The Ramsey sets and related sigma algebras and ideals

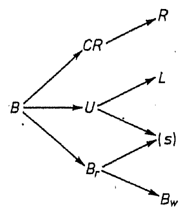
by

Jack B. Brown (Auburn, Ala.)

Abstract. We investigate the similarities and dissimilarities between the σ-algebra of completely Ramsey subsets of the space of infinite subsets of ω, the σ-algebras of Lebesgue-, Marczewski-, and universally-measurable sets, and the σ-algebras of sets with the Baire properties. We also study the relationships between the σ-ideals associated with these σ-algebras and other classes of singular sets such as the concentrated sets, the sets of strong measure zero, and the rarefied sets.

I. Introduction. We are interested in the classes of sets represented in the following diagram:

(I)



B represents the Borel sets. B_w represents the sets M with the Baire property in the wide sense (i.e. M is the symmetric difference between an open set and a first category set). B_r represents the sets M with the Baire property in the restricted sense (i.e. for every perfect set P, $M \cap P$ has property B_w relative to P). U represents the universally measurable sets.

L represents the Lebesgue measurable sets and (s) denotes the Marczewski [9] measurable sets (a set S has property (s) if it is true that every perfect subset of the space has a perfect subset which is a subset of or misses S). See [3] and [15] for more information on (s)-sets and functions. Properties B, B_r, B_w, U, and (s) have meaning in any Polish space. L has its usual meaning for subsets of the reals $\boldsymbol{R}$. The complete Ramsey property, CR, has meaning only in the space $[\omega]^\omega$ of all infinite subsets of ω, and it has a definition similar to that of the (s)-sets. The CR-sets were first defined by

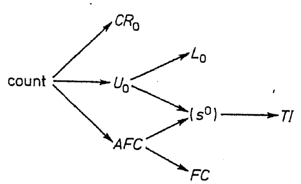
Galvin and Prikry in [6], where they proved that Borel subsets of $[\omega]^\omega$ have the Ramsey property R (this theorem was extended to analytic sets by Silver in [14], and Silver's proof was greatly simplified by Ellentuck in [5] and independently by Louveau in [8]).

$[\omega]^\omega$ is naturally imbedded in ω^ω as the class of all increasing sequences from ω, and the usual metric d_1 on ω^ω relativized to $[\omega]^\omega$ produces the topology for $[\omega]^\omega$. If x and y belong to ω^ω, then $d_1(x, y) = 1/n$, where n is the first integer such that $x_n \neq y_n$. It is more common to think of $[\omega]^\omega$ as being imbedded in $\{0, 1\}^\omega$, by taking the characteristic sequences associated with those increasing sequences indicated above. The metric d_2 usually associated with $\{0, 1\}^\omega$, relativized to $[\omega]^\omega$, also produces the usual topology for $[\omega]^\omega$. Thus, if x and y belong to $[\omega]^\omega$, then $d_2(x, y) = 1/\min\{x_n, y_n\}$, where n is the first integer such that $x_n \neq y_n$. We will see in Section II that there are significant differences between these two metrics for $[\omega]^\omega$ insofar as they pertain to "singular" sets.

We adopt present convention and say that an *Ellentuck set* or *E-set* is a set denoted by the symbol $[F, M]$, where F is a finite subset of ω, M is an infinite subset of ω, and $[F, M] = \{S \in [\omega]^\omega \colon F \subset S \subset F \cup M$ and $\max F < \min S \setminus F\}$. Then we say that a set $S \in CR$ if every E-set $[F, M]$ has an E-subset $[F, N]$ which is a subset of or misses S. A subset S of $[\omega]^\omega$ is a *Ramsey set*, or has *property* R, if there is an infinite subset M of ω such that the set $[M] = [\emptyset, M]$ of all infinite subsets of M is a subset of or misses S. All of these classes of sets form σ-algebras except the class R.

The σ-ideals associated with the σ-algebras discussed above are of course related as follows:

(II)



count, FC, and L_0 denote the countable, first category, and Lebesgue measure zero sets, respectively. The AFC (always first category) sets M are such that for every perfect set P, $M \cap P$ is FC relative to P. The U_0 (universal null) sets have measure zero relative to the completion of every non-atomic Borel measure on the space. The CR_0 (Ramsey null) sets S are such that every E-set $[F, M]$ has an E-subset $[F, N]$ which misses S. The (s^0) (Marczewski null) sets M are such that every perfect set P has a perfect subset Q which misses M. The TI-sets are the totally imperfect sets (i.e. the sets which have no perfect subsets).

Ellentuck [5] (and independently, Louveau [8]) considered the topology on $[\omega]^\omega$ one gets by using the E-sets as a basis. We will call this the *E-topology*, which is finer than the classical topology. It was shown that a subset S of $[\omega]^\omega$ is a CR-set if and only if it has the E-B_w-property, and that S is CR_0 if and only if it is E-FC, in which case it is

E-(nowhere dense) (see [12] for more on the E-topology). It is because of this that it is clear that the definition of the CR-sets can be reworded so as to be completely analogous to that of the (s)-sets.

THEOREM 1. *S is CR if and only if every E-set $[F, M]$ has an E-subset $[G, N]$ which is a subset of or misses S.*

Proof. This is a well-known result from the "folk-lore" of this subject, but we include a proof for completeness.

It is obvious that CR implies this latter property. If S has this latter property, one can let O_1 be the union of all the E-sets which are subsets of S, and let O_2 be the union of all the E-sets which miss S. O_1 and O_2 are disjoint E-open sets, and it follows from the assumption that $O_1 \cup O_2$ is dense in the space. Thus $S = O_1 \cup (S \setminus O_1 \cup O_2)$ is the union of an E-open set and an E-FC-set and is an E-B_w-set. ∎

Note. It is also true that S is CR_0 if and only every E-set $[F, M]$ has an E-subset $[G, N]$ which misses S.

II. Theorems and examples. We now turn to the question of describing the necessary examples to show that diagram (I) includes all possible implications that hold between the properties in question.

Galvin and Prikry [6] gave an example which was FC and L_0 but not R. In $[\omega]^\omega$, L and L_0 are taken to mean measurability under the usual product measure μ on $\{0, 1\}^\omega$. The paper [1] by Aniszczyk, Frankiewicz, and Plewik compares some of the properties of (I), and several interesting examples are given there. It is actually possible to describe as many as 14 examples which would show that any combination of the properties CR, L, (s), and B_w need not imply the others. The author has in fact checked that all 14 examples exist (some assuming CH and others in ZFC). Most of these examples are similar to previously described sets, so only the most interesting ones are given here.

Walsh [16] has recently given a ZFC example which shows that (s^0) implies neither B_w nor L. After the author discussed the 14 examples mentioned above with Corazza, Corazza showed [4] that several of them could be described in ZFC (rather than assuming CH). However, his example to show (s^0) does not imply B_w, L, or R turned out to need an extra set-theoretic assumption. Therefore, we will give our original CH example below. We will need the following lemma.

LEMMA 2. *Every perfect set P in $[\omega]^\omega$ has a perfect subset Q which is L_0 and CR_0.*

Proof. Suppose P is a perfect set. First get an L_0 perfect subset P_1 of P. If P_1 is CR_0, we are through. If P_1 is non-CR_0, use the following procedure (modeled after a trick in [1]). Consider an arbitrary E-subset $[F, M]$ of P_1. Assume that $[F, M] = [\langle f_0, \ldots, f_k \rangle, \langle m_0, m_1, \ldots \rangle]$, and that $f_k < m_0$. Let $O' = \langle m_1, m_3, \ldots \rangle$ and $E' = \langle m_0, \ldots, m_2, \ldots \rangle$. Let

$$K_0 = [F, O'] \cup [F, E'],$$

$$K_n = [F, O' \cup \langle m_0, \ldots, m_{2n-2} \rangle] \cup [F, E' \cup \langle m_1, \ldots, m_{2n-1} \rangle], \quad n > 0,$$

$$H = [F, M] \setminus (K_0 \cup K_1 \cup \ldots).$$

H is a dense relative G_δ in $[F, M]$ and $\mu(H) = \mu([F, M]) = 0$.

We now show that H is CR_0. Consider an arbitrary E-set $[G, N]$. Assume that $F \subset G \subset F \cup M$ (otherwise $[G, N \cap M]$ misses $[F, M]$). Likewise, we may assume $H \cap M$ is infinite (otherwise we are through). Therefore either $N \cap O'$ or $N \cap E'$ is infinite. Let N' be the infinite intersection. Then $[G, N']$ is a subset of some K_i and misses H. This shows that H is CR_0. Since H is a dense relative G_δ in the perfect set $[F, M]$, we can take a perfect nowhere dense subset Q of H. This is the desired set Q. ∎

EXAMPLE 3. CH implies that there exists an (s^0)-set which has none of the properties R, L, or B_w.

Proof. Assume CH, and let Ω denote the first uncountable ordinal. List the perfect sets, the E-sets, the perfect sets of positive measure, and the locally residual G_δ-sets as $\{P_\alpha: \alpha < \Omega\}$, $\{E_\alpha: \alpha < \Omega\}$, $\{M_\alpha: \alpha < \Omega\}$, and $\{R_\alpha: \alpha < \Omega\}$, respectively. A locally residual G_δ is a set of the form $O \backslash F$, where O is open and F is an FC-F_σ.

Using the lemma, take Q_0 to be a perfect subset of P_0 which is CR_0 and L_0. Pick a_0 and b_0 to be distinct elements of $E_0 \backslash Q_0$, c_0 and d_0 to be distinct elements of $M_0 \backslash (Q_0 \cup \{a_0, b_0\})$, and e_0 and f_0 to be distinct elements of $R_0 \backslash (Q_0 \cup \{a_0, b_0, c_0, d_0\})$. This completes step 0.

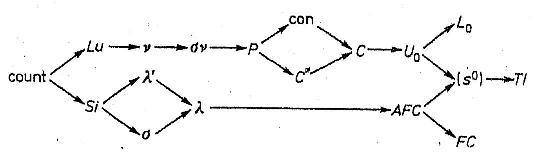
For $0 < \alpha < \Omega$, step α is as follows. Take Q_α to be a perfect subset of P_α which is CR_0 and L_0. Set

$$N_\alpha = \bigcup_{\beta < \alpha} (Q_\beta \cup \{a_\beta, b_\beta, c_\beta, d_\beta, e_\beta, f_\beta\}) \cup Q_\alpha.$$

Pick a_α and b_α to be distinct elements of $E_\alpha \backslash (Q_\alpha \cup N_\alpha)$, c_α and d_α to be distinct elements of $M_\alpha \backslash (Q_\alpha \cup N_\alpha \cup \{a_\alpha, b_\alpha\})$, and e_α and f_α to be distinct elements of $R_\alpha \backslash (Q_\alpha \cup N_\alpha \cup \{a_\alpha, b_\alpha, c_\alpha, d_\alpha\})$. Let $S = \{a_\alpha, c_\alpha, e_\alpha: \alpha < \Omega\}$. It is obvious that S has none of the properties R, L, or B_w because of the b_α, d_α, and f_α. To see that S has property (s^0), suppose P is an arbitrary perfect set. P appears as P_α for some α. At most countably many points of S chosen up through step α intersect Q_α, and none of the points of S chosen after stage α intersect Q_α, so there is a perfect subset Q of Q_α (and of P) which misses S. ∎

We now consider how the properties of diagram (II) relate to other "singularity" properties. In particular, we will be interested in those properties included in the following diagram of implications.

(III)



We will define only those properties discussed here (see [2], [7], and [10] for the others). For subsets X of a Polish space Y with metric d the properties are as follows: con(rel Y) = concentrated about a countable subset C of Y (i.e. every open set containing C contains all but countably many points of X). C(rel d) = strong measure zero relative to the metric d (i.e. for every sequence $t_1, t_2, \ldots$ of positive numbers, there exists a sequence $x_1, x_2, \ldots$ of elements of X such that $X \subset N(x_1, t_1) \cup N(x_2, t_2) \cup \ldots$, where $N(x, t) = \{y \in Y: d(x, y) < t\}$ denotes the t-neighborhood of x under d. λ = rarefied (i.e. every countable subset of X is a G_δ relative to X); λ'(rel Y) = the union of X and any countable subset of Y still has property λ.

THEOREM 4. $C(\text{rel}\, d_1) \Rightarrow CR_0$.

Proof. First notice that S has property C(rel d_1) if and only if it is true that if $n(1), n(2), \ldots$ is any increasing sequence of positive integers, then there exists a sequence $X_i = \langle x_{i,1}, x_{i,2}, \ldots \rangle$, $i = 1, 2, \ldots$, such that the neighborhoods $N(X_i, 1/n(i)) = [\langle x_{i,1}, \ldots, x_{i,n(i)} \rangle, \omega]$, $i = 1, 2, \ldots$, cover S. Now, consider an E-set $[F, M] = [\langle f_1, \ldots, f_k \rangle, \langle m_1, m_2, \ldots \rangle]$. Assume $f_k < m_1$ and that $[F, M]$ intersects S. Consider the sequence $k+2, k+4, k+8, k+16, \ldots$ There exists the sequence X_i, $i = 1, 2, \ldots$, such that the neighborhoods $N(X_i, 1/(k+2^i))$ described above cover S.

Let I be the set of positive integers i such that the first $k+2^i$ terms of X_i consist of F, followed by 2^i terms of M. Notice that if i is in I^c, then $N(X_i, 1/(k+2^i))$ misses $[F, M]$. If I is finite (or empty) and i exceeds all the elements of I, we can let N be the set of elements of M which exceed $k+2^i$. Then $[F, N]$ not only misses S, it misses the union of the neighborhoods described above. If I is infinite, construct an infinite subset N of M as follows. If i is the jth smallest element of I, the first $k+2^i$ terms of X_i consist of F, followed by 2^i terms of M. Let G_j consist of those 2^i terms of M. Put one element of G_1 in N (leaving the others out). Put one element of $G_2 \backslash G_1$ in N (leaving at least one other out). In general, put one element of $G_n \backslash (G_1 \cup \ldots \cup G_{n-1})$, in N (leaving at least one other out). $[F, N]$ misses the union of the neighborhoods described above. ∎

THEOREM 5. $\lambda'(\text{rel}\,\{0, 1\}^\omega) \Rightarrow CR_0$.

Proof. If $m \in \omega$, let $\langle m, \text{--}\rangle$ denote the set (or increasing sequence) of all elements of ω greater than or equal to m. The sets $[F, \langle m, \text{--}\rangle]$, where F is a finite subset of ω, form a basis for the topology on $[\omega]^\omega$. Let $E = [\omega]^{<\omega}$ be the collection of all finite subsets of ω. E is identified with the "left-endpoints" of $\{0, 1\}^\omega$ which terminate in zeros. Assume $S \cup E$ is λ, and let $O_1, O_2, \ldots$ be a sequence of open sets in $\{0, 1\}^\omega$, all containing E, such that $S \cap O_1 \cap O_2 \cap \ldots$ is empty. For each i, let $S_i = S \backslash (O_1 \cap \ldots \cap O_i)$. Each S_i is clearly CR_0, and $S = S_1 \cup S_2 \cup \ldots$, so S is CR_0, because CR_0 forms a σ-ideal. ∎

It is shown in Theorem 4 of [1] that an axiom weaker than CH implies the existence of a U_0-set which does not have property CR. We will combine the techniques of that proof and the trick in a 1941 paper of Rothberger [13] to show that CH implies the existence of a much stronger example.

EXAMPLE 6. CH implies that there exists a set $S \subset [\omega]^\omega$ which has properties con(rel $\{0, 1\}^\omega$) and λ'(rel ω^ω) but not property R.

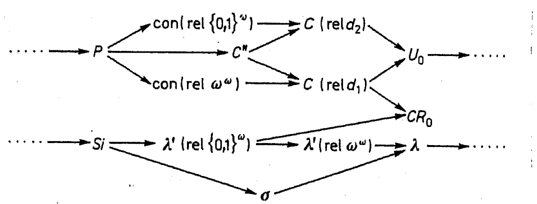
Proof. We will modify Rothberger's construction [13] of a set which is con and λ, applying a trick of Aniszczyk, Frankiewicz, and Plewik [1, Th. 4]. As usual, if f and g are elements of ω^ω, we say $f <^* g$ if $f_i < g_i$ for all but finitely many i, and $f < g$ if $f_i < g_i$ for all i. $((0, f))$ denotes $\{g\colon g <^* f\}$ and $((f, \infty))$ denotes $\{g\colon f <^* g\}$. It is well known that (1) every $((0, f))$ is σ-compact, (2) every σ-compact subset of ω^ω is a subset of some $((0, f))$, and (3) every $((f, \infty))$ is an F_σ. List $\omega^\omega = \{Y_\alpha\colon \alpha < \Omega\}$ and the increasing sequences $[\omega]^\omega = \{A_\alpha\colon \alpha < \Omega\}$. Let $g_0 = Y_0$, and look at $((0, g_0))$. Pick distinct p_0 and q_0 in $[A_0]$ so that both are $> g_0$. Continue the process. At stage α, first pick the first element of ω^ω, call it g_α, such that $\bigcup_{\beta<\alpha}((0, g_\beta)) \cup \{p_\beta, q_\beta\} \subset ((0, g_\alpha))$. Notice that $Y_\beta < g_\alpha$ for every $\beta < \alpha$. Then pick distinct p_α and q_α in $[A_\alpha]$ so that both are $> g_\alpha$ and Y_α. $S = \{p_\alpha\colon \alpha < \Omega\}$ is wellordered by $<^*$ and unbounded, so it is concentrated about $E = [\omega]^{<\omega}$ in $\{0, 1\}^\omega$ and has property λ.

It is actually the case that if B is any countable subset of ω^ω, then $S \cup B$ still has property λ, so that S has property λ' (rel ω^ω). To see this, suppose C is a countable subset of $S \cup B$. There will exist an α such that every element of $B \cup C$ precedes Y_α in the wellordering of ω^ω. $((p_\alpha, \infty))$ is an F_σ in ω^ω, so $S \cup B \setminus ((p_\alpha, \infty))$ is a countable relative G_δ-subset of $S \cup B$. It follows that C is a relative G_δ-subset of $S \cup B$. Thus S has property λ'(rel ω^ω).

On the other hand, we have made S a subset of $[\omega]^\omega$. Every A_α intersects S, but no A_α is a subset of S, so S does not have property R. ∎

We note that it would be impossible to prove the existence of the set of Example 6 in ZFC (see [1, pp. 483–4]).

Problems. Expand that part of diagram III concerning the concentrated sets, strong measure zero, and the rarefied sets as follows:



It is clear that the set S of Example 6 is con (rel $\{0, 1\}^\omega$) but not rel ω^ω, λ' (rel ω^ω) but not rel $\{0, 1\}^\omega$, and has property C(rel d_2) but not rel d_1 (also see [11]). Miller [10, Th. 5.7] used Rothberger's approach to build a set which is con and σ. We do not see how to make the set of Example 6 have property σ. It is conceivable that $\sigma \Rightarrow CR_0$. We also wonder how the property of being concentrated relative to *every* complete space Y in which X may be embedded relates to property P, and how the property of being λ' relative to *every* complete space Y in which X may be embedded relates to properties Si and σ. It is known [11] that having property C relative to every equivalent metric is equivalent to having property C''.

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DIVISION OF MATHEMATICS
FOUNDATIONS ANALYSIS AND TOPOLOGY
AUBURN UNIVERSITY
218 Parker Hall
Auburn, AL 36849-5310
U.S.A.

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