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Intersection properties of partitions of a cardinal

by

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Abstract. We study the properties P and R which are statements about families of functions, and are motivated by the characterization of Bernstein's property B (of families of sets) in terms of characteristic functions. In an earlier paper we applied constraints that were generalizations of those introduced by Erdös and Hajnal for families of sets.

Here we impose conditions that are of an opposite nature and have meaning only for families of functions. Positive results are obtained under weaker conditions, showing that these are more appropriate for families of functions.

Introduction. In this paper we study the properties P and R introduced in [2]. These are statements about families of functions, and are motivated by the characterization of Bernstein's property B (of families of sets A) in terms of characteristic functions χ_A. In [2] we imposed a condition, denoted by $C(2, \lambda)$, which is a direct generalization of the condition $C(2, \lambda)$ for sets, introduced by Erdös and Hajnal [1].

Here we look at families of functions all with the same domain (rather than of arbitrary domain), and constrained by intersection conditions that are in a sense opposite from those dealt with in [2]. The earlier intersection conditions require that like preimages are "well-spaced", while it seems more natural when considering families on a fixed domain to require that different preimages be separated.

We introduce the intersection condition $C[\eta, \lambda]$ on such a family, defined to mean that every intersection of the preimages of η different values is of size less than λ. Positive results are ensured even when the conditions are weaker than those of $C(\eta, \lambda)$, showing that $C[\eta, \lambda]$ is more appropriate for families of functions.

Background. A family of sets $\mathbf{A}$ is said to *possess property* B if there is a set T such that $A \cap T \neq \emptyset$ and yet $A \nsubseteq T$ for all sets A in $\mathbf{A}$. Equivalently:

$$\exists x(\chi_T(x) = \chi_A(x) = 1) \quad \text{and} \quad \exists y(\chi_T(y) \neq \chi_A(y) = 1).$$

Bernstein showed that a family of $\varkappa$ sets each of size $\varkappa$ always possess property B.

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Intersection conditions on A were introduced by Erdös and Hajnal in 1961 (see [1]) to guarantee that larger families of sets possess property B. The family A is said to satisfy the condition $C(\eta, \lambda)$ if $|\bigcap B| < \lambda$ for every B in $[A]^{\eta}$.

A strengthened version of property B was also considered by Erdös and Hajnal in [1]: A family of sets A has property $B(\mu)$ if it possesses property B and further the set T confirming property B is also a μ-transversal for A; that is, $1 \leqslant |T \cap A| < \mu$ for all sets A in A. The condition $C(\eta, \lambda)$ was again imposed on families of sets in order to ensure that property $B(\mu)$ holds. In this case it is easily shown that a $\varkappa$-size family of sets each of size $\varkappa$ and satisfying the condition $C(3, 1)$ need not possess property $B(\mu)$ for any non-trivial cardinal μ.

For this reason, only the effects of the conditions of the form $C(2, \lambda)$ were studied.

We study properties P and R which are statements about families of functions, and these are motivated by the characterisation of property B in terms of characteristic functions.

DEFINITIONS. A family of functions $A = \{\varphi_\alpha\colon \varkappa \to \theta;\ \alpha < \varrho\}$ is said to possess *property* P if there is a function $T\colon S \to \theta$ that satisfies

$$\mathrm{P}\colon\ \forall\alpha < \varrho\forall\beta < \theta\exists x, y\varepsilon\varkappa \cap S\big((T(x) = \varphi_\alpha(x) = \beta) \wedge (T(y) \neq \varphi_\alpha(y) = \beta)\big).$$

A is said to possess *property* R if T satisfies

$$\mathrm{R}\colon\ \forall\alpha < \varrho\exists\beta < \theta\exists x, y\varepsilon\varkappa \cap S\big((T(x) = \varphi_\alpha(x) = \beta) \wedge (T(y) \neq \varphi_\alpha(y) = \beta)\big).$$

The weaker versions of P and R are P′ and R′:

$$\mathrm{P}'\colon\ \forall\alpha < \varrho\forall\beta < \theta\exists x\varepsilon\varkappa \cap S\big((T(x) = \varphi_\alpha(x) = \beta)\big),$$

$$\mathrm{R}'\colon\ \forall\alpha < \varrho\exists\beta < \theta\exists x\varepsilon\varkappa \cap S\big((T(x) = \varphi_\alpha(x) = \beta)\big),$$

We say that a function $T\colon S \to \theta$ *witnesses property* P *for the function* φ if

$$\exists x\big(T(x) = \varphi(x) = \beta\big) \quad \text{and} \quad \exists y\big(T(y) \neq \varphi(y) = \beta\big)$$

for all $\beta < \theta$, and we say that T *witnesses property* P *for the family* A if T witnesses property P for every function φ in A. These definitions will have the obvious meanings when P is replaced by P′ or R. Note that we only consider families of functions that satisfy the *non-triviality condition* that every preimage of every function is large.

We also extend the μ-transversal property to families of functions:

A family of functions $A = \{\varphi_\alpha\colon S_\alpha \to \theta;\ \alpha < \varrho\}$ is said to possess the *μ-tranversal property* $Z(\mu)$ if there is a function $T\colon S \to \theta$ that satisfies

$$Z(\mu)\colon\ \forall\alpha < \varrho\forall\beta < \theta\big(|T^{-1}(\beta) \cap \varphi_\alpha^{-1}(\beta)| < \mu\big),$$

and T is said to *witness* $Z(\mu)$ *for* A. A is said to possess *property* $P(\mu)$ if there is a function $T\colon S \to \theta$ that witnesses both P and $Z(\mu)$ for A. Define $P'(\mu)$, $R(\mu)$ and $R'(\mu)$ similarly. $P(\mu)$ and $R(\mu)$ are called the *strong transversal properties* and $P'(\mu)$ and $R'(\mu)$ are called the *weak transversal properties*.

Preliminaries. The case for $\eta = 2$ is examined in Section 1. Unlike the situation with sets for $C(3, \lambda)$, we find in Section 2 that there are positive results when we impose conditions of the form $C[3, \lambda]$. There are also negative results that show that these positive results are the best possible. Finally, Section 3 shows that no positive results hold when we impose conditions $C[\eta, \lambda]$ with $\eta \geqslant 4$.

We shall consider families of functions of the form $A = \{\varphi_\alpha;\ \alpha < \varrho\}$ where each φ_α is in ${}^{\varkappa}\theta$. The family constructed in Theorem 1.2 of [2] shows that intersection conditions are necessary to ensure that a family of this form of size $2^{\varkappa}$ has property R for cardinals θ satisfying $3 \leqslant \theta \leqslant \varkappa$. For $\theta = 2$, property R is easily seen to be guaranteed without imposing intersection conditions.

The family constructed in Theorem 1.4 of [2] shows that intersection conditions are necessary to ensure that property P′ holds for all non-trivial cardinals θ.

We introduce the conditions which, by separating different preimages, will ensure further positive results.

DEFINITIONS. We say that the family of functions $\mathscr{A} = \{\varphi_\alpha\colon \alpha < \varrho\} \subseteq {}^{\varkappa}\theta$ satisfies the *condition* $C[2, \lambda]$ if

$$\forall\alpha_0, \alpha_1 < \varrho\forall\beta_0, \beta_1 < \theta(\beta_0 \neq \beta_1 \Rightarrow |\varphi_{\alpha_0}^{-1}(\beta_0) \cap \varphi_{\alpha_1}^{-1}(\beta_1)| < \lambda).$$

More generally, we define $C[\eta, \lambda]$ for cardinals η satisfying $\eta \leqslant \varkappa$. For sets A, B let $\mathrm{Inj}({}^{A}B)$ denote the set $\{f \in {}^{A}B;\ f \text{ is 1-1}\}$. Then $\mathscr{A}$ satisfies $C[\eta, \lambda]$ if

$$\forall f \in {}^{\eta}\varrho\forall g \in \mathrm{Inj}({}^{\eta}\theta)\big(\big|\bigcap\{\varphi_{f(\beta)}^{-1}(g(\beta));\ \beta < \eta\}\big| < \lambda\big).$$

We say that such a family where $|\varphi_\alpha^{-1}(\beta)| = \varkappa$ for all $\alpha < \varrho$ and $\beta < \theta$ is a *$[\varrho, \varkappa, \theta, \eta, \lambda]$-family* and let the relation $[\varrho, \varkappa, \theta, \eta, \lambda] \to P(\mu)$ mean that every $[\varrho, \varkappa, \theta, \eta, \lambda]$-family has property $P(\mu)$.

These definitions will have the obvious meanings when $P(\mu)$, $R(\mu)$ are replaced by $P'(\mu)$, $R'(\mu)$, P, R, or P′.

§ 1. **Families satisfying** $C[2, \lambda]$. We are ready to present the main positive results for families satisfying $C[2, \lambda]$. The later theorems of this section will show that these results are the best possible.

THEOREM 1.1. *If* $\aleph_0 \leqslant \lambda \leqslant \varkappa$ *and* $\theta < \lambda'$, *then*

$$[2^{\varkappa}, \varkappa, \theta, 2, \lambda] \to P(\lambda^+).$$

Proof. Suppose that the family $\mathscr{A} = \{\varphi_\alpha;\ \alpha < 2^{\varkappa}\}$ is a $[2^{\varkappa}, \varkappa, \theta, 2, \lambda]$-family. We shall construct a function $T\colon S_T \to \theta$ that witnesses property $P(\lambda^+)$. Choose any function from $\mathscr{A}$, say φ_0, and for each $\beta < \theta$ choose sets A_β, B_β such that $A_\beta, B_\beta \subseteq \varphi_0^{-1}(\beta)$ with $|A_\beta| = |B_\beta| = \lambda$ and $A_\beta \cap B_\beta = \emptyset$. We define T by its preimages. Let $\pi\colon \theta \leftrightarrow \theta$ be any permutation of θ with the property that $\pi(x) \neq x$ for all x in θ. For each $\beta < \theta$ put $T^{-1}(\beta) = A_\beta \cup B_{\pi(\beta)}$. We show that T witnesses property $P(\lambda^+)$ for $\mathscr{A}$.

Let $\alpha < 2^{\varkappa}$ and $\beta < \theta$. We claim $\varphi_\alpha^{-1}(\beta) \cap A_\beta \neq \emptyset$.

For suppose by way of contradiction that $\varphi_\alpha^{-1}(\beta)\cap A_\beta=\emptyset$. But $\varphi_0^{-1}(\beta)-\varphi_\alpha^{-1}(\beta) = \bigcup\{\varphi_0^{-1}(\beta)\cap\varphi_\alpha^{-1}(\gamma);\ \gamma\neq\beta\wedge\gamma<\theta\}$, and so $A_\beta\subseteq\bigcup\{\varphi_0^{-1}(\beta)\cap\varphi_\alpha^{-1}(\gamma);\ \gamma\neq\beta\wedge\gamma<\theta\}$. Because $|A_\beta|=\lambda$ and $\theta<\lambda'$, there must be $\gamma<\theta$ with $\gamma\neq\beta$ such that $|\varphi_0^{-1}(\beta)\cap\varphi_\alpha^{-1}(\gamma)|\geqslant\lambda$ and this contradicts $C[2,\lambda]$. A similar argument shows that $\varphi_\alpha^{-1}(\beta)\cap B_\beta\neq\emptyset$. Also, because $|T^{-1}(\beta)|=\lambda$ for all $\beta<\theta$, we have $|T^{-1}(\beta)\cap\varphi_\alpha^{-1}(\beta)|<\lambda^+$ for all $\alpha<2^\varkappa$. Hence the family $\mathscr{A}$ has property $\mathrm{P}(\lambda^+)$. ∎

Theorem 1.2 shows that the result of Theorem 1.1 is the best possible in the sense that property $\mathrm{P}(\lambda^+)$ cannot be strengthened to $\mathrm{P}(\lambda)$.

THEOREM 1.2. *Suppose* $\aleph_0\leqslant\lambda\leqslant\varkappa$ *and* $2\leqslant\theta\leqslant\varkappa$. *Then*

$$[(\varkappa^{<\lambda})^\theta,\ \varkappa,\ \theta,\ 2,\ \lambda]\nrightarrow\mathrm{R}'(\lambda).$$

Proof. We shall construct a $[(\varkappa^{<\lambda})^\theta,\varkappa,\theta,2,\lambda]$-family $\mathscr{A}$ which does not possess property $\mathrm{R}'(\lambda)$. Let $\varkappa=\dot{\bigcup}\{B_\beta;\ \beta<\theta\}$ be any disjoint partition of $\varkappa$ such that $|B_\beta|=\varkappa$ for all $\beta<\theta$. For each $\beta<\theta$, list the sets in $[B_\beta]^{<\lambda}$ by $[B_\beta]^{<\lambda}=\{C_{\beta\alpha};\ \alpha<\varkappa^{<\lambda}\}$. For each function $\varphi\colon\theta\to\varkappa^{<\lambda}$ define the induced function $\bar\varphi\colon\varkappa\to\theta$ by

$$\bar\varphi(x)=\begin{cases}\beta & \text{if } x\in B_\beta-C_{\beta\varphi(\beta)},\\ 0 & \text{if } x\in C_{\beta\varphi(\beta)}\wedge\beta\neq 0,\\ 1 & \text{if } x\in C_{0\varphi(0)}.\end{cases}$$

Put $\mathscr{A}=\{\bar\varphi;\ \varphi\in{}^\theta(\varkappa^{<\lambda})\}$ so $\mathscr{A}\subseteq{}^\varkappa\theta$. It is easily seen that $\mathscr{A}$ satisfies $C[2,\lambda]$, since for each $\beta<\theta$ we have $\bar\varphi^{-1}(\beta)\subseteq B_\beta\cup X_{\varphi\beta}$ where $X_{\varphi\beta}=\emptyset$ for $\beta\neq 0, 1$ and $|X_{\varphi\beta}|<\lambda$ for $\beta=0,1$. Also, each preimage $\varphi^{-1}(\beta)$ is of size $\varkappa$ because $|C_{\beta\varphi(\beta)}|<\lambda\leqslant\varkappa$. Suppose, for a contradiction, that there is a function $T\colon S_T\to\theta$ witnessing property $\mathrm{R}'(\lambda)$ for the family $\mathscr{A}$. We construct a function $\varphi\colon\theta\to\varkappa^{<\lambda}$ such that the induced function $\bar\varphi$ is not witnessed by T.

We define values $\varphi(\beta)$ for each $\beta\geqslant 2$. If $|T^{-1}(\beta)\cap B_\beta|<\lambda$, define $\varphi(\beta)$ by putting $T^{-1}(\beta)\cap B_\beta=C_{\beta\varphi(\beta)}$. Then $T^{-1}(\beta)\cap\bar\varphi^{-1}(\beta)=\emptyset$ since $\bar\varphi^{-1}(\beta)=B_\beta-C_{\beta\varphi(\beta)}$. If $|T^{-1}(\beta)\cap B_\beta|\geqslant\lambda$, define $\varphi(\beta)$ by putting $C_{\beta\varphi(\beta)}=\emptyset$. Then $|T^{-1}(\beta)\cap\bar\varphi^{-1}(\beta)| = |T^{-1}(\beta)\cap B_\beta|\geqslant\lambda$. In either case, T does not satisfy the requirements of property $\mathrm{R}'(\lambda)$ for the value β. Finally, it is easily seen that the values $\varphi(0)$ and $\varphi(1)$ may be defined so as to complete the proof. ∎

Theorem 1.3 deals with the case when $\theta\geqslant\lambda'$.

THEOREM 1.3. (i) *If* $\aleph_0\leqslant\lambda'\leqslant\theta<\lambda<\varkappa$, *then* $[\varkappa^\lambda,\varkappa,\theta,2,\lambda]\nrightarrow\mathrm{R}'(\lambda^+)$.

(ii) *If* $\aleph_0\leqslant\theta<\varkappa$, *then* $[\varkappa^\theta,\varkappa,\theta,2,2]\nrightarrow\mathrm{R}'(\theta^+)$.

Proof. (i) We construct a $[\varkappa^\lambda,\varkappa,\theta,2,\lambda]$-family which does not possess property $\mathrm{R}'(\lambda^+)$. Let $\varkappa=\dot{\bigcup}\{B_\beta;\ \beta<\theta\}$ be any disjoint partition of $\varkappa$ with $|B_\beta|=\varkappa$ for all $\beta<\theta$. For each $\beta<\theta$, list $[B_\beta]^{\leqslant\lambda}$ by $[B_\beta]^{\leqslant\lambda}=\{C_{\beta\alpha};\ \alpha<\varkappa^\lambda\}$ and for each $\alpha<\varkappa^\lambda$ list $C_{\beta\alpha}$ as follows. Let $\lambda=\Sigma\{\lambda_\sigma;\ \sigma<\lambda'\}$ where $\{\lambda_\sigma;\ \sigma<\lambda'\}$ is a strictly increasing sequence of cardinals each less than λ. If $|C_{\beta\alpha}|=\lambda$, put $C_{\beta\alpha}=\{x_{\beta\alpha}^{\gamma\sigma};\ \gamma<\lambda_\sigma,\ \sigma<\lambda'\}$. If $|C_{\beta\alpha}|<\lambda$, put $C_{\beta\alpha}=\{x_{\beta\alpha}^{\gamma 0};\ \gamma<|C_{\beta\alpha}|\}$. For each function $\varphi\colon\theta\to\varkappa^\lambda$ define the induced function $\bar\varphi\colon\varkappa\to\theta$ as follows. Let θ be partitioned by $\theta=\dot{\bigcup}\{A_\beta;\ \beta<\theta\}$ such that for each $\beta<\theta$, $\beta\notin A_\beta$ and $A_\beta=\{\tau_{\beta\varepsilon};\ \varepsilon<\theta\}$. Put

$$\bar\varphi(x)=\begin{cases}\beta & \text{if } x\in B_\beta-C_{\beta\varphi(\beta)},\\ \tau_{\beta\sigma} & \text{if } x=x_{\beta\varphi(\beta)}^{\gamma\sigma} \text{ for some } \gamma.\end{cases}$$

Define the family $\mathscr{A}$ by $\mathscr{A}=\{\bar\varphi;\ \varphi\in{}^\theta(\varkappa^\lambda)\}$. It is easily seen that $\mathscr{A}$ satisfies $C[2,\lambda]$ since for any $\beta<\theta$ and any $\bar\varphi$ in $\mathscr{A}$ there is $\sigma<\lambda'$ such that $|\bar\varphi^{-1}(\beta)-B_\beta|\leqslant\lambda_\sigma<\lambda$. Now, for a contradiction, suppose that there is a function $T\colon S_T\to\theta$ witnessing that $\mathscr{A}$ has property $\mathrm{R}'(\lambda^+)$. By considering whether there is $\beta<\theta$ such that $|T^{-1}(\beta)\cap B_\beta|>\lambda$, we may define $\varphi\colon\theta\to\varkappa^\lambda$ such that T does not witness property $\mathrm{R}'(\lambda^+)$ and we have the required contradiction.

(ii) We proceed as for part (i) except that we consider the set $[B_\beta]^{\leqslant\theta}$ for each $\beta<\theta$. List $[B_\beta]^{\leqslant\theta}$ by $[B_\beta]^{\leqslant\theta}=\{C_{\beta\alpha};\ \alpha<\varkappa^\theta\}$ and for each $\alpha<\varkappa^\theta$ list $C_{\beta\alpha}$ by $C_{\beta\alpha}=\{x_{\beta\alpha}^\varepsilon;\ \varepsilon<|C_{\beta\alpha}|\}$. For each $\beta<\theta$ let $\bar\beta$ denote the unique ordinal for which $\beta\in A_{\bar\beta}$. Let $A_\beta=\{\tau_{\beta\varepsilon};\ \varepsilon<\theta\}$ be any listing of A_β which has the property that

$$\tau_{\beta 0}=\begin{cases}\bar\beta & \text{if } \bar\beta\in A_\beta,\\ \text{arbitrary} & \text{if } \bar\beta\notin A_\beta.\end{cases}$$

Define $\bar\varphi\colon\varkappa\to\theta$ by

$$\bar\varphi(x)=\begin{cases}\beta & \text{if } x\in B_\beta-C_{\beta\varphi(\beta)},\\ \tau_{\beta\varepsilon+1} & \text{if } x=x_{\beta\varphi(\beta)}^\varepsilon \text{ for some } \varepsilon.\end{cases}$$

Put $\mathscr{A}=\{\bar\varphi;\ \varphi\in{}^\theta(\varkappa^\theta)\}\subseteq{}^\varkappa\theta$. It is easy to show that $\mathscr{A}$ satisfies $C[2,2]$. There are two cases as for the proof of part (i) and the argument follows through similarly. ∎

We now present theorems that investigate the case when $\theta=\varkappa$.

THEOREM 1.4. (i) $[2^\varkappa,\varkappa,\varkappa,2,3]\nrightarrow\mathrm{R}$ *for any* $\varkappa$.

(ii) $[2^\varkappa,\varkappa,\theta,2,\varkappa]\nrightarrow\mathrm{R}$ *if* $\varkappa'\leqslant\theta<\varkappa$.

Proof. (i) Let $\varkappa=\dot{\bigcup}\{B_\beta;\ \beta<\varkappa\}$ be any disjoint partition of $\varkappa$ with always $|B_\beta|=\varkappa$ and let $\varkappa=\dot{\bigcup}\{A_\beta;\ \beta<\varkappa\}$ be a disjoint partition of $\varkappa$ such that $\beta\notin A_\beta$. Define the family of functions $\mathscr{A}_0$ by

$$\mathscr{A}_0=\big\{\varphi\colon\varkappa\to\varkappa;\ \forall\beta<\varkappa\big((|\varphi^{-1}(\beta)|=\varkappa)\wedge(|\varphi^{-1}(\beta)-B_\beta|\leqslant 1)\big)\big\}.$$

So for all $\beta<\varkappa$ and every φ in $\mathscr{A}_0$ we have $\varphi^{-1}(\beta)=(\varphi^{-1}(\beta)\cap B_\beta)\cup X_{\varphi\beta}$ where $|X_{\varphi\beta}|\leqslant 1$ and $X_{\varphi\beta}\cap B_\beta=\emptyset$. It follows that $\mathscr{A}_0$ satisfies $C[2,3]$.

Define the family of functions $\mathscr{A}_1$ by

$$\mathscr{A}_1=\{\varphi\colon\varkappa\to\varkappa;\ \forall\beta<\varkappa\big((|\varphi^{-1}(\beta)|=\varkappa)\wedge\big(\forall\gamma<\varkappa(|\varphi^{-1}(\beta)\cap B_\gamma|\leqslant 1)$$
$$\wedge\big(|\varphi^{-1}(\beta)-\dot{\bigcup}\{B_\gamma;\ \gamma\in A_\beta\}|\leqslant 1\big)\big)\big\}.$$

So for all $\beta<\varkappa$ and every φ in $\mathscr{A}_1$ we have

$$\varphi^{-1}(\beta)=\big(\varphi^{-1}(\beta)\cap\dot{\bigcup}\{B_\gamma;\ \gamma\in A_\beta\}\big)\cup X_{\varphi\beta}$$

where $|X_{\varphi\beta}|\leqslant 1$ and $X_{\varphi\beta}\cap\dot{\bigcup}\{B_\gamma;\ \gamma\in A_\beta\}=\emptyset$. It follows that $\mathscr{A}_1$ satisfies $C[2,3]$.

Put $\mathscr{A}=\mathscr{A}_0\cup\mathscr{A}_1$. It can easily be shown that $\mathscr{A}$ satisfies $C[2,3]$.

Suppose for a contradiction that there is a function $T\colon S_T \to \varkappa$ that witnesses property R for the family $\mathscr{A}$. Put $E = \{\beta < \varkappa;\ |B_\beta - T^{-1}(\beta)| = \varkappa\}$. Firstly suppose that $|E| = \varkappa$, and construct $\varphi\colon \varkappa \to \varkappa$ as follows. For each β in E, put $B_\beta - T^{-1}(\beta) \subseteq \varphi^{-1}(\beta)$, and for each β not in E, put $B_\beta \cap T^{-1}(\beta) = \varphi^{-1}(\beta)$. List the remaining elements of $\varkappa$ by

$$\varkappa - [\bigcup \{B_\beta - T^{-1}(\beta);\ \beta \in E\} \cup \bigcup \{B_\beta \cap T^{-1}(\beta);\ \beta \notin E\} = \{v_\alpha;\ \alpha < \varrho\}$$

where ϱ is some cardinal less than or equal to $\varkappa$, and for each $\alpha < \varrho$ choose inductively values $\varphi(v_\alpha)$ from $E - [\{\varphi(v_\varepsilon);\ \varepsilon < \alpha\} \cup \{T(v_\alpha)\}]$. This completes the definition of φ and it is clear that $\varphi \in \mathscr{A}_0$, and that T does not witness property R for φ.

On the other hand, suppose $|E| < \varkappa$. Construct $\varphi\colon \varkappa \to \varkappa$ as follows. For each γ in $\varkappa - E$ choose $x_\gamma \in B_\gamma \cap T^{-1}(\gamma)$ and put $\varphi(x_\gamma) = \beta$ if $\gamma \in A_\beta$. Note that $\varphi(x_\gamma) \neq T(x_\gamma)$ since $\gamma \notin A_\gamma$. For all $\beta < \varkappa$ we now have that $|\varphi^{-1}(\beta)| = \varkappa$ since $|E| < \varkappa$. List the remaining elements of $\varkappa$ by $\varkappa - \{x_\gamma;\ \gamma \notin E\} = \{w_\alpha;\ \alpha < \varrho\}$ where ϱ is some cardinal no greater than $\varkappa$, and for each $\alpha < \varrho$ put $\beta(\alpha) = v$ *if* $w_\alpha \in B_\gamma$ and $\gamma \in A_v$. For each $\alpha < \varrho$ choose inductively values $\varphi(w_\alpha)$ from

$$\varkappa - [\{\varphi(w_\varepsilon);\ \varepsilon < \alpha\} \cup \{T(w_\alpha)\} \cup \{\beta(\alpha)\}].$$

This proves part (i).

(ii) We use a method similar to that in part (i). Let $\varkappa = \dot{\bigcup}\{B_\beta;\ \beta < \theta\}$ be a partition of $\varkappa$ such that $|B_\beta| = \varkappa$ for all $\beta < \theta$, and let $\theta = \dot{\bigcup}\{A_\beta;\ \beta < \theta\}$ be such that $\beta \notin A_\beta$. Define the family $\mathscr{A}_0$ by

$$\mathscr{A}_0 = \{\varphi\colon \varkappa \to \theta;\ \forall \beta < \theta\big((|\varphi^{-1}(\beta)| = \varkappa) \wedge (|\varphi^{-1}(\beta) - B_\beta| < \varkappa)\big)\},$$

and $\mathscr{A}_1$ by

$$\mathscr{A}_1 = \{\varphi\colon \varkappa \to \theta;\ \forall \beta < \theta\big((|\varphi^{-1}(\beta)| = \varkappa) \wedge (\forall \gamma < \theta(|\varphi^{-1}(\beta) \cap B_\gamma| < \varkappa))$$
$$\wedge (|\varphi^{-1}(\beta) - \dot{\bigcup}\{B_\gamma;\ \gamma \in A_\beta\}| < \varkappa)\big)\}.$$

Put $\mathscr{A} = \mathscr{A}_0 \cup \mathscr{A}_1$, and it follows that $\mathscr{A}$ satisfies $C[2, \varkappa]$.

Now suppose for a contradiction that T witnesses property R for $\mathscr{A}$ and put $E = \{\beta < \theta;\ |B_\beta - T^{-1}(\beta)| = \varkappa\}$.

The proof follows through by considering the cases $|E| = \theta$ and $|E| < \theta$. ■

Theorem 1.4 leaves open the truth of the relation $[2^\varkappa, \varkappa, \varkappa, 2, 2] \to \mathrm{R}$.

We now consider the property P′. Theorem 1.5 presents a stronger negation under the weak constraint imposed by $C[2, \varkappa]$. Theorem 1.6 negates property P′ under the strong constraint $C[2, 2]$.

THEOREM 1.5 (GCH). *Let $\varkappa$ be a regular cardinal. Then* $[2^\varkappa, \varkappa, \varkappa, 2, \varkappa] \nrightarrow \mathrm{P}'$.

Proof. We shall actually prove a stronger result, namely that there is a family $\mathscr{A} = \{\psi_\alpha;\ \alpha < 2^\varkappa\}$ of functions $\psi_\alpha\colon \varkappa \to \varkappa$ for which there is no $T\colon \varkappa \to \varkappa$ satisfying

$$\forall \alpha(|\{\beta;\ \psi_\alpha^{-1}(\beta) \cap T^{-1}(\beta) \neq \emptyset\}| \geqslant 2).$$

Thus for every function $T\colon \varkappa \to \varkappa$ there is some α such that $\psi_\alpha^{-1}(\beta) \cap T^{-1}(\beta) \neq \emptyset$ for at most one β. Recall that for property P′ we need only consider T whose domain is equal to $\varkappa$.

We construct the family $\mathscr{A}$ inductively as follows. Noting that, by the GCH, the family of functions $\{T;\ T \in {}^\varkappa\varkappa\}$ is of size $\varkappa^+$, we list this family by $\{T_\alpha;\ \alpha < \varkappa^+\}$ and for each $\alpha < \varkappa^+$ we construct a function $\psi_\alpha\colon \varkappa \to \varkappa$ which agrees with T_α at no more than one value, and also satisfies $C[2, \varkappa]$ with respect to the previous functions (of which there are no more than $\varkappa$). So it suffices to prove the following statement:

Given $T\colon \varkappa \to \varkappa$ and $\{\varphi_\alpha;\ \alpha < \varkappa\} \subseteq {}^\varkappa\varkappa$ that statisfies $C[2, \varkappa]$ and $\forall \alpha, \beta < \varkappa$ $(|\varphi_\alpha^{-1}(\beta)| = \varkappa)$, then there is a function $\varphi\colon \varkappa \to \varkappa$ such that for all $\beta, \mu, \gamma < \varkappa$ with $\beta \neq \mu$ we have $|\varphi^{-1}(\beta)| = \varkappa$ and $|\varphi^{-1}(\mu) \cap \varphi_\gamma^{-1}(\beta)| < \varkappa$ and $|\{\beta < \varkappa;\ \varphi^{-1}(\beta) \cap T^{-1}(\beta) \neq \emptyset\}| \leqslant 1$.

We leave this to the reader, noting that the proof follows by considering whether there are ordinals σ, $\sigma < \varkappa$ for which

$$|(\varkappa - T^{-1}(v)) - \bigcup\{\varphi_\gamma^{-1}(\beta);\ \gamma, \beta < \sigma \wedge \beta \neq v\}| < \varkappa. \ ■$$

THEOREM 1.6. *For any cardinal $\varkappa$,* $[2^\varkappa, \varkappa, \varkappa, 2, 2] \nrightarrow \mathrm{P}'$.

Proof. Let $\varkappa = \dot{\bigcup}\{B_\beta;\ 0 < \beta < \varkappa\}$ be any disjoint partition of $\varkappa$ with $|B_\beta| = \varkappa$ for all β and list each B_β by $B_\beta = \{x_{\beta\alpha};\ \alpha < \varkappa\}$. For each function $\varphi\colon (\varkappa - \{0\}) \to \varkappa$ define an induced function $\bar{\varphi}\colon \varkappa \to \varkappa$ by

$$\bar{\varphi}(x) = \begin{cases} \beta & \text{if } x = x_{\beta\alpha} \wedge \alpha \neq \varphi(\beta), \\ 0 & \text{if } x = x_{\beta\alpha} \wedge \alpha = \varphi(\beta). \end{cases}$$

Put $\mathscr{A} = \{\bar{\varphi};\ \varphi \in {}^\varkappa\varkappa\}$. For each $\bar{\varphi}$ in $\mathscr{A}$ and for all $\beta < \varkappa$ it is easily checked that $|\bar{\varphi}^{-1}(\beta)| = \varkappa$ and that $\mathscr{A}$ satisfies $C[2, 2]$. Suppose for a contradiction that there is a function $T\colon \varkappa \to \varkappa$ that witnesses that $\mathscr{A}$ has property P′. By considering whether there is non-zero β such that $B_\beta \subseteq T^{-1}(0)$ we can construct a φ such that T does not witness P′. ■

§ 2. Families satisfying $C[3, \lambda]$. First we present the main positive result for this case.

THEOREM 2.1. *Suppose $\theta < \lambda' \leqslant \lambda < \varkappa$. Then* $[2^\varkappa, \varkappa, \theta, 3, \lambda] \to \mathrm{R}$.

Proof. Let $\mathscr{A} = \{\varphi_\alpha;\ \alpha < 2^\varkappa\} \subseteq {}^\varkappa\theta$ be any $[2^\varkappa, \varkappa, \theta, 3, \lambda]$-family. We may assume that $\mathscr{A}$ does not satisfy $C[2, \varkappa]$, otherwise Theorem 1.1 applies. So assume that $|\varphi_0^{-1}(0) \cap \varphi_1^{-1}(1)| = \varkappa$. Decompose this set by $\varphi_0^{-1}(0) \cap \varphi_1^{-1}(1) = A \dot{\cup} B$ where $|A| = \varkappa$ and $|B| = \lambda$. Then this induces a decomposition of $\varkappa$, being $\varkappa = A \dot{\cup} B \dot{\cup} C$ where $|C| = \varkappa$ (since $\varphi_0^{-1}(1) \subseteq C$). Define $T\colon \varkappa \to \theta$ by

$$T(x) = \begin{cases} 0 & \text{if } x \in B, \\ 1 & \text{if } x \in A \dot{\cup} C. \end{cases}$$

It follows easily that T witnesses property R for $\mathscr{A}$. ■

The next theorem (without proof) shows that the result of Theorem 2.1 is best possible in the sense that property R cannot be strengthened.

THEOREM 2.2. *If $2 \leqslant \theta \leqslant \varkappa$, then* $[2^\varkappa, \varkappa, \theta, 3, 1] \nrightarrow \mathrm{P}$. ■

When $\theta < \varkappa$ the questions for property P′ are left open, as are those for property R (except when $\theta < \lambda' \leqslant \lambda < \varkappa$).

§ 3. Families satisfying weak conditions. We now examine the question for the cases when the parameter η in the condition $C[\eta, \lambda]$ is greater than three. The proofs are left to the reader.

THEOREM 3.1. (i) $[2^{\varkappa}, \varkappa, \theta, 4, 1] \nleftrightarrow \mathrm{R}$ if $3 \leqslant \theta \leqslant \varkappa$;

(ii) $[2^{\varkappa}, \varkappa, \theta, 4, 1] \nleftrightarrow \mathrm{P}'$ if $2 \leqslant \theta \leqslant \varkappa$. ■

Theorem 3.1 shows that under these weaker conditions *all* questions are solved in the negative. This appears to correspond to the case $C(3, \lambda)$ in [2].

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More on locally atomic models

by

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Abstract.

THEOREM. *Assume T is a stable theory with* $\varkappa(T) \leqslant \aleph_1$. *If* $|T| < \mathrm{cov}\,\boldsymbol{K} + \mathfrak{b} + \min(\mathrm{cov}\,\boldsymbol{L}, \mathfrak{d})$, *then any* $A \subseteq \mathfrak{C}$ *can be extended to a model of T locally atomic over A.*

This improves Theorem 2.2 from [N2], in which we have a stronger assumption that $\varkappa(T) = \aleph_0$ (i.e. T is superstable). The coefficients bounding $|T|$ above are defined in terms of measure and category on the real line, and can vary between $\aleph_1$ and $2^{\aleph_0}$.

§ 1. Introduction. Throughout, we use the same standard terminology as in [N2]. In particular, T is a fixed first-order theory in language L, $\mathfrak{C}$ is the monster model of T, i.e. a very saturated model of T of high cardinality, such that all models of T under consideration are elementary submodels of $\mathfrak{C}$. For a formula $\theta \in L(\mathfrak{C})$, $[\theta]$ is the class of types containing θ. A is a set of parameters from $\mathfrak{C}$. $L(A)$ is the set of formulas with parameters from A, $S(A)$ is the set of complete 1-types over A. $p \in S(A)$ is locally isolated if for every $\varphi(x, \bar{y}) \in L$ there is a $\psi \in p$ such that $\psi \vdash p|\varphi$, i.e. for every $\bar{a} \in A$, $\psi(x)$ implies either $\varphi(x, \bar{a})$ or $\neg\varphi(x, \bar{a})$. A model M of T containing A is locally atomic over A if for each $\bar{a} \in M$, $\mathrm{tp}(\bar{a}/A)$ is locally isolated. The notion of local isolation, invented by Shelah, is fundamental in stability theory. It is one of the main tools to construct models of stable theories in the non-totally transcendental case.

To understand the paper, no deep understanding of stability theory is necessary. In particular, the reader does not have to know what $\varkappa(T)$ is, provided he is willing to accept Lemma 1, (2)→(3) without proof.

Now we explain what the real line coefficients $\mathfrak{b}$, $\mathfrak{d}$, $\mathrm{cov}\,\boldsymbol{K}$ and $\mathrm{cov}\,\boldsymbol{L}$ are.

$$\mathfrak{b} = \min\{|A|\colon\ A \subseteq {}^{\omega}\omega \ \&\ \forall f \in {}^{\omega}\omega\ \exists g \in A\ \exists^{\infty} n\ f(n) < g(n)\},$$
$$\mathfrak{d} = \min\{|A|\colon\ A \subseteq {}^{\omega}\omega \ \&\ \forall f \in {}^{\omega}\omega\ \exists g \in A\ \forall n\ f(n) < g(n)\}.$$

Thus $\mathfrak{b}$ is the minimal power of an unbounded family of reals, and $\mathfrak{d}$ is the minimal power of a dominating family of reals. If $\boldsymbol{I} \subseteq \mathscr{P}(X)$ then we define

$$\mathrm{cov}\,\boldsymbol{I} = \min\{|A|\colon\ A \subseteq \boldsymbol{I} \ \&\ \bigcup A = X\}.$$

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