $j < b(n) - 1$ then we put $\sigma^j_{n,t} = \sigma^{j-1}_{n,t}{}^\frown \langle g_{j-1}(n, \sigma^{j-1}_{n,t}) \rangle$. By $|T| < \mathfrak{b}$ there is g_j such that for every $\chi \in L$, $n < \omega$, if $j < b(n) - 1$ then for all but finitely many $t < \omega$, $g_j(n, \sigma^j_{n,t}) \geq g_\chi(n, \sigma^j_{n,t})$.

Having found g_j's for every $\chi \in L$ define $f_\chi \in {}^\omega\omega$ by

$$f_\chi(n) = \min\{t\colon (\forall u \geq t)(\forall j < b(n) - 1)(g_j(n, u) \geq g_\chi(n, \sigma^j_{n,u}))\}.$$

Choose $f \in {}^\omega\omega$ such that for every $\chi \in L$, for all but finitely many n, $f(n) \geq f_\chi(n)$. For $n < \omega$ define σ_n as $\sigma^{b(n)-1}_{n,f(n)} \in {}^{b(n)}\omega$. By the above construction it follows that for every $\chi \in L$, for all n large enough we have $\sigma_n \in A^n_\chi$. So if we put $\varphi_n = \Phi^n_{\sigma_n}$, then for each $\chi \in L$, χ $\aleph_1$-splits below φ_n for all but finitely many n, thus contradicting (∗).

Case 3. (∗∗) We proceed as in case 2, i.e. find a sequence ψ_α, $\alpha < \omega_1$, which $\aleph_1$-splits below θ'. Suppose we have found ψ_α, $\alpha < \beta$, which $\aleph_1$-splits below θ'. Consider the countable set of formulas $\{\Phi^n_\sigma\colon n < \omega,\ \sigma \in {}^{b(n)}\omega\}$. By (∗∗) there is $\chi \in L$ such that χ $\aleph_1$-splits below Φ^n_σ for each n, σ. So for $\psi_\beta = \chi$, $\langle \psi_\alpha, \alpha \leq \beta \rangle$ $\aleph_1$-splits below θ'.

The model-theoretical keypoint of the proof is Lemma 2, giving a translation of local isolation into a chain condition. Suppose that in some model of ZFC we want to find a stable theory T with $\varkappa(T) \leq \aleph_1$ and a set of parameters A such that there is no model M of T containing A and locally atomic over A. By Lemma 3, for some $\theta(x) \in L(A)$ we have to ensure that no sequence θ_n, $n < \omega$, satisfies the conclusion of Lemma 3 (1), and simultaneously we have to falsify conditions (1) and (2) from Lemma 1. Lemma 4 shows that sometimes there may be that many sequences θ_n, $n < \omega$, that this task is impossible to render.

COROLLARY 5. *If T is stable, $\varkappa(T) \leq \aleph_1$, $|T| < \operatorname{cov} \boldsymbol{K} + \mathfrak{b} + \min\{\operatorname{cov} \boldsymbol{L}, \mathfrak{d}\}$, $Q(x) \in L$, and $M \precneqq$ are models of T with $Q(M) = Q(N)$, then there is $N' \succneqq N$ with $Q(N') = Q(N)$.*

Proof. See [La], [N2].

The above corollary was the reason why the author became interested in locally atomic models. This corollary gives a nice proof of the two-cardinal theorem for stable theories (of small power). Primarily it was proved by Lachlan [La] for countable stable T. Several trials to improve Lachlan's proof have been made in [B], [Ls] and [H]. V. Harnik has proved Corollary 5 without assumptions on $|T|$ or $\varkappa(T)$, but instead adding additional assumption that M (or N) is $|T|$-compact, and using $F^1_{|T|}$-isolated types instead of $F^1_{\aleph_0}$ (= locally)-isolated types. For the definition of $F^1_\varkappa$ see [Sh]. The author has obtained some similar consistency results on $F^1_\varkappa$-atomic models for $\aleph_0 < \varkappa < |T|$, which require however some additional forcing technique.

QUESTION. Can we replace $\operatorname{cov} \boldsymbol{K} + \mathfrak{b} + \min\{\operatorname{cov} \boldsymbol{L}, \mathfrak{d}\}$ in the theorem just by $\mathfrak{d}$?

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INSTYTUT MATEMATYCZNY PAN
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
ul. Kopernika 18
51-617 Wrocław
Poland

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On the additivity of the fixed point property for 1-dimensional continua

by

Roman Mańka (Warszawa)

Abstract. Two rational arcwise connected continua X_0 and Y_0 with the fixed point property are constructed such that $X_0 \cap Y_0$ is contractible and $X_0 \cup Y_0$ does not have the fixed point property. The problem of the additivity of the fixed point property for 1-dimensional continua is summarized in the remark at the end of the paper.

1. Introduction. It is known that if X and Y are 1-dimensional continua with the fixed point property and $X \cap Y$ is an AR, then $X \cup Y$ has the fixed point property ([6], p. 1292). The aim of the present paper is to show that, roughly speaking, nothing more can be proved on the additivity of the fixed point property, for non-planar 1-dimensional continua, answering simultaneously a problem raised in [5] (p. 237).

To formulate the main result of the paper precisely, recall that a continuum X is said to be *rational* if X has a base of neighbourhoods with countable boundaries, and *arcwise connected* if any two points of X can be joined by an arc in X. Any homeomorphic image of a cone over a convergent sequence of points together with its limit is called a *harmonic brush*; such a brush is contractible and has the fixed point property (see, for instance, [1], p. 20). The main result will be the following

THEOREM. *There exist two rational arcwise connected continua X_0 and Y_0 with the fixed point property such that $X_0 \cap Y_0$ is a straightline harmonic brush and $X_0 \cup Y_0$ does not have the fixed point property.*

The continua X_0 and Y_0 will be *uniquely arcwise connected*, i.e. for any two points of X_0 or Y_0 there will be exactly one arc between them, so that the results from [5] can be applied. All other topological notions used, but not defined in the present paper, can be found in [4].

The continua X_0 and Y_0 will be constructed almost wholly on the plane E^2 (except an arc of Y_0 lying in E^3 outside E^2). A basic role in their description will be played by certain geometrical functions on E^2, which we shall define now.

Namely, for the points

(1.1) $\quad a = (1, 1), \quad c = (0, 2) \quad$ and $\quad d_{2n-1} = (0, 2+2^{-n}), \quad n = 1, 2, \ldots,$

we take into account the following functions, defined for all $p \in E^2$, $k = -1, 0, 1, 2, \ldots$ and $n = 1, 2, \ldots$: