Embedding of Boolean algebras in $P(\omega)/\text{fin}$

by

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Abstract. A model of ZFC set theory is constructed in which $\aleph_1$ (the continuum hypothesis) is assumed, then each Boolean algebra of cardinality $\leq \aleph_1$ can be embedded into $P(\omega)/\text{fin}$. What happens if CH fails? Kunen [K] proved that after adding $\omega_2$ Cohen reals to a model of CH, there is an algebra of cardinality $\aleph_1$ which cannot be embedded into $P(\omega)/\text{fin}$. In [F] a model of ZFC $+$ MA is constructed in which $\aleph_1 = 2^{\aleph_0}$ and the powerset algebra $P(\omega_1)$ cannot be embedded into $P(\omega)/\text{fin}$. There is also a model in which $\aleph_1 = \omega_2$ and the measure algebra does not embed into $P(\omega)/\text{fin}$. On the other hand, Laver shows (in [L]) that, consistently with a large value for $\omega_1$, each linear ordering of cardinality $\leq \omega_1$ embeds in $\omega^\omega$ (ordered by eventual dominance).\(^1\)

Using Laver's technique we prove here the following:

**Theorem.** There is a model of ZFC set theory with arbitrarily large $\aleph_1$ in which each Boolean algebra of cardinality $\leq \aleph_1$ can be embedded into $P(\omega)/\text{fin}$. In addition, the Martin Axiom for $\sigma$-linked orderings holds in the model.

In particular, it follows that the conclusion of the Parovichenko theorem is not equivalent to CH.\(^1\)

**Proof.** In the ground model $V$ we fix a regular cardinal $\kappa > \omega_1$, with $2^{\kappa} = \kappa$, and force with a $\kappa$-stage finite support iteration $\langle P_\alpha, Q_\alpha : \alpha < \kappa \rangle$ so that the resulting extension $V[G]$ will have the required properties.

In a usual way choose names $\check{Q}_\alpha$, $\alpha < \kappa$ and $\alpha$ odd, so that

$P_\alpha \vdash \neg \check{Q}_\alpha$ is $\sigma$-linked"\(^3\)

and $V[G] \models$ MA ($\sigma$-linked).

\(^1\) The consistency of ZFC $+$ $\neg$CH $+$ each Boolean algebra of size $\leq \aleph_1$ can be embedded into $P(\omega)/\text{fin}$ has been pointed out without proof by Baumgartner in his article Application of the proper forcing axiom in Handbook of Set-Theoretic Topology.
Embedding of Boolean algebras

At even stages $\alpha$, $Q_\alpha$ adds a subset $A_\alpha \subseteq \omega$ which fills a certain gap in $P(\omega)/\text{fin}$. To be more precise, let $C(\alpha)$ consist of finite zero-one functions $s$ with $dm(s) \leq \alpha$. Set

$$D(\alpha) = \{s \in C(\alpha); \forall \xi \in dm(s) \; \xi \text{ is even}\}.$$ 

Each $s \in D(\alpha)$ determines a Boolean combination

$$A(s) = \bigcap_{\xi \in dm(s)} (\bigcup\{A_\xi \cap A(\xi)\})_{\xi > 0}.$$ 

Now, in $V^{P(\omega)}(\alpha)$ a pair $\langle S, T \rangle$ is called a generic gap if $S, T \subseteq D(\alpha)$ and the following two properties hold:

1. $A(\alpha) \cap A(\alpha) \in \text{fin}$, for each $s \in S$ and $t \in T$.
2. If $A(\alpha) \subseteq \bigcup_{s \in S} A(s)$, then $s \in S$; similarly for $T$.

Standard reasoning shows there is a sequence $\langle S_\alpha, T_\alpha \rangle$ such that $\langle S_\alpha, T_\alpha \rangle$ is a $P_\alpha$-name of a generic gap in $V^{P(\omega)}(\alpha)$ and each $< \alpha$-generated gap occurs in the sequence, for arbitrarily large $\alpha < \kappa$.

Now, $Q_\alpha$ consists of $q = \langle x_q, a_q, b_q, m_q \rangle$, where $x_q \subseteq S_\alpha$, $y_q \subseteq T_\alpha$ are finite, $a_q$ is a zero-one sequence of length $m_q$, and

$$\bigcup_{n \leq m_q} A(n) \cup A(n) \subseteq m_q.$$ 

The ordering on $Q_\alpha$ is defined thus:

$$p < q \Leftrightarrow \forall \alpha \in A_s \exists \alpha' \in A_\alpha (a_\alpha(0) = 1)$$

and for each $i$ with $m_q < i < m_q$, we have

$$a_q(i) = 0, \quad \text{for } i \in \bigcup\{A(s); s \in x_q\},$$

$$a_q(i) = 1, \quad \text{for } i \in \bigcup\{A(s); t \in y_q\}.$$ 

If $H \subseteq Q_\alpha$ is a generic filter, then

$$A_s = \{s \in H; A_s(0) = 1\}$$

fills the gap $\langle H_s, T_s \rangle$, i.e. we have

$$A_s \subseteq A_\alpha, \quad \text{for each } s \in S_\alpha,$$

$$A_\alpha \cap A_\alpha \in \text{fin}, \quad \text{for each } t \in T_\alpha.$$ 

Remark. Let us note the following property of $Q_\alpha$. Let $s \in D(\alpha)$. If $A(s) \subseteq A_\alpha$, then $s \in S_\alpha$. Similarly, if $A(s) \cap A_\alpha \neq \emptyset$, then $s \in T_\alpha$. Indeed, if $\alpha \vdash A(s) \cap A_\alpha \neq \emptyset$, then

$$A(s) \cap A_\alpha \subseteq \bigcup\{A(s); s \in x_q\}$$

for otherwise we have a $q < s$ and an $i \in A(s) \cap A_\alpha$ with $a_q(i) = 1$. Thus, $s \in S_\alpha$, by the condition (2) above.

It remains to show that $P_\alpha$ has the c.c.c. Indeed, assuming that for the moment, it is clear that $\kappa = \alpha$ and MA (e-linked) hold in the resulting model $V[A]$. Suppose that

$$B = \{b_\xi; \xi < \alpha\}$$

is a Boolean algebra and let $b_\alpha$ be such that $B_{b_\alpha}$ is generated by $B_\alpha$ and $B_\alpha$ for $\alpha < \kappa$. In view of the Sikorski extension theorem it suffices to find a function $f: \kappa \to \kappa$ (with even values) such that for each $s \in C(\alpha)$ we have

$$b(s) = 0 \quad \text{iff } A(s) \in \text{fin},$$

(where $b(s) = \bigcap_{\xi \in \alpha} \bigcap_{\eta \leq \eta} \bigcap \{b_\xi \cap B_{b_\eta}\}$) and $\tau(f(s)) = s(f(b))$, for $b \in dm(s)$.

Assume that $f$ has already been defined for all $\beta < \alpha$ so that $\tau$ holds for each $s \in C(\alpha)$ and $b_\alpha$ not in the subalgebra generated by $\{b_\beta; f(\beta) < \alpha\}$. Let

$$S = \{s'; s \in C(\alpha) \text{ and } b(s) = b_\alpha\},$$

$$T = \{s'; s \in C(\alpha) \text{ and } b(s) = b_\alpha\}.$$ 

Now, $\langle S, T \rangle$ generates a generic gap $\langle S_\alpha, T_\alpha \rangle$ at some stage $\gamma > \sup \{f(\beta); \beta < \alpha\}$ and we define $f(\alpha) = \gamma$. To check (e), let $\sup dm(s) = \alpha$ and for example $s(\alpha) = 1$. Then $b(\alpha) = 0$ implies $b(s(\alpha)) = b_\alpha$ and hence $(s(\alpha))' \in S$, which gives $A((s(\alpha))') \subseteq A_\alpha$, i.e.

$$A(s(\alpha))' = \emptyset.$$ 

Conversely, from $A(s(\alpha))' \in \text{fin}$ follows $A((s(\alpha))') \subseteq A_\alpha$ and hence $(s(\alpha))' \in S$, by the Remark above. This means

$$A((s(\alpha))') \subseteq A(s(\alpha))' \cdots \subseteq A(s(\alpha))' \subseteq \cdots$$

for some $s_1, \ldots, s_\gamma \in S$. By the inductive assumption

$$b(s_1) = b(s_2) + \cdots + b(s_\gamma).$$

Since $b(s_1), \ldots, b(s_\gamma) \leq b_\alpha$ we obtain $b(s(\alpha)) \leq b_\alpha$ and hence $b(\alpha) = 0$.

The rest of the paper contains the proof of the countable chain condition.

Let $E_\alpha \subseteq P_\alpha$ consist of all $p \in P_\alpha$ having the following properties:

3. For each even $\xi \in \text{supp}(p)$, there are $\alpha_\xi, \alpha_\eta, \eta_\xi$ such that $p(\alpha_\xi) = p(\eta) = \langle x_\xi, y_\xi, \alpha_\eta \rangle$, for each $s \in x_\xi \cup y_\xi$ with $dm(s) \subseteq \text{supp}(p)$.

4. There is a number $l = l(p)$ such that for each even $\xi \in \text{supp}(p)$, we have $l(\alpha_\xi) = l(p)$ (i.e. all the $\alpha_\xi$ are of the same length).

For odd $\alpha$ we have $P_\alpha \vdash "Q_\alpha$ is $\sigma$-linked", and hence we can choose a $P_\alpha$-name $h_\alpha$ such that

$$P_\alpha \vdash "h_\alpha; Q_\alpha \in \omega \text{ and } \forall n \in h_\alpha (n + 1) \in (\alpha) \text{ is linked}".$$

Let $E^*_\alpha$ consist of all $p \in E_\alpha$ which in addition satisfy

5. For each odd $\xi \in \text{supp}(p)$, there is an $n < \omega$ such that

$$p(\xi) = h_\xi; \eta = \omega \text{ and } n \in (\eta) \text{ is linked}.$$

Lemma 1. For each $p \in P_\alpha$ and $\eta \in \omega$, there is a $q \leq p$ such that $q \in E^*_\alpha$ and $l(q) \geq \eta$.

Proof. By induction on $\alpha$. Since supports are finite, it suffices to consider only the case $\alpha = \beta + 1$ and $\beta$ is a support. If $\beta$ is odd, we find a $q \leq p$ and an $n \in \omega$ with $\eta' = h_\beta(p(\beta)) = n$. By the inductive assumption, there is a $q' \leq q$ such that $q' \in E^*_\alpha$ and
Repeating this for each \( r \in y \) such that \( s|\beta \cup t \) is a function, we find a \( q' \in E_y \) with \( q' \preceq \beta | \beta \) and
\[ l(q') = l(p) \]
and
\[ q' \vdash \{ \beta \} (p) = \{ q \} \]
and \( \alpha_m (q) \subseteq \text{supp}(q') \) for each \( s \in x_y \cup y_y \). By the inductive assumption, there is a \( q' \in E \)
with \( l(q') \geq \max \{ m, l(x_y) \} \). In particular,

\[ q' \vdash \{ \beta \} (s) \subseteq \{ \alpha \} \]

and \( \alpha_m (q') \subseteq \text{supp}(q') \) for each \( s \in x_y \cup y_y \). Hence, we can define an \( a \in a_y \) with \( l(a) = l(q') \) such that \( q' \vdash \{ \beta \} (s) \subseteq \{ \alpha \} \cap \{ a \} \).

Now, if \( q' \in P_y \), then \( q' \) has the required properties.

**Lemma 2.** Let \( p \in E \), \( s \in D(\beta) \) and \( \beta = \max \text{dim}(s) \). If \( p \vdash \{ \beta \} \subseteq \text{fin} \), then there is \( a \) such that \( q \preceq \beta | \beta \) and \( q \vdash \{ \beta \} (s) \subseteq \{ \alpha \} \cap \{ a \} \).

Proof. Induction on \( \alpha \). From the assumptions of the lemma it follows that
\[ p(\beta + 1) \vdash \{ \beta \} \subseteq \text{fin} \]
and hence we may assume \( \alpha = \beta + 1 \). Suppose
\[ p(\beta) \vdash \{ q \} \subseteq \{ \alpha \} \]
and let \( s(\beta) = 1 \). Thus, we have
\[ p(\beta + 1) \vdash \{ \beta \} \subseteq \text{fin} \]
and hence \( p(\beta) \vdash \{ \beta \} \subseteq \text{fin} \). In particular,
\[ p(\beta) \vdash \{ \beta \} \subseteq \text{fin} \], for each \( r \in y \).

We have to show
\[ p(\beta) \vdash \{ \beta \} \subseteq \text{fin} \], for each \( r \in y \).

If \( s|\beta \cup t \) is not a function, then we have
\[ p(\beta) \vdash \{ \beta \} \subseteq \text{fin} \], for each \( r \in y \).

If \( s|\beta \cup t \) is a function, then \( A(\beta | \beta \) is a function and we can apply the inductive assumption; there is a \( q \in E \) with \( q \preceq \beta | \beta \) and \( l(q) = l(p) \) such that \( s(\beta) = \max \text{dim}(s) \) and
\[ q_1 \vdash \{ \beta \} (s) \subseteq \{ \alpha \} \cap \{ a \} \]
then \( q_1 \vdash \{ \beta \} (s) \subseteq \{ \alpha \} \cap \{ a \} \). In either case we obtain
\[ q_1 \vdash \{ \beta \} (s) \subseteq \{ \alpha \} \cap \{ a \} \].

Hence, if \( q \beta = q' \) and \( q \beta = \{ x, \alpha \} \cup \{ a \}, \alpha, y \} \) then \( q \) has the required properties. The case \( s(\beta) = 0 \) is similar.

**Lemma 3.** Assume that \( p, q \in E \), \( l(q) = l(p) \) and for each \( x \in \text{supp}(p) \)
\[ (6) \] if \( x \) is odd and \( p(\xi) \vdash \{ x \} \in \{ \alpha \} \), then \( p(\xi) \vdash \{ x \} \in \{ \alpha \} \).
\[ \text{and} \]
\[ (7) \] if \( x \) is even and \( p(\xi) \vdash \{ x \} \in \{ \alpha \} \), then \( p(\xi) \vdash \{ x \} \in \{ \alpha \} \).

Then \( p \) and \( q \) are compatible. In fact, there is an \( r \in \rho, q \) with \( r \in E \) and \( l(r) = l(p) = l(q) \).

Proof. Induction on \( r \). The only essential case is \( r = \beta + 1 \) and \( \beta \subseteq \text{fin} \). Suppose that \( \beta \) is odd. By the inductive assumption there is an \( r' \leq \beta | \beta \) with \( r' \in E \) and \( l(r') = l(p) = l(q) \).

It follows that there is an \( r^* \) such that
\[ r' \vdash \{ \beta \} \subseteq \{ q \} \]
and let \( r' \leq \beta | \beta \) be a function. Suppose \( r \leq p, q \) and \( l(r) = l(p) = l(q) \). Then \( r \in E \), since \( r' \) need not determine the value \( h(q) \).

Now let \( \beta \) be even. Then we have
\[ \vdash \{ \beta \} \subseteq \{ q \} \]
and \( \beta \subseteq \text{fin} \). By the inductive assumption there is an \( r' \in E \) with \( r' \leq \beta | \beta \) and \( l(r') = l(p) = l(q) = l(q) \).

It follows that
\[ r' \vdash \{ \beta \} \subseteq \{ q \} \], for each \( x \in x \) and \( r \in y \).

Now, we may apply Lemma 2 repeatedly to \( r' \) and to each \( A(\beta) \) such that \( \beta \subseteq \text{fin} \) is a function. The properties of the resulting \( r^* \) ensure that there is an \( r \in E \) such that \( r \beta = r^* \) and \( r \beta = \{ x \cup y \} \cup \{ x \cup y \} \).

In particular, \( r \leq p, q \) and \( l(r) = l(p) = l(q) \), which finishes the proof of the lemma.

Suppose now \( C \subseteq E \) is an uncountable antichain. By Lemma 1 we may assume \( C \subseteq E \). Applying the \( A \)-system lemma to \( \text{supp}(p) \) then \( p \in C \) and some elementary operations we find an uncountable \( C_0 \subseteq C \) so that any \( p, q \in C_0 \) are as in Lemma 3, a contradiction. Thus, each \( P_y \) has the c.c.c. and the proof is complete.

**Remark.** It is not difficult to see that, in the model constructed, each automorphism of a given algebra \( B \) can be, after embedding, extended to an automorphism of \( P(\alpha) \).
On bounded paradoxical subsets of the plane

by

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Abstract. We give a precise lower bound for the number of pieces required in a bounded paradoxical subset of the plane.

Intuitively, a subset $X$ of a metric space is said to be paradoxical if it admits a partition $X = A \cup B$ such that each of the sets $A$, $B$ can be subdivided into finitely many pieces which can be reassembled via isometries to produce $X$; if $A$ is subdivided into $m$ pieces and $B$ is subdivided into $n$ pieces, the set $X$ is said to be $(m, n)$-paradoxical.

The Sierpiński–Mazurkiewicz paradox is that there is a $(1, 1)$-paradoxical subset of the plane [MS]. Hadwiger, Debrunner and Klee [HDK, p. 80] have shown that a bounded $(m, n)$-paradoxical subset of the plane must satisfy $m+n > 2$. A bounded $(1, 3)$-paradoxical subset of the plane has recently been constructed by Just [J].

Our main purpose here is to show that there is no bounded $(1, 2)$-paradoxical subset of the plane. This improves the result of Hadwiger, Debrunner and Klee, and renders optimal the recent construction of Just. We also construct here a bounded $(2, 2)$-paradoxical subset of the plane.

**Definition 1.** $X$ is an $(m, n)$-paradoxical subset of the plane if $X$ is nonempty, and there are subsets $C_1, \ldots, C_m$, $D_1, \ldots, D_n$ of $X$ and planar isometries $G_1, \ldots, G_m$, $H_1, \ldots, H_n$, such that $P_1 = \{C_i\}$, $P_2 = \{D_j\}$ and $P_3 = \{G_i(C_i) \cup H_j(D_j)\}$ are each partitions of $X$.

**Definition 2.** Let $X$ be an $(m, n)$-paradoxical subset of the plane whose paradoxical decomposition is witnessed by subsets $C_1, \ldots, C_m$, $D_1, \ldots, D_n$ and planar isometries $G_1, \ldots, G_m$, $H_1, \ldots, H_n$. Write $\mathcal{G} = \{C_1, \ldots, C_m\}$, $\mathcal{D} = \{D_1, \ldots, D_n\}$, $\mathcal{G} = \{G_1, \ldots, G_m\}$, and $\mathcal{H} = \{H_1, \ldots, H_n\}$. We define the associated directed graph $\Gamma = \Gamma(\mathcal{G}, \mathcal{D}, \mathcal{G}, \mathcal{H})$ of the decomposition. $\Gamma$ is an infinite directed graph with vertex set $V(\Gamma) = X$. The set of darts (i.e. directed edges) of $\Gamma$ consists of all pairs $(x, G_k(x))$ and $(x, H_k(x))$, where $x \in C_i \cap D_j$.

It is helpful, when drawing diagrams, to label each dart of $\Gamma$ with the planar isometry that determined its second coordinate.

Observe that every $x$ in $V(\Gamma)$ has invalency 1 and outvalency 2.

**Lemma 1.** Let $\Gamma$ be an infinite directed graph with invalency 1 at each vertex, and suppose furthermore that $\Gamma$ is connected. Then $\Gamma$ contains at most one cycle.