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On bounded paradoxical subsets of the plane

by

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Abstract. We give a precise lower bound for the number of pieces required in a bounded paradoxical subset of the plane.

Intuitively, a subset X of a metric space is said to be *paradoxical* if it admits a partition $X = A \cup B$ such that each of the sets A, B can be subdivided into finitely many pieces which can be reassembled via isometries to produce X; if A is subdivided into m pieces and B is subdivided into n pieces, the set X is said to be *(m, n)-paradoxical.*

The Sierpiński–Mazurkiewicz paradox is that there is a (1,1)-paradoxical subset of the plane [MS]. Hadwiger, Debrunner and Klee [HDK, p. 80] have shown that a *bounded* (m, n)-paradoxical subset of the plane must satisfy $m+n > 2$. A bounded (1, 3)-paradoxical subset of the plane has recently been constructed by Just [J].

Our main purpose here is to show that there is no bounded (1,2)-paradoxical subset of the plane. This improves the result of Hadwiger, Debrunner and Klee, and renders optimal the recent construction of Just. We also construct here a bounded (2,2)-paradoxical subset of the plane.

Definition 1. X is an *(m, n)-paradoxical* subset of the plane if X is nonempty, and there are subsets $C_1, \ldots, C_m$, $D_1, \ldots, D_n$ of X and planar isometries $G_1, \ldots, G_m$, $H_1, \ldots, H_n$, such that $P_1 = \{C_i\}$, $P_2 = \{D_j\}$ and $P_3 = \{G_i(C_i)\} \cup \{H_j(D_j)\}$ are each partitions of X.

Definition 2. Let X be an (m, n)-paradoxical subset of the plane whose paradoxical decomposition is witnessed by subsets $C_1, \ldots, C_m$, $D_1, \ldots, D_n$ and planar isometries $G_1, \ldots, G_m$, $H_1, \ldots, H_n$. Write $\mathscr{C} = \{C_1, \ldots, C_m\}$, $\mathscr{D} = \{D_1, \ldots, D_n\}$, $\mathscr{G} = \{G_1, \ldots, G_m\}$, and $\mathscr{H} = \{H_1, \ldots, H_n\}$. We define the *associated directed graph* $\Gamma = \Gamma(\mathscr{C}, \mathscr{D}, \mathscr{G}, \mathscr{H})$ of the decomposition. Γ is an infinite directed graph with vertex set $V(\Gamma) = X$. The set of darts (i.e. directed edges) of Γ consists of all pairs $(x, G_i(x))$ and $(x, H_j(x))$, where $x \in C_i \cap D_j$.

It is helpful, when drawing diagrams, to label each dart of Γ with the planar isometry that determined its second coordinate.

Observe that every x in $V(\Gamma)$ has invalency 1 and outvalency 2.

Lemma 1. *Let Γ be an infinite directed graph with invalency* 1 *at each vertex, and suppose furthermore that Γ is connected. Then Γ contains at most one cycle.*

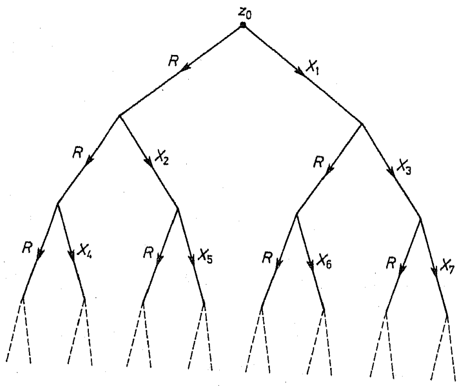
Proof. Easy.

THEOREM 1. *There is no bounded* (1,2)-*paradoxical subset of the plane.*

Our proof is based on the method of Hadwiger, Debrunner and Klee [HDK, p. 80].

Proof. Suppose X is a (1,2)-paradoxical subset of the plane. Then there are subsets D_1, D_2 of X, and planar isometries R, F and G, such that $P_1 = \{X\}$, $P_2 = \{D_1, D_2\}$ and $P_3 = \{R(X), F(D_1), G(D_2)\}$ are partitions of X. We show that X is not bounded.

Let Γ be the associated directed graph of this decomposition. If every component of Γ contains a cycle, let c be a vertex in some cycle, and choose z_0 such that (c, z_0) is a dart of Γ not in the cycle. Otherwise, let z_0 be any vertex of an acyclic component. Let Δ be the smallest subgraph of Γ which contains the vertex z_0 and every directed path which has initial vertex z_0. Then Δ is acyclic by Lemma 1. The labelled subgraph Δ is shown in the figure below, where each $X_i \in \{F, G\}$,



Since $R(z_0)$, $R^2(z_0)$, $R^3(z_0)$, … are the final vertices of distinct paths of Δ (having initial vertex z_0), they are all different. Therefore, R is not a reflection or a rotation of finite order.

Now suppose X is bounded, then so is the set $\{R(z_0), R^2(z_0), R^3(z_0), \ldots\}$. Therefore, R is not a translation or a glide reflection, so R must be a rotation of infinite order. Let O be the fixed point of R, and let D be the smallest closed disc with center O that contains all the vertices of Δ.

Note that the entire boundary of D is contained in the topological closure of the vertex set of Δ. (Since R has infinite order, each R-orbit is dense in a circle centered at O, so $\mathrm{cl}(V(\Delta))$ is a union of circles centered at O. By the minimality of D, one such circle coincides with ∂D so $\partial D \subset \mathrm{cl}(V(\Delta))$.)

The proof now breaks into six essentially different cases.

Case 1. $F(O) \neq O$ and $G(O) \neq O$. Clearly, the boundary of a disc of radius r cannot be covered by two discs also of radius r, unless at least one of the covering discs coincides with the covered disc. Thus $F^{-1}(D) \cup G^{-1}(D)$ does not cover ∂D. Since $\partial D \subset \mathrm{cl}(V(\Delta))$, there is a vertex v of Δ not in $F^{-1}(D) \cup G^{-1}(D)$, but this contradicts the fact that one of the points $F(v)$, $G(v)$ is a vertex of Δ.

Case 2. $F(O) \neq O$ and G is a rotation fixing O. Since $F^{-1}(D)$ does not cover ∂D and $\partial D \subset \mathrm{cl}(V(\Delta))$, there is a vertex v of Δ not in $F^{-1}(D)$. Since the point $F(v)$ is not a vertex of Δ, the point $G(v)$ is a vertex of Δ. Let $k \geq 1$. Exactly one of the points $FR^k(v)$, $GR^k(v)$ is a vertex of Δ. If $GR^k(v) \in V(\Delta)$, then $GR^k(v)$ and $R^k G(v)$ are the final vertices of different paths having initial vertex v, so they should be different. But GR^k and $R^k G$ are equal as planar isometries. Hence $GR^k(v)$ is not a vertex of Δ. Therefore $FR^k(v) \in V(\Delta)$. This is true for all $k \geq 1$. But now we have

$$v \in \mathrm{cl}\big(\{R^k(v)\}_{k=1}^{\infty}\big) \subset \mathrm{cl}\big(F^{-1}(V(\Delta))\big) \subset \mathrm{cl}\big(F^{-1}(D)\big) = F^{-1}(D)$$

contradicting our choice of v.

Case 3. $F(O) \neq O$ and G is a reflection fixing O. Since $F^{-1}(D)$ does not cover ∂D and $\partial D \subset \mathrm{cl}(V(\Delta))$, there is a vertex v of Δ not in $F^{-1}(D)$, and $G(v)$ is a vertex of Δ as in case 2. Let $k \geq 1$. Exactly one of the points $FR^k G(v)$, $GR^k G(v)$ is a vertex of Δ. If $GR^k G(v)$ is a vertex of Δ, then so is $R^k GR^k G(v)$; $R^k GR^k G(v)$ is the final vertex of a path having initial vertex v and so should be different from v, but $R^k GR^k G$ is equal to the identity as a planar isometry. Hence $GR^k G(v)$ is not a vertex of Δ so $FR^k G(v)$ is. This is true for all $k \geq 1$, but now we have

$$v \in \mathrm{cl}\big(\{R^k G(v)\}_{k=1}^{\infty}\big) \subset \mathrm{cl}\big(F^{-1}(V(\Delta))\big) \subset \mathrm{cl}\big(F^{-1}(D)\big) = F^{-1}(D)$$

(since v and $G(v)$ are equidistant from O), and this Δ contradicts the choice of v.

Case 4. F and G are rotations fixing O. For some $X, Y, Z \in \{F, G\}$, the points $R^2X(z_0)$, $RYR(z_0)$ and $ZR^2(z_0)$ are vertices of Δ. These are the final vertices of three different paths having initial vertex z_0, so they should be all different, but at least two of the maps R^2X, RYR, ZR^2 are equal as planar isometries.

Case 5. F and G are reflections fixing O. For some $U, V, W, X, Y, Z \in \{F, G\}$, the points $R^2UV(z_0)$, $RWXR(z_0)$, $YZR^2(z_0)$, and $R^2(z_0)$ are vertices of Δ, and should be all different. If $U = V$, then $R^2UV = R^2$, so we must have $U \neq V$ and similarly $W \neq X$, $Y \neq Z$. Thus the three rotations UV, WX, YZ must lie in the set $\{FG, GF\}$. So at least two of the maps R^2UV, $RWXR$, YZR^2 are equal.

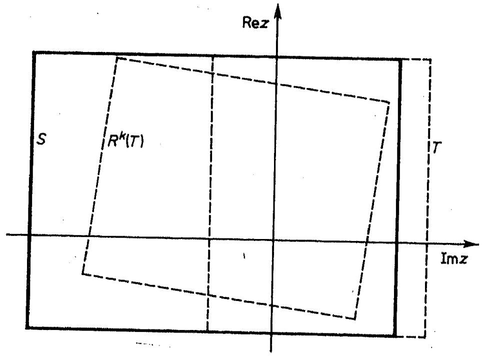
Case 6. F is a rotation fixing O and G is a reflection fixing O. For some $U, V, W, X, Y, Z \in \{F, G\}$, the points $R^2VU(z_0)$, $RWRU(z_0)$, $RU(z_0)$, $XR(z_0)$, $RYXR(z_0)$, $ZRXR(z_0)$, and $R^2(z_0)$ are vertices of Δ. These points and z_0 should be all different. If $U = G$, then $V = F$ (otherwise $R^2VU = R^2$) and $W = F$ (otherwise $RWRU$ is the identity), but then $R^2VU = RWRU$. If $U = F$ then $X = G$ (otherwise $RU = XR$), so $Y = F$ (otherwise $RYXR = R^2$) and $Z = F$ (otherwise $ZRXR$ is the identity), but then $RYXR = ZRXR$.

THEOREM 2. *There is a bounded* (2, 2)-*paradoxical subset X of the plane.*

Proof. We define the planar isometries R and F, and a subset E of C, exactly as in [MS]: $R(z) = e^i z$, $F(z) = z+1$ and E consists of the point O together with all the images of O under the action of the semigroup generated by R and F.

The (2, 2)-paradoxical subset X will be contained in E and also in the rectangular subset $S = [-8, 4]+i\,[-3, 6]$ of the complex plane.

Let $T = [-2, 5] + i[-3, 6]$. Fix $k > 0$ such that $R^k(T) \subset S$. The rotation R^k is then approximately a quarter turn counter-clockwise. See the figure below.



Let $P = [-8, -2] + i[-3, 6]$, and let $Q = (-2, 4] + i[-3, 6]$. Then $\{P, Q\}$ is a partition of S. Note that Q, $Q+1$, $P+6$ and $P+7$ are all subsets of T, so since $R^k(T) \subset S$, it follows that $R^k(Q)$, $R^kF(Q)$, $R^kF^6(P)$ and $R^kF^7(P)$ are all subsets of S. We define $X = \bigcup_{n=0}^{\infty} X_n$ as follows.

Let $X_0 = \{0\}$, $X_1 = \{R^kF(0)\}$. Define $X_2, X_3, \ldots$ inductively. Suppose $n \geqslant 1$, and we have defined X_n. For each point z in X_n, assume inductively that $z \in S = P \cup Q$.

If $z \in P$, put $R^kF^6(z)$ and $R^kF^7(z)$ into X_{n+1}.

If $z \in Q$, put $R^k(z)$ and $R^kF(z)$ into X_{n+1}.

This completes the definition of X. To show that X is (2, 2)-paradoxical we argue that $P_1 = P_2 = \{X \cap P, X \cap Q\}$ and $P_3 = \{R^kF^6(X \cap P), R^k(X \cap Q), R^kF^7(X \cap P), R^kF(X \cap Q)\}$ are partitions of X. $P_1 = P_2$ is clearly a partition of X. It is clear from the construction that $X = \bigcup P_3$ so it remains to show that P_3 is disjoint. Since e^i is transcendental, each point x in E other than 0 has a unique representation $L_n \ldots L_2L_1F(0)$ where $L_i \in \{F, R\}$, $n \in \{0, 1, \ldots\}$. In particular, suppose $x \in X = \bigcup P_3$. Then

$x \in R^kF^6(X \cap P) \Rightarrow x = R^kF^6R\ldots F(0)$; $x \in R^k(X \cap Q) \Rightarrow x = 0$ or $x = R^kR\ldots F(0)$;

$x \in R^kF^7(X \cap P) \Rightarrow x = R^kF^7\ldots F(0)$; $x \in R^kF(X \cap Q) \Rightarrow x = R^kF(0)$ or $x = R^kFR\ldots F(0)$.

By uniqueness of the representations on the right, P_3 is disjoint.

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