Embedding of Boolean algebras in $P(\omega)/\mathrm{fin}$

by

J. Baumgartner (Hanover, N.H.), **R. Frankiewicz** (Warszawa)
and **P. Zbierski** (Warszawa)

Abstract. A model of ZFC set theory is constructed in which $\mathfrak{c}$ (the continuum) is large, MA (σ-linked) holds and each compact 0-dimensional space of weight $\mathfrak{c}$ is a continuous image of ω^*.

A well-known theorem of Parovichenko (see e.g. [C–N]) says that, if CH (the continuum hypothesis) is assumed, then each Boolean algebra of cardinality $\leqslant \mathfrak{c} = 2^{\omega}$ can be embedded into $P(\omega)/\mathrm{fin}$. What happens if CH fails? Kunen [K] proved that after adding ω_2 Cohen reals to a model of CH, there is an algebra of cardinality $\mathfrak{c}$ which cannot be embedded into $P(\omega)/\mathrm{fin}$. In [F] a model of ZFC+MA is constructed in which $\mathfrak{c} = 2^{\omega_1}$ and the powerset algebra $P(\omega_1)$ cannot be embedded into $P(\omega)/\mathrm{fin}$. There is also a model in which $\mathfrak{c} = \omega_2$ and the measure algebra does not embed into $P(\omega)/\mathrm{fin}$. On the other hand, Laver shows (in [L]) that, consistently with a large value for $\mathfrak{c}$, each linear ordering of cardinality $\leqslant \mathfrak{c}$ embeds in ω^{ω} (ordered by eventual dominance)([1]).

Using Laver's technique we prove here the following:

THEOREM. *There is a model of* ZFC *set theory with arbitrarily large* $\mathfrak{c}$ *in which each Boolean algebra of cardinality* $\leqslant \mathfrak{c}$ *can be embedded into* $P(\omega)/\mathrm{fin}$. *In addition, the Martin Axiom for σ-linked orderings holds in the model.*

In particular, it follows that the conclusion of the Parovichenko theorem is not equivalent to CH.

Proof. In the ground model V we fix a regular cardinal $\varkappa > \omega_1$, with $2^{<\varkappa} = \varkappa$, and force with a $\varkappa$-stage finite support iteration $\langle P_\alpha, Q_\alpha\colon \alpha < \varkappa\rangle$ so that the resulting extension $V[G]$ will have the required properties.

In a usual way choose names Q_α, $\alpha < \varkappa$ and α odd, so that

$$P_\alpha \Vdash \text{“}Q_\alpha \text{ is } \sigma\text{-linked”}$$

and $V[G] \models \mathrm{MA}$ (σ-linked).

([1]) The consistency of ZFC+$\neg$CH+each Boolean algebra of size $\leqslant \mathfrak{c}$ can be embedded into $P(\omega)/\mathrm{fin}$ has been pointed out without proof by Baumgartner in his article *Application of the proper forcing axiom* in *Handbook of Set-Theoretic Topology.*

At even stages α, Q_α adds a subset $A_\alpha \subseteq \omega$ which fills a certain gap in $P(\omega)/\text{fin}$. To be more precise, let $C(\alpha)$ consist of finite zero-one functions s with $dm(s) \subseteq \alpha$. Set

$$D(\alpha) = \{s \in C(\alpha)\colon \forall \xi \in dm(s)\ \xi \text{ is even}\}.$$

Each $s \in D(\alpha)$ determines a Boolean combination

$$A(s) = \bigcap_{s(\xi)=0} A_\xi \cap \bigcap_{s(\eta)=1} (\omega \backslash A_\eta).$$

Now, in $V^{(P_\alpha)}$ a pair $\langle S, T\rangle$ is called a *generic gap* if $S, T \subseteq D(\alpha)$ and the following two properties hold:

(1) $A(s) \cap A(t) \in \text{fin}$, for each $s \in S$ and $t \in T$.

(2) If $A(s) \subseteq_* A(s_1) \cup \ldots \cup A(s_n)$, for some $s \in D(\alpha)$ and $s_1, \ldots, s_n \in S$, then $s \in S$; similarly for T.

Standard reasoning shows that there is a sequence $\langle S_\alpha, T_\alpha\rangle$ such that $\langle S_\alpha, T_\alpha\rangle$ is a P_α-name of a generic gap in $V^{(P_\alpha)}$ and each $<\varkappa$-generated gap occurs in the sequence, for arbitrarily large $\alpha < \varkappa$.

Now, Q_α consists of $q = \langle x_q, a_q, y_q, m_q\rangle$, where $x_q \subseteq S_\alpha$, $y_q \subseteq T_\alpha$ are finite, a_q is a zero-one sequence of length m_q and

$$\bigcup_{s \in x_q} A(s) \cap \bigcup_{t \in y_q} A(t) \subseteq m_q.$$

The ordering on Q_α is defined thus:

$$p \leqslant q \quad \text{iff} \quad x_p \supseteq x_q,\ a_p \supseteq a_q,\ y_p \supseteq y_q,\ m_p \geqslant m_q$$

and for each i with $m_q \leqslant i < m_p$ we have

$$a_p(i) = 0, \quad \text{for } i \in \bigcup \{A(s)\colon s \in x_q\},$$

$$a_p(i) = 1, \quad \text{for } i \in \bigcup \{A(t)\colon t \in y_q\}.$$

If $H \subseteq Q_\alpha$ is a generic filter, then

$$A_\alpha = \{i \in \omega\colon \exists q \in H\ [a_q(i) = 0]\}$$

fills the gap $\langle S_\alpha, T_\alpha\rangle$, i.e. we have

$$A_s \subseteq_* A_\alpha, \quad \text{for each } s \in S_\alpha,$$

$$A_t \cap A_\alpha \in \text{fin}, \quad \text{for each } t \in T_\alpha.$$

Remark. Let us note the following property of Q_α. Let $s \in D(\alpha)$. If $A(s) \subseteq_* A_\alpha$, then $s \in S_\alpha$. Similarly, if $A(t) \cap A_\alpha =_* \emptyset$, then $t \in T_\alpha$. Indeed, $q \Vdash A(s) \backslash n \subseteq A_\alpha$ implies

$$A(s) \backslash \max\{n, m_q\} \subseteq \bigcup \{A(s')\colon s' \in x_q\}$$

for otherwise we have a $q' \leqslant q$ and an $i \in A(s) \backslash n$ with $a_{q'}(i) = 1$. Thus, $s \in S_\alpha$, by the condition (2) above.

It remains to show that $P_\varkappa$ has the c.c.c. Indeed, assuming that for the moment, it is clear that $\mathfrak{c} = \varkappa$ and MA (σ-linked) hold in the resulting model $V[G]$. Suppose that

$B = \{b_\alpha\colon \alpha < \varkappa\}$ is a Boolean algebra and let b_α be such that $B_{\alpha+1}$ is generated by B_α and b_α, for $\alpha < \varkappa$. In view of the Sikorski extension theorem it suffices to find a function $f\colon \varkappa \to \varkappa$ (with even values) such that for each $s \in C(\varkappa)$ we have

$$(*) \qquad b(s) = 0 \quad \text{iff} \quad A(s^f) \in \text{fin},$$

(where $b(s) = (\prod_{s(\xi)=0} b_\xi)\cdot(\prod_{s(\eta)=1} b_\eta)$) and $s^f(f(\beta)) = s(\beta)$, for $\beta \in dm(s)$.

Assume that f has already been defined for all $\beta < \alpha$ so that $(*)$ holds for each $s \in C(\alpha)$ and b_α not in the subalgebra generated by $\{b_\beta\colon \beta < \alpha\}$. Let

$$S = \{s^f\colon s \in C(\alpha) \text{ and } b(s) \leqslant b_\alpha\},$$

$$T = \{s^f\colon s \in C(\alpha) \text{ and } b(s)\cdot b_\alpha = 0\}.$$

Now, $\langle S, T\rangle$ generates a generic gap $\langle S_\gamma, T_\gamma\rangle$ at some stage $\gamma > \sup\{f(\beta)\colon \beta < \alpha\}$ and we define $f(\alpha) = \gamma$. To check $(*)$, let $\max dm(s) = \alpha$ and for example $s(\alpha) = 1$. Then $b(s) = 0$ implies $b(s|\alpha) \leqslant b_\alpha$ and hence $(s|\alpha)^f \in S$, which gives $A((s|\alpha)^f) \subseteq_* A_\gamma$, i.e. $A(s^f) =_* \emptyset$.

Conversely, from $A(s^f) \in \text{fin}$ follows $A((s|\alpha)^f) \subseteq_* A_\gamma$ and hence $(s|\alpha)^f \in S_\gamma$ by the Remark above. This means

$$A((s|\alpha)^f) \subseteq_* A(s_1^f) \cup \ldots \cup A(s_n^f)$$

for some $s_1^f, \ldots, s_n^f \in S$. By the inductive assumption

$$b(s|\alpha) \leqslant b(s_1) + \ldots + b(s_n).$$

Since $b(s_1), \ldots, b(s_n) \leqslant b_\alpha$ we obtain $b(s|\alpha) \leqslant b_\alpha$ and hence $b(s) = 0$.

The rest of the paper contains the proof of the countable chain condition.

Let $E_\alpha \subseteq P_\alpha$ consist of all $p \in P_\alpha$ having the following properties:

(3) For each even $\xi \in \text{supp}(p)$, there are x_ξ, a_ξ, y_ξ such that $p|\xi \Vdash p(\xi) = \langle x_\xi, x_\xi, y_\xi\rangle$, for each $s \in x_\xi \cup y_\xi$ with $dm(s) \subseteq \text{supp}(p)$.

(4) There is a number $l = l(p)$ such that for each even $\xi \in \text{supp}(p)$, we have $l(a_\xi) = l(p)$ (i.e. all the a's are of the same length).

For odd α we have $P_\alpha \Vdash$ "Q_α is σ-linked", and hence we can choose a P_α-name h_α such that

$$P_\alpha \Vdash \text{“}h_\alpha\colon Q_\alpha \to \omega \text{ and } \forall n\ h^{-1}(n) \text{ is linked”}.$$

Let E_α^* consist of all $p \in E_\alpha$ which in addition satisfy

(5) For each odd $\xi \in \text{supp}(p)$, there is an $n < \omega$ such that

$$p|\xi \Vdash \text{“}h_\xi(p(\xi)) = n\text{”}.$$

Lemma 1. *For each $p \in P_\alpha$ and $m \in \omega$, there is a $q \leqslant p$ such that $q \in E_\alpha^*$ and $l(q) \geqslant m$.*

Proof. By induction on α. Since supports are finite, it suffices to consider only the case $\alpha = \beta+1$ and $\beta \in \text{supp}(p)$. If β is odd, we find a $q' \leqslant p|\beta$ and an $n \in \omega$ with $q' \Vdash$ "$h_\beta(p(\beta)) = n$". By the inductive assumption, there is a $q'' \leqslant q'$ such that $q'' \in E_\beta^*$ and

$l(q'') \geqslant m$. Now, if $q \in \boldsymbol{P}_\alpha$ is defined by $q|\beta = q''$ and $q(\beta) = p(\beta)$, then $q \leqslant p$, $q \in E_\alpha^*$ and $l(q) \geqslant m$. If β is even, then we find a $q' \leqslant p|\beta$ and $\langle x_\beta, a_\beta, y_\beta\rangle$ so that

$$q' \Vdash \text{“}p(\beta) = \langle x_\beta, a_\beta, y_\beta\rangle\text{”}$$

and $dm(s) \subseteq \operatorname{supp}(q')$, for each $s \in x_\beta \cup y_\beta$. By the inductive assumption, there is a $q'' \in E_\beta^*$ with $l(q'') \geqslant \max\{m, l(a_\beta)\}$. In particular,

$$q'' \Vdash \text{“}A(s) \cap A(t) \subseteq l(a_\beta)\text{”}, \quad \text{for } s \in x_\beta \text{ and } t \in y_\beta,$$

and q'' determines the segments $A_\xi \cap l(q'')$, for each $\xi \in dm(s)$ and $s \in x_\beta \cup y_\beta$. Hence, we can define an $a \supseteq a_\beta$ with $l(a) = l(q'')$ such that $q'' \Vdash \text{“}\langle x_\beta, a, y_\beta\rangle \leqslant \langle x_\beta, a_\beta, y_\beta\rangle\text{”}$.

Now, if $q \in \boldsymbol{P}_\alpha$ is such that $q|\beta = q''$ and $q(\beta) = \langle x_\beta, a, y_\beta\rangle$, then q has the required properties.

LEMMA 2. *Let $p \in E_\alpha$, $s \in D(\alpha)$ and $\beta = \max dm(s)$. If $p \Vdash \text{“}A(s) \in \text{fin”}$, then there is a $q \leqslant p$ with $q \in E_\alpha$ and $l(q) = l(p)$, such that if $q|\beta \Vdash \text{“}q(\beta) = \langle x_\beta, a_\beta, y_\beta\rangle\text{”}$, then*

$$s|\beta \in x_\beta \quad \textit{if } s(\beta) = 1,$$

$$s|\beta \in y_\beta \quad \textit{if } s(\beta) = 0.$$

Proof. Induction on α. From the assumptions of the lemma it follows that $p|(\beta+1) \Vdash \text{“}A(s) \in \text{fin”}$ and hence we may assume $\alpha = \beta+1$. Suppose

$$p|\beta \Vdash p(\beta) = \langle x, a, y\rangle$$

and let $s(\beta) = 1$. Thus, we have

$$p|(\beta+1) \Vdash \text{“}A(s|\beta) \subseteq_* A_\beta\text{”}$$

and hence $p|\beta \Vdash \text{“}s|\beta \in S_\beta\text{”}$. In particular,

$$p|\beta \Vdash \text{“}A(s|\beta) \cap A(t) \in \text{fin”}, \quad \text{for each } t \in y.$$

We have to show

$$p|\beta \Vdash \text{“}A(s|\beta) \cap A(t) \subseteq l(p)\text{”}, \quad \text{for } t \in y.$$

If $s|\beta \cup t$ is not a function, then we have

$$\boldsymbol{P}_\beta \Vdash \text{“}A(s|\beta) \cap A(t) = \emptyset\text{”}.$$

If $s|\beta \cup t$ is a function, then $A(s|\beta) \cap A(t) = A(s|\beta \cup t)$ and we can apply the inductive assumption: there is a $q_t \in E_\beta$ with $q_t \leqslant p|\beta$ and $l(q_t) = l(p)$ such that if $\beta_t = \max dm(s|\beta \cup t)$ and

$$q_t|\beta_t \Vdash \text{“}q_t(\beta) = \langle x_t, a_t, y_t\rangle\text{”}$$

then $(s|\beta \cup t)|\beta_t$ is in x_t or y_t according to the value at β_t. In either case we obtain

$$q_t \Vdash \text{“}A(s|\beta \cup t) \subseteq l(q_t) = l(p)\text{”}.$$

Repeating this for each $t \in y$ such that $s|\beta \cup t$ is a function, we find a $q' \in E_\beta$ with $q' \leqslant p|\beta$, $l(q') = l(p)$ and

$$q' \Vdash A(s|\beta) \cup A(t) \subseteq l(p), \quad \text{for each } t \in y.$$

Hence, if $q|\beta = q'$ and $q(\beta) = \langle x \cup \{s|\beta\}, a, y\rangle$, then q has the required properties. The case $s(\beta) = 0$ is similar.

LEMMA 3. *Assume that $p, q \in E_\alpha^*$, $l(p) = l(q)$ and for each $\xi \in \operatorname{supp}(p) \cap \operatorname{supp}(q)$:*

(6) if ξ is odd and $p|\xi \Vdash \text{“}h_\xi(p(\xi)) = n\text{”}$, $q|\xi \Vdash \text{“}h_\xi(q(\xi)) = m\text{”}$, then $n = m$.

(7) if ξ is even and $p|\xi \Vdash \text{“}p(\xi) = \langle x_\xi, a_\xi, y_\xi\rangle\text{”}$, $q|\xi \Vdash \text{“}q(\xi) = \langle u_\xi, b_\xi, v_\xi\rangle\text{”}$, then $a_\xi = b_\xi$.

Then p and q are compatible. In fact, there is an $r \leqslant p, q$ with $r \in E_\alpha$ and $l(r) = l(p) = l(q)$.

Proof. Induction on α. The only essential case is $\alpha = \beta+1$ and $\beta \in \operatorname{supp}(p) \cap \operatorname{supp}(q)$.

Suppose that β is odd. By the inductive assumption there is an $r' \leqslant p|\beta, q|\beta$ with $r' \in E_\beta$ and $l(r') = l(p)$. Since

$$r' \Vdash \text{“}h_\beta(p(\beta)) = h_\beta(q(\beta))\text{”}$$

it follows that there is an r^* such that

$$r' \Vdash \text{“}r^* \leqslant p(\beta), q(\beta)\text{”}.$$

Now, if $r|\beta = r'$ and $r(\beta) = r^*$, then $r \in E_\alpha$, $r \leqslant p, q$ and $l(r) = l(p)$ (it may happen that $r \notin E_\alpha^*$, since r' need not determine the value $h_\beta(r^*)$).

Let now β be even. Then we have

$$p|\beta \Vdash \text{“}p(\beta) = \langle x, a, y\rangle\text{”}, \quad q|\beta \Vdash \text{“}q(\beta) = \langle u, a, v\rangle\text{”}.$$

By the inductive assumption there is an $r' \in E_\beta$ with $r' \leqslant p|\beta, q|\beta$ and $l(r) = l(p|\beta)$. It follows that

$$r' \Vdash \text{“}A(s) \cap A(t) \in \text{fin”}, \quad \text{for each } s \in x \cup u \text{ and } t \in y \cup v.$$

Now, we may apply Lemma 2 repeatedly to r' and to each $A(s \cup t)$ such that $s \cup t$ is a function. The properties of the resulting r^* ensure that there is an $r \in \boldsymbol{P}_\alpha$ such that $r|\beta = r^*$, $r(\beta) = \langle x \cup u, a, y \cup v\rangle$ and $r \in E_\alpha$. In particular, $r \leqslant p, q$ and $l(r) = l(p) = l(q)$, which finishes the proof of the lemma.

Suppose now that $C \subseteq \boldsymbol{P}_\alpha$ is an uncountable antichain. By Lemma 1 we may assume $C \subseteq E_\alpha^*$. Applying the Δ-system lemma to $\{\operatorname{supp}(p)\colon p \in C\}$ and some elementary operations we find an uncountable $C_0 \subseteq C$ so that any $p, q \in C_0$ are as in Lemma 3, a contradiction. Thus, each $\boldsymbol{P}_\alpha$ has the c.c.c. and the proof is complete.

Remark. It is not difficult to see that, in the model constructed, each automorphism of a given algebra B can be, after embedding, extended to an automorphism of $P(\omega)/\text{fin}$.

References

[C–N] W. W. Comfort and S. Negrepontis, *The Theory of Ultrafilters*, Springer, Berlin–Heidelberg–New York 1974.
[F] R. Frankiewicz, *Some remarks on embeddings of Boolean algebras and topological spaces*, Fund. Math. 126 (1984), 63–68.
[K] K. Kunen, *Inaccessibility properties of cardinals*, Doctoral Dissertation, Stanford 1968.
[L] R. Laver, *Linear orders in ω^ω under eventual dominance*, preprint.

DEPARTMENT OF MATHEMATICS
DARTMOUTH COLLEGE
Hanover, New Hampshire 03755
U.S.A.

INSTYTUT MATEMATYCZNY PAN
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
Śniadeckich 8, 00-950 Warszawa
Poland

INSTYTUT MATEMATYKI UW
DEPARTMENT OF MATHEMATICS
WARSAW UNIVERSITY
PKiN, IXp., 00-901 Warszawa
Poland

Received 22 May 1989

On bounded paradoxical subsets of the plane

by

Glen Aldridge Sherman (Toronto, Ont.)

Abstract. We give a precise lower bound for the number of pieces required in a bounded paradoxical subset of the plane.

Intuitively, a subset X of a metric space is said to be *paradoxical* if it admits a partition $X = A \cup B$ such that each of the sets A, B can be subdivided into finitely many pieces which can be reassembled via isometries to produce X; if A is subdivided into m pieces and B is subdivided into n pieces, the set X is said to be *(m, n)-paradoxical.*

The Sierpiński–Mazurkiewicz paradox is that there is a (1,1)-paradoxical subset of the plane [MS]. Hadwiger, Debrunner and Klee [HDK, p. 80] have shown that a *bounded* (m, n)-paradoxical subset of the plane must satisfy $m+n > 2$. A bounded (1, 3)-paradoxical subset of the plane has recently been constructed by Just [J].

Our main purpose here is to show that there is no bounded (1,2)-paradoxical subset of the plane. This improves the result of Hadwiger, Debrunner and Klee, and renders optimal the recent construction of Just. We also construct here a bounded (2,2)-paradoxical subset of the plane.

Definition 1. X is an *(m, n)-paradoxical* subset of the plane if X is nonempty, and there are subsets $C_1, \ldots, C_m$, $D_1, \ldots, D_n$ of X and planar isometries $G_1, \ldots, G_m$, $H_1, \ldots, H_n$, such that $P_1 = \{C_i\}$, $P_2 = \{D_j\}$ and $P_3 = \{G_i(C_i)\} \cup \{H_j(D_j)\}$ are each partitions of X.

Definition 2. Let X be an (m, n)-paradoxical subset of the plane whose paradoxical decomposition is witnessed by subsets $C_1, \ldots, C_m$, $D_1, \ldots, D_n$ and planar isometries $G_1, \ldots, G_m$, $H_1, \ldots, H_n$. Write $\mathscr{C} = \{C_1, \ldots, C_m\}$, $\mathscr{D} = \{D_1, \ldots, D_n\}$, $\mathscr{G} = \{G_1, \ldots, G_m\}$, and $\mathscr{H} = \{H_1, \ldots, H_n\}$. We define the *associated directed graph* $\Gamma = \Gamma(\mathscr{C}, \mathscr{D}, \mathscr{G}, \mathscr{H})$ of the decomposition. Γ is an infinite directed graph with vertex set $V(\Gamma) = X$. The set of darts (i.e. directed edges) of Γ consists of all pairs $(x, G_i(x))$ and $(x, H_j(x))$, where $x \in C_i \cap D_j$.

It is helpful, when drawing diagrams, to label each dart of Γ with the planar isometry that determined its second coordinate.

Observe that every x in $V(\Gamma)$ has invalency 1 and outvalency 2.

Lemma 1. *Let Γ be an infinite directed graph with invalency 1 at each vertex, and suppose furthermore that Γ is connected. Then Γ contains at most one cycle.*