Proof. The proof is almost identical to the proof of Lemma 4. Without loss of generality every φ_i, ψ_j satisfies (*) and also for all $i<j$ and all n we assume that $\varphi_i(n) \leqslant \varphi_j(n)$ and $\psi_i(n) \geqslant \psi_j(n)$. Again we choose an ascending sequence of integers $N_1 < N_2 < \ldots$ such that if $n > N_k$ then $(\psi_k)^k(n) > 2^k \varphi_k(n)$. The function γ is defined by $\gamma(n) = \max\{\gamma(n-1), \psi_k^k(n)\}$ if $N_{k^2} \leqslant n < N_{(k+1)^2}$. Obviously $\gamma \preccurlyeq \psi_i$ for every $i \in N$. Also if $n > N_{k^2}$, then

$$\gamma^k(n) \geqslant (\psi_m)^{mk}(n) \quad \text{if } N_{m^2} < n < N_{(m+1)^2}.$$

Therefore

$$\gamma^k(n) \geqslant (\psi_m)^{m^2}(n) \geqslant (\psi_{m^2})^{m^2}(n) \geqslant 2^{m^2} \varphi_{m^2}(n) \geqslant 2^k \varphi_k(n),$$

which implies that $\varphi_n \preccurlyeq \gamma$ for all $n \in N$. ∎

The main theorem now follows as a corollary of all the work we have done.

THEOREM. *There exists a continuum of topologically distinct orbits.*

Proof. According to Lemmas 4 and 5 the space $(\mathscr{F}, \preccurlyeq)$ contains $\boldsymbol{R}$ as an ordered subset. Any two different elements of $\boldsymbol{R}$ correspond to non-equivalent orbits according to Lemma 3.

We conclude with an unsolved problem. Consider an irrational rotation ϱ_α on the circle S^1. The flow $\Sigma(S^1, \varrho_\alpha)$ is called an irrational flow on the torus. It is well known when $\Sigma(S^1, \varrho_\alpha)$ and $\Sigma(S^1, \varrho_\beta)$ are equivalent. Consider one orbit Γ_α from $\Sigma(S^1, \varrho_\alpha)$ and Γ_β from $\Sigma(S^1, \varrho_\beta)$. It is highly unsatisfactory that the following question has not been answered yet:

QUESTION. Are there α and β such that Γ_α and Γ_β are not homeomorphic?

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References

[A] J. M. Aarts, *The structure of orbits in dynamical systems*, Fund. Math. 129 (1987), 39–58.
[AF] J. M. Aarts and Z. Frolík, *Homeomorphisms of Q and the orbit classification of flows*, Rend. Circ. Mat. Palermo Suppl. II-18 (1988), 15–25.
[AM] J. M. Aarts and M. Martens, *Flows on one-dimensional spaces*, Fund. Math. 131 (1988), 53–67.
[D] F. M. Dekking, *Combinatorial and statistical properties of sequences generated by substitutions*, Thesis, University of Nijmegen, 1980.
[E] R. Engelking, *Dimension Theory*, PWN, Warszawa 1978.
[F] N. E. Foland, *The structure of the orbits and their limit sets in flows*, Pacific J. Math. 13 (1963), 563–570.
[H] O. Hájek, *Dynamical Systems in the Plane*, London 1968.
[I] M. Irwin, *Smooth Dynamical Systems*, New York 1980.
[K] K. Kuratowski, *Introduction to Set Theory and Topology*, Pergamon, Oxford 1962.
[MH] M. Morse and G. A. Hedlund, *Symbolic dynamics*, Amer. J. Math. 60 (1938), 816–866.
[W] H. Whitney, *Regular families of curves*, Ann. of Math. 34 (1933), 244–270.

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Natural continuity space structures on dual Heyting algebras

by

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Abstract. Every dual Heyting algebra carries three intrinsic "generalized quasi-metrics": $d(x, y) = x - y$, $d^*(x, y) = y - x$, and $d^s(x, y) = (x-y)+(y-x)$, where $x-y$ denotes the relative dual pseudocomplement. Formally, these are "continuity functions" satisfying the triangle inequality. In a dual Heyting algebra, a set of positives is a dual ideal P whose meet is 0. We investigate properties of the topologies, $T(A_P)$, $T(A_P^*)$, $T(A_P^s)$, which arise from the continuity spaces so defined. For example, $T(A_P)$ and $T(A_P^*)$ are completely distributive, and $T(A_P^s)$ is a zero-dimensional Hausdorff topology. Furthermore, we show that for any coframe, that is, for any complete dual Heyting algebra A:

(1) $T(A_P)$ is the Scott topology iff P satisfies the ascending chain condition.
(2) $T(A_P^*)$ is the dual Scott topology iff P satisfies the local descending chain condition.
(3) $T(A_P^s)$ is the order topology (Lawson topology, interval topology) iff P is locally finite iff $T(A_P^s)$ is compact.

1. Motivation and preliminaries. Continuity spaces are among the many generalizations of metric spaces found in the literature. In [18] it is shown that all topologies arise in a natural way from continuity spaces. More to the point for us, it was shown in [12] that the hull-kernel topology long studied on spaces of prime ideals (see, e.g., [10], [11], [13], [16], [25]) arises from a continuity space in which the distance between two ideals I, J is their set-theoretic difference $d(I, J) = J \setminus I$. The "converse" continuity space, in which the distance $I \setminus J$ is used in place of $J \setminus I$, gives rise to the Scott topology on the power set of the underlying ring. Further, its "symmetrization", in which $J \setminus I$ is replaced by $(J \setminus I) \cup (I \setminus J)$, gives rise to the patch topology, which here agrees with the Lawson topology (see [9], [13]).

The above construction can be generalized from power sets to arbitrary dual Heyting algebras, alias (dual) Brouwerian lattices (cf. [1], [22], [24]). We shall carry out this construction below and describe just when it actually does give rise to Scott, dual Scott and Lawson topologies, respectively. The "structure spaces" of [12] are special cases of the "lattice continuity spaces" studied in the sequel.

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Henceforth, A denotes a bounded lattice with least element 0 and greatest element ∞, reminiscent of the standard example of the extended nonnegative real line. Join and meet of a subset B are written $\bigvee B$ and $\bigwedge B$, respectively. However, it will be convenient to use the symbol $+$ (rather than $\vee$) for the binary join and the symbol $\cdot$ (instead of $\wedge$) for the binary meet.

Now, a *lattice continuity space* is a quadruple $X = (X, d, A, P)$, where

X is a set,

A is a bounded distributive lattice,

$d\colon X \times X \to A$, called the *distance* or *continuity function*, satisfies

$$d(x, x) = 0 \quad \text{and} \quad d(x, z) \leqslant d(x, y) + d(y, z),$$

P, called the *set of positives*, is a meet-dense dual ideal of A.

As usual, *meet-density* means that each $x \in A$ is a meet of elements from P.

If, in addition, $d(x, y) = d(y, x)$ then d and X are called *symmetric*, and if $d(x, y) \neq 0$ for $x \neq y$ then d and X are called *separated.*

Notice that any bounded distributive lattice is a set of positives in itself; this choice leads to *Alexandrov* (*discrete*) *topologies*, i.e. topologies in which arbitrary intersections of open sets are open.

The *converse* of X is $X^* = (X, d^*, A, P)$, where

$$d^*(x, y) = d(y, x).$$

(In [18] this is called the *dual*, but we avoid this terminology to prevent confusion with other notions of duality studied below.) The *symmetrization* of X is $X^s = (X, d^s, A, P)$, where

$$d^s(x, y) = d(x, y) + d(y, x).$$

Given a (lattice) continuity space, its *associated quasi-uniformity* and *associated topology* are defined by generalizing classical constructions on metric spaces:

$$N_p = \{(x, y)\colon d(x, y) \leqslant p\} \quad (p \in P),$$

$$U(X) = \{U \subseteq X \times X : N_p \subseteq U \text{ for some } p \in P\},$$

$$N_p(x) = \{y \in X\colon d(x, y) \leqslant p\} \quad (p \in P,\ x \in X),$$

$$T(X) = \{T \subseteq X\colon \text{if } x \in T \text{ then } N_p(x) \subseteq T \text{ for some } p \in P\}.$$

In accordance with Fletcher and Lindgren [8], we call a quasi-uniformity *transitive* if it is generated by a set of quasi-orderings, i.e. reflexive and transitive relations. Our first theorem summarizes and refines some of the main facts from [18] (see also [17]).

1.1. THEOREM. *For any lattice continuity space* X,

(1) $U(X)$ *and* $U(X^*)$ *are transitive quasi-uniformities;*

(2) $U(X^s)$ *is the transitive uniformity generated by* $U(X)$ *and* $U(X^*)$;

(3) $T(X)$ *is the topology induced by* $U(X)$, *and the neighborhoods* $N_p(x) = \{y \in X\colon d(x, y) \leqslant p\}$ *form an open base for* $T(X)$;

(4) $T(X^*)$ *is the topology induced by* $U(X^*)$, *and the neighborhoods* $N_p^*(x) = \{y \in X\colon d^*(x, y) \leqslant p\}$ *form an open base for* $T(X^*)$;

(5) $T(X^s)$ *is the topology induced by* $U(X^s)$ *and generated by* $T(X)$ *and* $T(X^*)$. *Furthermore,* $T(X^s)$ *is zero-dimensional, since the neighborhoods* $N_p^s(x) = \{y \in X\colon d^s(x, y) \leqslant p\}$ *form a base of clopen sets.*

Conversely, all transitive quasi-uniformities and all (*zero-dimensional*) *topologies arise in this way from suitable* (*symmetric*) *lattice continuity spaces.*

Proof. The asserted properties of $U(X)$, $T(X)$ etc. are easily verified. For example, down-directedness of the set of positives guarantees that $U(X)$ is in fact a filter, and the triangle inequality ensures that each N_p is a transitive relation. Moreover, $N_p^s = \{(x, y)\colon d^s(x, y) \leqslant p\}$ is an equivalence relation. Now consider any set $\boldsymbol{R}$ of quasi-orderings on a set X. Certainly the power set of $\boldsymbol{R}$ is a complete Boolean lattice A, and the collection P of all cofinite subsets of $\boldsymbol{R}$ is a set of positives in A. The function

$$d\colon X \times X \to A \quad \text{with} \quad d(x, y) = \{R \in \boldsymbol{R}\colon (x, y) \notin R\}$$

satisfies $d(x, x) = \varnothing$ and $d(x, z) \subseteq d(x, y) \cup d(y, z)$. Hence $X = (X, d, A, P)$ is a Boolean lattice continuity space, and $U \in U(X)$ means $N_p \subseteq U$ for some $p \in P$, that is, $\bigcap F \subseteq U$ for some finite subset F of $\boldsymbol{R}$. Hence $U(X)$ is the quasi-uniformity generated by $\boldsymbol{R}$. Of course, if $\boldsymbol{R}$ consists of equivalence relations only, then d is symmetric (but not conversely!) Furthermore, it is well known that every topology is induced by a suitable transitive quasi-uniformity (namely the so-called *Pervin quasi-uniformity*; see e.g. [8]), and that every zero-dimensional topology arises from a transitive uniformity. ∎

We shall need some more order- and lattice-theoretic definitions and notations (for background, the reader is referred to [1], [2], [6], [9], [12]). Let $(Q, \leqslant)$ (or simply Q) denote a partially ordered set (*poset*) and $\tilde{Q} = (Q, \geqslant)$ its dual.

$$(y] = \{x \in Q\colon x \leqslant y\}$$

is the *principal ideal* and

$$[y) = \{x \in Q\colon x \geqslant y\}$$

is the *principal dual ideal* generated by $y \in Q$. Furthermore,

$$\downarrow Y = \bigcup\{(y]\colon y \in Y\}$$

denotes the *lower set* and

$$\uparrow Y = \bigcup\{[y)\colon y \in Y\}$$

the *upper set*, respectively, generated by Y. The upper sets of Q form [illegible]xand-rov-discrete topology $\alpha(Q)$.

Q satisfies the *Ascending Chain Condition* (ACC) iff each nonempty subset of Q [illegible] a maximal element. The *Descending Chain Condition* (DCC) is defined dually. If each closed, bounded *interval* $[x, z] = [x) \cap (z]$ satisfies the ACC (DCC) then Q is said to satisfy the *local* ACC (*local* DCC). If, moreover, each of these intervals is finite then Q is said to be *locally finite*. For example, the chain ω of nonnegative integers is locally finite and satisfies the DCC but not the ACC. Notice that a distributive lattice satisfies both (local) chain conditions iff it is (locally) finite.

Let A be any lattice (not necessarily complete). We call an element x of A *compact* iff for every up-directed subset Y of A possessing a join with $x \leqslant \bigvee Y$, there exists some $y \in Y$ with $x \leqslant y$ (dual notion: *cocompact*). If each element of A is a join of compact elements, then A is said to be *compactly generated.* A compactly generated complete lattice is called *algebraic.* It is easy to see that a bounded lattice satisfies the ACC iff it is complete and each of its elements is compact; of course, any such lattice is algebraic.

As already mentioned earlier, the major results of this paper are for *dual Heyting algebras*, sometimes also called (*dual*) *Brouwerian lattices.* These are bounded lattices endowed with a *relative pseudocomplementation*, that is, with a binary operation "$-$" satisfying the equivalence

$$x-y \leqslant z \Leftrightarrow x \leqslant y+z.$$

The following rules are valid in any dual Heyting algebra and will be applied without comment:

$$x-y=0 \Leftrightarrow x \leqslant y \Rightarrow z-y \leqslant z-x,$$
$$x-z \leqslant (x-y)+(y-z),$$
$$(x-y)-y = x-y, \quad (x-y)+y = x+y,$$
$$x-\bigwedge Y = \bigvee (x-Y) \quad \text{where } x-Y = \{x-y\colon\ y \in Y\},$$

provided Y has a meet. Every dual Heyting algebra is a distributive lattice. Moreover, the complete dual Heyting algebras are precisely the so-called *coframes*, i.e. complete lattices satisfying the infinite distributive law

$$x+\bigwedge Y = \bigwedge (x+Y) \quad \text{where } x+Y = \{x+y\colon\ y \in Y\}.$$

Of course, the coframes are the lattice-theoretical duals of *frames*, i.e. complete lattices in which the dual identity holds, viz.

$$x\cdot\bigvee Y = \bigvee (x\cdot Y) \quad \text{where } x\cdot Y = \{x\cdot y\colon\ y \in Y\}.$$

Notice that every distributive algebraic lattice, and in particular every distributive complete lattice satisfying the ACC, is a frame but not necessarily a coframe. But, on the other hand, a frame satisfying the DCC must also be a coframe.

A stronger condition than the DCC has proved of particular relevance for modern order theory: A poset Q is said to be *well-partially ordered* if it satisfies the following equivalent conditions:

(W1) For each sequence (x_n) in Q, there are indices $m<n$ such that $x_m \leqslant x_n$.

(W2) Q satisfies the DCC and has no infinite antichains.

(W3) The lattice $\alpha(Q)$ of all upper sets satisfies the ACC.

(W4) The lattice $\alpha(\tilde{Q})$ of all lower sets satisfies the DCC.

(W5) Each upper set in Q is *finitely generated* (i.e. of the form $\uparrow F$ for some finite subset F of Q).

We later use this well-known characterization of algebraic coframes:

1.2. Lemma. *The following statements on a complete lattice A are equivalent*:

(a) *A is an algebraic coframe.*

(b) *A is a coframe in which each element is a meet of completely meet-irreducible elements (i.e. elements $q \in A$ such that $q = \bigwedge Y$ implies $q \in Y$).*

(c) *A is isomorphic to $\alpha(Q)$ for some poset Q.*

Moreover, these conditions are self-dual, that is, they are fulfilled by A iff they are fulfilled by the dual lattice $\tilde{A}$.

1.3. Corollary. *A complete lattice is a coframe with* ACC *iff it is isomorphic to $\alpha(Q)$ for some well-partially ordered set Q.*

In Section 4, we shall need a certain strengthening of compactness. Let us call an element x of a lattice A *hypercompact* if the complement of the principal dual ideal $[x)$ is a finitely generated lower set, i.e. $A\backslash[x) = \downarrow F$ for some finite set $F \subseteq A$. The dual notion is *hypercocompact.* Thus in a well-partially ordered lattice every element is hypercocompact.

1.4. Lemma. *Every hypercompact element is compact. In coframes the converse is also true.*

Proof. The first statement is clear since finitely generated lower sets are closed under directed joins. For the second assertion, assume A is a coframe and x a compact element of A. An application of Zorn's Lemma shows that $A\backslash[x) = \downarrow M$, where M is the set of maximal elements of $A\backslash[x)$. For each finite subset F of M, put $x_F = x-\bigwedge F$. These elements x_F form an up-directed set with least upper bound x (since for $x \not\leqslant z$ there is an $m \in M$ with $z \leqslant m$, and then $x_{\{m\}} = x-m \not\leqslant z$). By compactness, x must coincide with $x-\bigwedge F$ for some finite $F \subseteq M$. Now the assumption $F \neq M$ leads to a contradiction, because for $m \in M\backslash F$, the maximality of m in $A\backslash[x)$ would entail $x \leqslant \bigwedge(m+F) = m+\bigwedge F$, so $x = x-\bigwedge F \leqslant m$. Hence $\downarrow M = \downarrow F$ is finitely generated, and x is hypercompact. ∎

The completeness assumption in 1.4 (which provides the hypothesis for Zorn's Lemma) is essential:

1.5. Example. The set

$$A = \{(-1/n,\ -1/n)\colon\ n \in N\} \cup \{(\pm 1/n,\ 0)\colon\ n \in N\},$$

ordered componentwise by the usual $\leqslant$, is a non-complete dual Heyting algebra. The element $x = (-1,\ 0)$ is compact but not hypercompact. Indeed, $A\backslash[x)$ has no maximal elements at all. Notice that the normal completion $NA \simeq A \cup \{(0,\ 0)\}$ is a coframe in which x is no longer compact.

2. Topologies on lattices. In the sequel we let A denote a lattice, although most of the definitions and results easily extend to posets. Here are a dozen of the most important "intrinsic" topologies, defined in terms of the order relation on A (cf. [5], [9]):

(1) The *upper topology* $\upsilon(A)$, generated by the complements of principal ideals.

(2) The *lower topology* $\upsilon(\tilde{A})$ (denoted by $\omega(A)$ in [9]), generated by the complements of principal dual ideals.

(3) The *interval topology* $\iota(A) = \upsilon(A) \vee \upsilon(\tilde{A})$, generated by $\upsilon(A)$ and $\upsilon(\tilde{A})$.

(4) The *Scott topology* $\sigma(A)$, consisting of all upper sets $U \subseteq A$ such that whenever $D \subseteq A$ is up-directed and has a join $\bigvee D \in U$, then D meets U.

(5) The *dual Scott topology* $\sigma(\tilde{A})$.

(6) The *Lawson topology* $\lambda(A) = \sigma(A) \vee \upsilon(\tilde{A})$.

(7) The *dual Lawson topology* $\lambda(\tilde{A})$.

(8) The *bi-Scott topology* $B(A) = B(\tilde{A}) = \sigma(A) \vee \sigma(\tilde{A})$.

(9) The *order topology* $\Omega(A) = \Omega(\tilde{A})$, in which a set T is open iff for each up-directed Y and down-directed Z with $\bigvee Y = \bigwedge Z \in T$, there are $y \in Y$ and $z \in Z$ such that $[y, z] \subseteq T$ (see [3]).

(10) The *upper Alexandrov topology* $\alpha(A)$, generated by the principal dual ideals.

(11) The *lower Alexandrov topology* $\alpha(\tilde{A})$, generated by the principal ideals.

(12) The *discrete topology* $\boldsymbol{P}(A) = \{Y\colon\ Y \subseteq A\} = \alpha(A) \vee \alpha(\tilde{A})$.

These topologies, together with the indiscrete topology $\{\emptyset, A\}$, are ordered by inclusion according to the diagram after 3.5. If A is a complete lattice, they form a meet-subsemilattice of the lattice of topologies on A — a fact whose proof is a nontrivial exercise. However, there are dual Heyting algebras A for which $\iota(A) \cap \alpha(A)$ is distinct from $\upsilon(A)$: see Example 1.5, where $[x) \in \iota(A) \cap \alpha(A) \setminus \upsilon(A)$.

An arbitrary topological space $(X, \boldsymbol{T})$ carries a natural quasi-order, the *specialization* $\leqslant_{\boldsymbol{T}}$ defined by $x \leqslant_{\boldsymbol{T}} y$ iff x belongs to the closure of $\{y\}$. A topology $\boldsymbol{T}$ on A is *compatible* if $\leqslant_{\boldsymbol{T}}$ agrees with the given order relation on A. It is easy to see that this is the case if and only if $\upsilon(A) \subseteq \boldsymbol{T} \subseteq \alpha(A)$. Thus $\upsilon(A)$ is the coarsest, $\alpha(A)$ the finest compatible topology on A, and the Scott topology, as well as any other topology between $\upsilon(A)$ and $\alpha(A)$, is compatible with the order on A. Dually, the topologies $\upsilon(\tilde{A})$, $\sigma(\tilde{A})$ and $\alpha(\tilde{A})$ are compatible with the dual order of A. In contrast to these "one-sided" topologies, the "two-sided" topologies $\iota(A)$, $\lambda(A)$, $\lambda(\tilde{A})$, $B(A)$, $\Omega(A)$ and $\boldsymbol{P}(A)$ are all compatible with the identity relation $=$, because they are T_1-topologies. Unfortunately, there are various different notions of compatibility floating around in the literature. The present one is in accordance with [14], but not with [9], for example.

We also study some convergence relations on a lattice A which are well known at least in the complete case (see e.g. [1], [3], [4], [5]). A filter $\boldsymbol{F}$ on A is said to be *S-convergent* to a point $x \in A$ if there exists an up-directed set Y possessing a join with $x \leqslant \bigvee Y$ and $[y) \in \boldsymbol{F}$ for all $y \in Y$; and $\boldsymbol{F}$ is *O-convergent* to x if it is $\boldsymbol{S}$-convergent to x in A and also in $\tilde{A}$. If A is complete, then

$$\boldsymbol{F}\ \boldsymbol{S}\text{-converges to } x \text{ in } A \text{ iff } x \leqslant \liminf \boldsymbol{F} = \bigvee\{\bigwedge F\colon\ F \in \boldsymbol{F}\},$$

$$\boldsymbol{F}\ \boldsymbol{S}\text{-converges to } x \text{ in } \tilde{A} \text{ iff } x \geqslant \limsup \boldsymbol{F} = \bigwedge\{\bigvee F\colon\ F \in \boldsymbol{F}\},$$

$$\boldsymbol{F}\ \boldsymbol{O}\text{-converges to } x \text{ in } A \text{ iff } x = \liminf \boldsymbol{F} = \limsup \boldsymbol{F}.$$

As usual, the corresponding convergence structures for nets are defined by calling a net convergent to x iff the associated filter does.

By a *continuous* lattice we mean a lattice A (not necessarily complete) such that for each $x \in A$ there exists a least ideal possessing a join above x. For the following results, see [3], [4], [5] and [9]:

2.1. THEOREM. *Let A be any lattice.*

(1) *The Scott topology is the finest topology $\boldsymbol{T}$ on A such that $\boldsymbol{S}$-convergence implies $\boldsymbol{T}$-convergence.*

(2) *A is a continuous lattice iff $\boldsymbol{S}$-convergence is topological* (*thus equal to convergence in the Scott topology*).

(3) *The order topology is the finest topology $\boldsymbol{T}$ on A such that $\boldsymbol{O}$-convergence implies $\boldsymbol{T}$-convergence.*

(4) *If A and $\tilde{A}$ are continuous lattices then $\boldsymbol{O}$-convergence is topological* (*thus equal to convergence in the order topology*). *The converse does not hold.*

By a *$\boldsymbol{C}$-topology* we mean a topology $\boldsymbol{T}$ on a set X such that every point has a neighborhood base consisting of *cores*, that is, sets of the form

$$[x)_{\boldsymbol{T}} = \bigcap\{T \in \boldsymbol{T}\colon\ x \in T\} \qquad (x \in X).$$

The cores are the principal dual ideals with respect to specialization and are always compact (for us compactness does not require Hausdorff separation). Any topology with minimal base, and in particular, every Alexandrov topology is a $\boldsymbol{C}$-topology. The Scott *topology on the unit interval is an example of a* $\boldsymbol{C}$-topology which has no minimal base. It is an interesting fact that the $\boldsymbol{C}$-topologies are exactly those which are completely distributive lattices (cf. [4], [14]).

2.2. THEOREM. *Let A be any lattice.*

(1) *Every compatible $\boldsymbol{C}$-topology on A includes the Scott topology.*

(2) *A is continuous iff $\sigma(A)$ is a $\boldsymbol{C}$-topology.*

(3) *A is compactly generated iff $\sigma(A)$ has a minimal base.*

Proof. For (1), observe that for each $y \in A$, the set of all points whose core is a neighborhood of y is up-directed and has join y. Hence, if $y \in U \in \sigma(A)$ then U must contain some core neighborhood of y.

For (2), refer to [4] (see also [9] and [14]).

For (3), use the fact that a compatible topology has a minimal (equivalently, a least) base iff the open principal dual ideals form a base, and that $[x)$ is σ-open iff x is compact. ■

Given any topology $\boldsymbol{T}$ on A, let us denote by $\boldsymbol{T}^{\vee}$ the topology generated by the $\boldsymbol{T}$-open sets and all complements of cores. Thus $\boldsymbol{T}^{\vee} = \boldsymbol{T} \vee \upsilon(\tilde{A})$ whenever $\boldsymbol{T}$ is compatible with the order of A; for example, $\iota(A) = \upsilon(A)^{\vee}$ and $\lambda(A) = \sigma(A)^{\vee}$. Of course, $\boldsymbol{T}^{\vee}$ is always contained in the so-called *patch topology* which is generated by the $\boldsymbol{T}$-open sets and the complements of the $\boldsymbol{T}$-compact saturated (i.e. upper) sets (cf. [9, VII-1.16] and [13]). Recall that an ordered topological space is said to be *totally order-disconnected* if for any two distinct points there is a clopen lower set containing one of these points but not the other.

2.3. LEMMA. *If $\boldsymbol{T}$ is a compatible $\boldsymbol{C}$-topology on A then $\boldsymbol{T}^{\vee}$ is the patch topology, and if $\boldsymbol{T}$ has a minimal base then $(A, \boldsymbol{T}^{\vee})$ is totally order-disconnected.*

Proof. In order to show that the patch topology is contained in $\boldsymbol{T}^{\vee}$, it suffices to verify that any compact upper set U is finitely generated (hence a union of finitely many principal dual ideals). But this follows from the fact that each point in U has a core neighborhood contained in U, so by compactness, U is [illegible] by a finite number of these cores.

Now assume T has a minimal base. For distinct points x, y, say $x \not\leq y$, the set $A\backslash(y]$ is open (by compatibility), so there exists a T-open core $[z) \subseteq A\backslash(y]$ containing x. Hence $A\backslash[z)$ is a $T^{\vee}$-clopen lower set containing y but not x. ∎

In accordance with [9, II-3.9], we mean by a *monotone convergence space* a topological T_0-space such that each subset which is up-directed by the specialization order has a supremum and converges to it (as a net). Every *sober* space, that is, every T_0-space whose point closures are the only irreducible closed sets, is a monotone convergence space (cf. [9, II-3.17] and [26]).

2.4. THEOREM. *For a compatible C-topology T on a lattice A, the following four statements are equivalent:*

(a) $T = \sigma(A)$.

(b) *T-convergence coincides with S-convergence.*

(c) *T is consistent with the order of A, i.e. $\upsilon(A) \subseteq T \subseteq \sigma(A)$.*

(d) *Each monotone increasing net in A possessing a supremum converges to it.*

Furthermore, the subsequent six statements are equivalent:

(e) *A is complete, and $T = \sigma(A)$.*

(f) *A is complete, and $T^{\vee} = \lambda(A)$.*

(g) *$T^{\vee}$ is compact.*

(h) *T is strongly sober, i.e. every ultrafilter has a greatest T-limit.*

(i) *T is compact and sober.*

(j) *(A, T) is a compact monotone convergence space.*

Proof. (a)⇔(b): Apply 2.1(2) and 2.2(2).

(a)⇒(c): This is clear.

(c)⇒(d): If $(x_i\colon i \in I)$ is a monotone increasing net in A and $x = \bigvee\{x_i\colon i \in I\}$ then for each T-neighborhood U of x there is some $i \in I$ with $[x_i) \subseteq U$ (since $\{x_i\colon i \in I\}$ is up-directed and U is σ-open), a fortiori $x_j \in U$ for all $j \geq i$.

(d)⇒(a): By 2.2(1), $\sigma(A)$ is contained in T. Conversely, if U is a T-open neighborhood of x then U is an upper set, by compatibility of T. Furthermore, if Y is an up-directed subset of A with $\bigvee Y \in U$ then Y (considered as a monotone net) T-converges to $\bigvee Y$, so $[y) \subseteq U$ for some $y \in Y$. Hence U is σ-open.

(e) ⇒(f): This is clear because $\lambda(A) = \sigma(A)^{\vee}$.

(f)⇒(g): [See 9, III-1.9].

(g)⇒(e): Again by 2.2(1), we have $\sigma(A) \subseteq T$. For the converse inclusion, see [15]. Hence $\sigma(A)$ agrees with T, and $\lambda(A)$ with the compact topology $T^{\vee}$. Thus the coarser interval topology is also compact, and this implies that A is complete.

(e)⇒(h): lim sup F is the greatest S-limit of any filter F on A, and as we have seen before, S-convergence coincides with convergence in the topology $\sigma(A) = T$.

(h)⇒(i): See [15].

(i)⇒(j): See above.

(j)⇒(e): Apply the implication (d)⇒(a). By compactness, A must have a least element. Hence every finite and every up-directed subset has a join, and consequently A is complete. ∎

There are two well-known and related categorical duality theories for bounded distributive lattices, both extending the classical duality between Boolean lattices and Boolean spaces (see [23]). One is the *Stone duality* [24] and the other the *Priestley duality* [21]. In both cases, the categorical dual of a bounded distributive lattice A is a topological space whose underlying set is the "*spectrum*", i.e. the collection of all prime ideals of A.

The topology of the Stone dual is called the *spectral topology* or *hull-kernel topology*. It is obtained by relativizing the lower topology on the algebraic lattice $I(A)$ of all ideals, which in turn is the trace of the lower topology on the power set $P(A)$. The spectrum is meet-dense in $I(A)$, so the hull-kernel topology is isomorphic to $I(A)$. Up to homeomorphism, the *Stone spaces* (alias *spectral spaces*) arising in this way are precisely those compact (strongly) sober spaces whose compact open sets form a base B which is closed under finite intersection. The initial distributive lattice is then recognized from its Stone dual as an isomorphic copy of B (for details, see [16]).

The Priestley topology is that inherited from the Lawson topology on $I(A)$; it is the join of the Stone topologies of A and $\tilde{A}$. Since it "forgets" the given order on A, a Priestley space is to be understood as an *ordered* topological space, where the order relation on the spectrum is set inclusion. The *Priestley spaces* arising in this fashion are, up to isomorphism, the compact totally order-disconnected spaces. The collection of all clopen lower sets is isomorphic to the initial lattice A, and that of all open lower sets is simply the Stone topology. Of course, each Priestley space is Hausdorff and zero-dimensional. The direct passage between Stone spaces and Priestley spaces is supplied by associating to the Stone topology its patch topology. (For a survey, see [21].)

Summarizing previous results and definitions, we are now in a position to give some useful characterizations of consistent topologies with minimal base.

2.5. THEOREM. *The following statements are equivalent for a topology T on a lattice A:*

(a) *A is algebraic, and $T = \sigma(A)$.*

(b) *T is consistent with the order on A, has a minimal base, and A is complete.*

(c) *T is a compatible C-topology on A, and (A, T) is a Stone space.*

(d) *T is a compatible C-topology on A, and $(A, T^{\vee})$ is a Priestley space.*

Proof. (a)⇒(b): The principal dual ideals generated by compact elements form the least base of $\sigma(A)$.

(b)⇒(c): A minimal base consists of open cores, and these are compact. Moreover, by compatibility of T, the base of all open cores is closed under finite intersection (because A is complete). Furthermore, the compact open sets are precisely the finite unions of open cores; consequently, they form another base closed under finite intersection. By 2.4, T is sober.

(c)⇒(a): By 2.4, T must coincide with $\sigma(A)$. The open cores of $\sigma(A)$ are precisely the principal dual ideals generated by compact elements of A. Since each core $[x)$ is the intersection of open cores, it follows that each $x \in A$ is a join of compact elements. Also by 2.4, A must be complete.

The equivalence of (c) and (d) follows from the aforementioned one-to-one correspondence between Stone spaces and Priestley spaces, using the fact that $T^\vee$ is the patch topology of T (see 2.3). ∎

The categorical duals of the spaces in 2.5 are described in [20].

The final result of this section shows that a compatible C-topology T on A is uniquely determined by the patch topology $T^\vee$ (and the underlying order).

2.6. Lemma. *Let T be a compatible C-topology on A and $\tilde{T}$ any topology on A consisting of lower sets. Then $T = (T \vee \tilde{T}) \cap \alpha(A)$. In particular, the $T^\vee$-open upper sets are precisely the T-open sets.*

Proof. The inclusion $T \subseteq (T \vee \tilde{T}) \cap \alpha(A)$ is clear by compatibility of T. For the other inclusion, assume U is an upper set which is open in $T \vee \tilde{T}$. For $x \in U$, we find $y \in A$, $V \in T$ and $W \in \tilde{T}$ with $x \in V \subseteq [y)$ and $x \in [y) \cap W \subseteq U$. Since W is a lower set, it must contain the point y, so $y \in [y) \cap W \subseteq U$ and therefore $x \in V \subseteq [y) \subseteq U$. This shows that U is T-open. ∎

3. Sets of positives in dual Heyting algebras. Throughout this section, A denotes a dual Heyting algebra and P a set of positives in A. This simply means that P is a dual ideal with $\bigwedge P = 0$, and it is easy to see that in this case,

$$x = \bigvee (x-P) = \bigwedge (x+P) \quad \text{for all } x \in A.$$

The map $d\colon A \times A \to A$ defined by

$$d(x, y) = x - y$$

satisfies $d(x, x) = 0$ and the triangle inequality. Henceforth, d always denotes this special type of "Brouwerian quasi-metric". The *symmetric difference* $d^s(x, y) = (x-y)+(y-x)$ is a well-known tool at least in the case of Boolean lattices (see [7] for example).

We denote the "Brouwerian" continuity space (A, d, A, P) by A_P, its converse (A, d^*, A, P) by A_P^*, and its symmetrization (A, d^s, A, P) by A_P^s. The defining neighborhoods of the induced topologies are intervals. Indeed, from the equivalences

$$d(x, y) \leqslant p \Leftrightarrow x-p \leqslant y \quad \text{and} \quad d^*(x, y) \leqslant p \Leftrightarrow y \leqslant x+p$$

we infer:

3.1. Lemma. *If $x \in A$ and $p \in P$ then*

$$N_p(x) = [x-p), \quad N_p^*(x) = (x+p] \quad \text{and} \quad N_p^s(x) = [x-p, x+p].$$

This together with 1.1 yields:

3.2. Corollary. *Let $x \in A$. Then:*

(1) *$\{[x-p)\colon p \in P\}$ is an open base for the $T(A_P)$-neighborhoods of x.*

(2) *$\{(x+p]\colon p \in P\}$ is an open base for the $T(A_P^*)$-neighborhoods of x.*

(3) *$\{[x-p, x+p]\colon p \in P\}$ is a clopen base for the $T(A_P^s)$-neighborhoods of x.*

3.3. Lemma. *For any set P of positives in A:*

(1) *Convergence in A_P implies S-convergence.*

(2) *Convergence in A_P^* implies dual S-convergence.*

(3) *Convergence in A_P^s implies O-convergence.*

Proof. (1) If a filter F converges to x in A_P then $N_p(x) = [x-p) \in F$ for each $p \in P$, and the latter implies that F S-converges to x since $x-P$ is an up-directed set with join x.

Using the equation $x = \bigwedge(x+P)$, we get (2) by a similar (but not entirely dual) argument.

(3) is an immediate consequence of (1) and (2). ∎

Now we are in a position to establish the main properties of the three natural topologies induced by the continuity spaces A_P, A_P^* and A_P^s:

3.4. Theorem. *For any set P of positives in a dual Heyting algebra A:*

(1) *$T(A_P)$ is a compact C-topology compatible with the order of A. Thus*

$$\sigma(A) \subseteq T(A_P) \subseteq \alpha(A)$$

and $\{[x-p)\colon x \in A,\ p \in P\}$ is the smallest base of $T(A_P)$.

(2) *$T(A_P^*)$ is a compact C-topology compatible with the dual order of A. Thus*

$$\sigma(\tilde{A}) \subseteq T(A_P^*) \subseteq \alpha(\tilde{A})$$

and $\{(x+p]\colon x \in A,\ p \in P\} = \{(p]\colon p \in P\}$ is the smallest base of $T(A_P^*)$.

(3) *$T(A_P^s)$ is a zero-dimensional Hausdorff topology on A with*

$$\Omega(A) \subseteq T(A_P^s) \subseteq P(A)$$

and $\{[x-p, x+p]\colon x \in A,\ p \in P\}$ is a clopen base for $T(A_P^s)$. Moreover, the ordered topological space $(A, T(A_P^s))$ is totally order-disconnected.

Proof. (1) Since the base in 3.2(1) consists of principal dual ideals, each $T(A_P)$-open set is an upper set, i.e. $T(A_P) \subseteq \alpha(A)$. Now compactness follows from the fact that the only open upper set containing the least element is the entire space A. From 2.1(1) and 3.3(1) we infer the inclusion $\sigma(A) \subseteq T(A_P)$, whence $\upsilon(A) \subseteq T(A_P) \subseteq \alpha(A)$. Thus $T(A_P)$ is compatible with the order of A. In other words, $(x]$ is the closure and $[x)$ is the core of $x \in A$ with respect to $T(A_P)$. In particular, the base $\{[x-p)\colon x \in A, p \in P\}$ consists of open cores and is therefore contained in any other base of $T(A_P)$.

(2) is shown analogously, noting that $x+P \subseteq P$ for each $x \in A$.

(3) From 2.1(3) and 3.3(3) we know that $\Omega(A) \subseteq T(A_P^s)$. If $x \nleqslant y$ then we find $p \in P$ such that $x \nleqslant y+p$, so $(y+p]$ is a lower set containing y but not x, and $(y+p]$ is clopen in $T(A_P^s)$. Thus $T(A_P^s)$ is totally order-disconnected. ∎

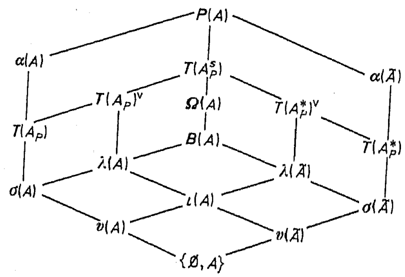
Using Lemma 2.6 and its dual, we see that the three topologies in question are related as follows:

3.5. Corollary. *For any set P of positives in a dual Heyting algebra A:*

$$T(A_P^s) = T(A_P) \vee T(A_P^*), \quad T(A_P) = T(A_P^s) \cap \alpha(A), \quad T(A_P^*) = T(A_P^s) \cap \alpha(\tilde{A}).$$

It is worth mentioning that the set of positives, P, is uniquely determined by $T(A_P^*)$ but not by $T(A_P)$. Also notice that in contrast to $T(A_P)$ and $T(A_P^*)$, the topology $T(A_P^s)$

has no minimal base unless it is discrete. The following diagram summarizes the positions of these topologies among other order-related topologies:



The maximal choice $P = A$ (which is certainly a set of positives) leads to the corresponding maximal topologies: $\boldsymbol{T}(A_P) = \alpha(A)$, $\boldsymbol{T}(A_P^*) = \alpha(\tilde{A})$, $\boldsymbol{T}(A_P^s) = \boldsymbol{P}(A)$. This shows that in general, $\boldsymbol{T}(A_P^s)$ is not compact, and it is one of our main purposes to find necessary and sufficient conditions under which it will be compact. In Section 4, we shall answer this and the question when the lower extremes are attained, i.e. when: $\boldsymbol{T}(A_P) = \sigma(A)$, $\boldsymbol{T}(A_P^*) = \sigma(\tilde{A})$, $\boldsymbol{T}(A_P^s) = \Omega(A)$. For related work on continuous posets, see [19].

The following results on the continuity of the lattice operations in these topologies are probably well known (see e.g. [18]):

3.6. Lemma. *$(A, \boldsymbol{T}(A_P))$ and $(A, \boldsymbol{T}(A_P^*))$ are topological lattices, and $(A, \boldsymbol{T}(A_P^s))$ is a topological Heyting algebra. In fact, the lattice operations are uniformly continuous with respect to d, d^*, d^s, and the relative pseudo-complementation is uniformly continuous with respect to d^s.*

Proof. Use the following three inequalities:

(1) $$(x+x')-(y+y') \leqslant (x-y)+(x'-y'),$$

since $x+x' \leqslant y+(x-y)+y'+(x'-y')$;

(2) $$xx'-yy' \leqslant (x-y)+(x'-y'),$$

since $xx' \leqslant ((x-y)+y+(x'-y'))\cdot((x-y)+y'+(x'-y')) = (x-y)+(x'-y')+yy'$;

(3) $$(x-x')-(y-y') \leqslant (x-y)+(y'-x'),$$

since $x \leqslant y'+(x-y') \leqslant x'+(y'-x')+(x-y)+(y-y')$. ∎

Cauchy nets in symmetric continuity spaces are defined as expected: A net $(x_i\colon i \in I)$ is *Cauchy* iff for each $p \in P$ there is an $i \in I$ such that if $j, k \geqslant i$ then $d(x_j, x_k) \leqslant p$. Cauchy nets and limits are easily characterized in A_P^s:

3.7. Lemma. *For any $p \in P$:*

$$d^s(x, y) \leqslant p \quad \textit{iff} \quad x+p = y+p.$$

Thus $(x_i\colon i \in I)$ is a Cauchy net in A_P^s iff for all $p \in P$ there is an $i \in I$ such that $x_i+p = x_j+p$ for $j \geqslant i$ (i.e. each net (x_i+p) is eventually constant). Similarly, $(x_i\colon i \in I)$ converges to $x \in A$ in A_P^s iff for all $p \in P$ there is an $i \in I$ such that $x+p = x_j+p$ for $j \geqslant i$.

Since $d^s(x, y) = 0$ implies $x-y = y-x = 0$, hence $x = y$, the symmetric continuity space A_P^s is always separated, and so is the corresponding uniformity $\boldsymbol{U}(A_P^s)$. The following result relates the completeness of the uniform space $(A, \boldsymbol{U}(A_P^s))$ to the lattice completeness of A.

3.8. Theorem. *Let A be a coframe and P a set of positives in A. Then the uniform space $(A, \boldsymbol{U}(A_P^s))$ is separated and complete.*

Proof. If $(x_i\colon i \in I)$ is a Cauchy net, 3.7 provides for each $p \in P$ an index $i(p)$ such that $x_{i(p)}+p = x_j+p$ whenever $j \geqslant i(p)$. Let $x = \bigwedge\{x_{i(p)}+p\colon p \in P\}$; we show that (x_i) converges to x by demonstrating that $x+q = x_{i(q)}+q$ for all $q \in P$ and then applying 3.7. By the infinite distributive law, we have $x+q = \bigwedge\{x_{i(p)}+p+q\colon p \in P\} \leqslant x_{i(q)}+q$. On the other hand, for each $p \in P$, we may choose a $j \in I$ with $j \geqslant i(p)$ and $j \geqslant i(q)$ to obtain $x_{i(q)}+q = x_j+q \leqslant x_j+p+q = x_{i(p)}+p+q$, whence $x_{i(q)}+q \leqslant \bigwedge\{x_{i(p)}+p+q\colon p \in P\} = x+q$. ∎

We conclude this section with the remark that completeness of the lattice A is not necessary for completeness of the uniform space $(A, \boldsymbol{U}(A_P^s))$. We have already mentioned that the choice $P = A$ leads to discrete uniform spaces, and these are certainly complete. Consider, for example, the rational chain $A = \{\pm 1/n\colon n \in \boldsymbol{N}\}$. Being a bounded chain, it is a dual Heyting algebra, but, of course, it is not complete. However, the uniform space $(A, \boldsymbol{U}(A_A^s))$ is complete, and $\boldsymbol{T}(A_A^s) = \iota(A) = \Omega(A) = \boldsymbol{P}(A)$.

4. When are the associated topologies minimal?

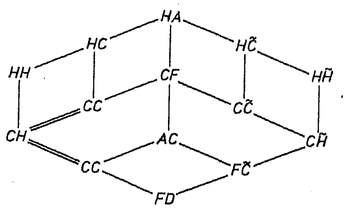
For the answer to this question, let us consider various classes of Heyting algebras. Some of them will serve as sources for sets of positives.

- ***HA***: dual Heyting algebras,
- ***HC***: dual Heyting algebras in which all elements are compact,
- ***H$\tilde{C}$***: dual Heyting algebras in which all elements are cocompact,
- ***HH***: dual Heyting algebras in which all elements are hypercompact,
- ***H$\tilde{H}$***: dual Heyting algebras in which all elements are hypercocompact,
- ***CF***: coframes,
- ***AC***: algebraic coframes,
- ***CC***: coframes with ACC = coframes in which all elements are compact,
- ***C$\tilde{C}$***: coframes with DCC = coframes in which all elements are cocompact,
- ***CH***: coframes in which all elements are hypercompact,
- ***C$\tilde{H}$***: coframes in which all elements are hypercocompact,
- ***F$\tilde{C}$***: frames with DCC (these are also coframes),
- ***FD***: finite distributive lattices (these are both frames and coframes).

From 1.2 and 1.4 we infer:

4.1. Lemma. *$\boldsymbol{CC} = \boldsymbol{CH}$, $\boldsymbol{F\tilde{C}} \subseteq \boldsymbol{C\tilde{H}} \subseteq \boldsymbol{C\tilde{C}}$ and $\boldsymbol{F\tilde{C}} \subseteq \boldsymbol{AC}$.*

Hence the above classes of Heyting algebras are ordered by inclusion according to the following diagram:



Notice also that $\boldsymbol{FD} = \boldsymbol{CC} \cap \boldsymbol{C\tilde{C}}$.

By a *strong homomorphism* between lattices A and B, we mean a map f from A into B which has both a *lower adjoint* $g\colon B \to A$ and an *upper adjoint* $h\colon B \to A$, i.e.

$$f(a) \geqslant b \Leftrightarrow a \geqslant g(b) \quad \text{and} \quad f(a) \leqslant b \Leftrightarrow a \leqslant h(b)$$

(cf. [9, 0-3]). Notice that a strong homomorphism preserves all existing joins and meets, and conversely, every complete homomorphism between complete lattices is strong. An easy verification shows that the previously introduced classes of Heyting algebras have the following useful closure properties:

4.2. Lemma. *Let $\boldsymbol{K}$ be any of the classes in Diagram* 4.1. *Then*:

(1) *$\boldsymbol{K}$ is closed under the formation of finite products.*

(2) *If a bounded lattice A is strongly embedded in some $B \in \boldsymbol{K}$ then A also belongs to $\boldsymbol{K}$.*

(3) *If A is in $\boldsymbol{K}$ then so is every interval of A.*

(4) *$\boldsymbol{K}$ is closed under images of strong homomorphisms.*

Given a subclass $\boldsymbol{K}$ of HA and $A \in \boldsymbol{HA}$, let $\boldsymbol{K}_A$ denote the set of all $p \in A$ such that the principal dual ideal $[p)$ belongs to $\boldsymbol{K}$.

4.3. Lemma. *Suppose $\boldsymbol{K}$ satisfies* 4.2 (1)–(3). *Then $\boldsymbol{K}_A$ is a dual ideal.*

Proof. If $[p)$ is in $\boldsymbol{K}$ then so is $[r)$ for each $r \in [p)$, by (3). Hence $\boldsymbol{K}_A$ is an upper set. If p and q belong to $\boldsymbol{K}_A$ then so does $p \cdot q$, since the embedding $f\colon [p \cdot q) \to [p) \times [q)$ with $f(r) = (r+p, r+q)$ is a strong homomorphism with upper adjoint

$$h\colon [p) \times [q) \to [p \cdot q), \quad h(x, y) = x \cdot y$$

and lower adjoint

$$g\colon [p) \times [q) \to [p \cdot q), \quad g(x, y) = x \cdot y + (x-p) + (y-q). \ \blacksquare$$

Denoting the class of all coframes with cocompact least element by $\boldsymbol{CC}^{\circ}$, we see that for a coframe A, $\boldsymbol{CC}^{\circ}_A$ is the set of all cocompact elements of A. Consequently, $\boldsymbol{CC}^{\circ}_A$ must be contained in any set of positives, and on the other hand, we have $\boldsymbol{C\tilde{C}}_A \subseteq \boldsymbol{CC}^{\circ}_A$. Furthermore, $\boldsymbol{CC}^{\circ}_A$ is always closed under finitary meets, but it need not be a dual ideal.

4.4. Examples. In each of the subsequent four examples, A is the only set of positives.

(1) In the following product A of two complete chains, $\boldsymbol{C\tilde{C}}_A$ is properly contained in $\boldsymbol{CC}^{\circ}_A$, and the latter fails to be an upper set:

$$A = \{0, 1\} \times (\{0, \tfrac{1}{2}\} \cup \{\tfrac{1}{2} + 1/n\colon n \geqslant 2\}) = \boldsymbol{CC}_A,$$
$$\boldsymbol{CC}^{\circ}_A = A \backslash \{(0, \tfrac{1}{2}), (1, \tfrac{1}{2})\},$$
$$\boldsymbol{FD}_A = \boldsymbol{C\tilde{C}}_A = \boldsymbol{CC}^{\circ}_A \backslash \{(0, 0), (1, 0)\}.$$

(2) If we adjoin a greatest element ∞ to the product $\{0, 1\} \times \omega$, we obtain a coframe which satisfies the DCC but is not a frame. In this example:

$$\boldsymbol{C\tilde{H}}_A = \boldsymbol{C\tilde{C}}_A = \boldsymbol{CC}^{\circ}_A = A, \quad \boldsymbol{F\tilde{C}}_A = (\{1\} \times \omega) \cup \{\infty\}, \quad \boldsymbol{FD}_A = \boldsymbol{CC}_A = \{\infty\}.$$

(3) Again adjoining a greatest element ∞, this time to the collection of all finite subsets of an infinite set, another coframe with DCC is obtained. In this example, $\boldsymbol{C\tilde{C}}_A = A$, while $\boldsymbol{C\tilde{H}}_A = \boldsymbol{F\tilde{C}}_A = \boldsymbol{FD}_A = \{\infty\}$.

(4) If a coframe is well-partially ordered then it belongs to $\boldsymbol{C\tilde{H}}$. The following example shows that the converse is not true. Let T be a countable binary tree and $A = \{{\downarrow}F\colon F \text{ is a finite subset of } T\} \cup \{T\}$. Then A is a coframe satisfying the DCC and $\boldsymbol{C\tilde{H}}_A = A$. But neither T nor A are well-partially ordered.

We now describe some of the ideals $\boldsymbol{K}_A$ in terms of suitable intrinsic topologies.

4.5. Lemma. *Let A be a dual Heyting algebra and $p \in A$.*

(1) *$p \in \boldsymbol{HC}_A$ iff for each $x \in A$, $[x-p) \in \sigma(A)$, i.e. $x-p$ is compact.*

(2) *$p \in \boldsymbol{H\tilde{C}}_A$ iff for each $x \in A$, $(x+p] \in \sigma(\tilde{A})$, i.e. $x+p$ is cocompact.*

(3) *$p \in \boldsymbol{HH}_A$ iff for each $x \in A$, $[x-p) \in \upsilon(A)$, i.e. $x-p$ is hypercompact.*

(4) *$p \in \boldsymbol{H\tilde{H}}_A$ iff for each $x \in A$, $(x+p] \in \upsilon(\tilde{A})$, i.e. $x+p$ is hypercocompact.*

(5) *$p \in \boldsymbol{HC}_A \cap \boldsymbol{H\tilde{C}}_A$ iff for each $x \in A$, $[x-p, x+p] \in B(A)$, respectively, $\Omega(A)$.*

Proof. (1) $p \in \boldsymbol{HC}_A$ means that each element of $[p)$ is compact in $[p)$. If $Y \subseteq A$ is up-directed and $x-p \leqslant \bigvee Y$, then $x+p \leqslant \bigvee Y + p = \bigvee (Y+p)$, so for some $y \in Y$, we must have $x+p \leqslant y+p$, i.e. $x-p \leqslant y$. This shows that $x-p$ is compact, or equivalently, that $[x-p) \in \sigma(A)$. Conversely, if this holds for all $x \in A$, and Y is an up-directed subset of $[p)$ with $q \leqslant \bigvee Y$ for some $q \in [p)$, then $\bigvee Y \in [q-p) \in \sigma(A)$. Thus we find some $y \in Y$ with $q-p \leqslant y$, i.e. $q \leqslant y+p = y$. Hence $p \in \boldsymbol{HC}_A$.

The proof of (2) is similar to that of (1).

(3) If $p \in \boldsymbol{HH}_A$ then for each $x \in A$, there is a finite set $F \subseteq [p)$ such that $[x+p) = [p) \backslash {\downarrow}F$. For arbitrary $y \in A$, one obtains the following equivalences:

$$x-p \leqslant y \Leftrightarrow x \leqslant y+p \Leftrightarrow x+p \leqslant y+p$$
$$\Leftrightarrow y+p \nleqslant z \text{ for all } z \in F \Leftrightarrow y \in A \backslash {\downarrow}F.$$

Conversely, assume $x-p$ is hypercompact, i.e. $[x-p) \in \upsilon(A)$ for all $x \in A$. For $q \in [p)$, choose a finite set F with $[q-p) = A \backslash {\downarrow}F$. Then $E = F \cap [p)$ is a finite subset of

$[p)$, and for arbitrary $y \in [p)$, one obtains the following equivalences:

$$y \in [p)\backslash[q) \Leftrightarrow q \not\leqslant y \Leftrightarrow q \not\leqslant y+p \Leftrightarrow q-p \not\leqslant y$$
$$\Leftrightarrow y \leqslant z \text{ for some } z \in F \Leftrightarrow y \in [p) \cap \downarrow E.$$

Hence each $q \in [p)$ is hypercompact in $[p)$.

[4] Suppose $p \in \boldsymbol{H}\tilde{\boldsymbol{H}}_A$, $x \in A$ and $q = x+p$. Then $[p)\backslash(q] = \uparrow E$ for some finite set $E \subseteq [p)$. Thus $F = E - p$ is finite, too, and $(q] = A\backslash\uparrow F \in \upsilon(\tilde{A})$; indeed, we have:

$$y \in A\backslash\uparrow F \Leftrightarrow e-p \not\leqslant y \quad \text{for } e \in E \Leftrightarrow e \not\leqslant y+p \text{ for } e \in E$$
$$\Leftrightarrow y+p \in [p)\backslash\uparrow E = [p, q] \Leftrightarrow y \leqslant q.$$

Conversely, if $(q] \in \upsilon(\tilde{A})$ for $q = x+p$ then there is a finite $F \subseteq A$ such that $q \in A\backslash\uparrow F \subseteq (q]$, and since $A\backslash\uparrow F$ is a lower set, this implies $(q] = A\backslash\uparrow F$. Hence $E = F+p$ is a finite subset of $[p)$ with $[p, q] = [p)\backslash\uparrow E$. Thus $p \in \boldsymbol{H}\tilde{\boldsymbol{H}}_A$.

(5) If $p \in \boldsymbol{HC}_A \cap \boldsymbol{H}\tilde{\boldsymbol{C}}_A$ then by (1) and (2), $[x-p, x+p] \in \sigma(A) \vee \sigma(\tilde{A}) = B(A) \subseteq \Omega(A)$. Conversely, if $[x-p, x+p]$ belongs to $\Omega(A)$ for all $x \in A$, then similar arguments as before show that each up-directed subset of $[p)$ possessing a join has a greatest element and that each down-directed subset of $[p)$ possessing a meet has a least element. Hence all elements of $[p)$ are compact and cocompact in $[p)$. ∎

Now we are prepared to answer the question in the title of this section. Combining 3.4 with 4.5, we arrive at

4.6. THEOREM. *Let A be any dual Heyting algebra and P a set of positives.*

(1) $\boldsymbol{T}(A_P) = \sigma(A)$ *iff each element of P is compact in P.*

(2) $\boldsymbol{T}(A_P^*) = \sigma(\tilde{A})$ *iff each element of P is cocompact in P.*

(3) $\boldsymbol{T}(A_P) = \upsilon(A)$ *iff each element of P is hypercompact in P.*

(4) $\boldsymbol{T}(A_P^*) = \upsilon(\tilde{A})$ *iff each element of P is hypercocompact in P.*

(5) $\boldsymbol{T}(A_P^s) = \Omega(A)$ *iff each element of P is compact and cocompact in P.*

For *complete* lattices, the above results can be improved essentially. Therefore we shall assume from now on that *A is a coframe.*

4.7. THEOREM. *The following statements about a set P of positives in a coframe A are equivalent*:

(a) *P satisfies the* ACC.

(b) $\boldsymbol{T}(A_P) = \upsilon(A)$.

(c) $\boldsymbol{T}(A_P) = \sigma(A)$.

(d) *$\boldsymbol{T}(A_P)$ is (strongly) sober.*

(e) *$\boldsymbol{T}(A_P)^\vee$ is compact.*

(f) *$(A, \boldsymbol{T}(A_P))$ is a Stone space.*

(g) *$\boldsymbol{S}$-convergence on A coincides with convergence in A_P.*

(h) *Each monotone increasing net in A is Cauchy (convergent) in A_P^s.*

(i) *Each monotone increasing net in A converges to its supremum in A_P.*

(j) *$(A, \boldsymbol{T}(A_P)^\vee)$ is a Priestley space.*

Each of these conditions implies that A is algebraic.

Proof. (a)⇔(b)⇔(c): Apply 1.4 and 4.6(1).

(c)⇔(d)⇔(e)⇔(f)⇔(g)⇔(i)⇔(j): These equivalences follow from 2.4, 2.5 and 3.4(1).

Clearly (i) implies (h).

(h)⇒(a): If Y is an up-directed subset of P then it is a Cauchy net in A_P^s, and by 3.7, it must have a greatest element. ∎

Next we establish some conditions under which there is a set of positives with the properties discussed in 4.7:

4.8. PROPOSITION. *The following statements on a coframe A are equivalent*:

(a) *A is isomorphic to $\alpha(Q)$ for some poset Q whose principal ideals are well-partially ordered sets.*

(b) *A (respectively, $\tilde{A}$) is algebraic and $\boldsymbol{CC}_A^\circ \subseteq \boldsymbol{CC}_A$.*

(c) $\bigwedge \boldsymbol{CC}_A = 0$.

(d) *$\boldsymbol{CC}_A$ is the greatest set of positives satisfying the* ACC.

(e) *$\uparrow \boldsymbol{CC}_A^\circ$ is the smallest set of positives satisfying the* ACC.

(f) *A has a set of positives satisfying the* ACC.

Proof. (a)⇒(b): By 1.2, A and $\tilde{A}$ are both algebraic. Each $p \in A$ which corresponds to the complement of a principal ideal $(q]$ in Q under the given isomorphism between A and $\alpha(Q)$, belongs to $\boldsymbol{CC}_A$. In fact, $[p)$ is isomorphic to the collection of all upper sets in Q containing $Q\backslash(q]$, hence dually isomorphic to that of all lower sets in $(q]$. Since $(q]$ is well-partially ordered, the latter satisfies the DCC, so $[p)$ satisfies the ACC. Also note that the sets $Q\backslash\downarrow F$ for finite $F \subseteq Q$ are precisely the cocompact members of $\alpha(Q)$. Using the equation $Q\backslash\downarrow F = \bigcap\{Q\backslash(q]\colon q \in F\}$ and the fact the $\boldsymbol{CC}_A$ is closed under finite meets, we finally obtain $\boldsymbol{CC}_A^\circ \subseteq \boldsymbol{CC}_A$.

(b)⇒(c): If A is an algebraic coframe then by 1.2 it is dually algebraic; thus $\bigwedge \boldsymbol{CC}_A^\circ = 0$, a fortiori $\bigwedge \boldsymbol{CC}_A = 0$ (as $\boldsymbol{CC}_A^\circ \subseteq \boldsymbol{CC}_A$).

(c)⇒(d): By 4.3, $\boldsymbol{CC}_A$ is a dual ideal, thus by (c), a set of positives. Clearly, any set of positives satisfying the ACC must be contained in $\boldsymbol{CC}_A$.

(b)⇒(e): Again, $\bigwedge \boldsymbol{CC}_A^\circ = 0$ and $\boldsymbol{CC}_A^\circ \subseteq \boldsymbol{CC}_A$ imply that $\uparrow \boldsymbol{CC}_A^\circ$ is the smallest set of positives satisfying the ACC.

The implications (d)⇒(f), (e)⇒(f) and (f)⇒(c) are clear.

(c)⇒(a): Since $\boldsymbol{CC}_A$ is a dual ideal with infimum 0, we have $\boldsymbol{CC}_A^\circ \subseteq \boldsymbol{CC}_A$, and $x = x - \bigwedge \boldsymbol{CC}_A = \bigvee (x - \boldsymbol{CC}_A)$ for each $x \in A$. By 4.5(1), $x - \boldsymbol{CC}_A$ consists of compact elements. Thus A is algebraic, and 1.2 yields an isomorphism between A and some upper set lattice $\alpha(Q)$. Now an argument similar to that for (a)⇒(b), using the inclusion $\boldsymbol{CC}_A^\circ \subseteq \boldsymbol{CC}_A$, shows that each principal ideal of Q must be well-partially ordered. ∎

In contrast to 4.7, the equation $\boldsymbol{T}(A_P^*) = \upsilon(\tilde{A})$ is not equivalent to $\boldsymbol{T}(A_P^*) = \sigma(\tilde{A})$ without additional assumptions on A. The coframe from 4.4 (3) satisfies the DCC, so the only set of positives is A. In this example, $\boldsymbol{T}(A_A^*) = \sigma(\tilde{A}) = \alpha(\tilde{A})$, while $\upsilon(\tilde{A})$ is properly coarser, as $\{\emptyset\} \notin \upsilon(\tilde{A})$.

4.9. THEOREM. *The following statements on a set P of positives in a coframe A are equivalent*:

(a) *P satisfies the local* DCC.
(b) $P = CC^{\circ}_{A}$.
(c) $T(A^{*}_{P}) = \sigma(\tilde{A})$.
(d) $T(A^{*}_{P})$ *is (strongly) sober.*
(e) $T(A^{*}_{P})^{\vee}$ *is compact.*
(f) $(A,\ T(A^{*}_{P}))$ *is a Stone space.*
(g) *S-convergence on $\tilde{A}$ coincides with convergence in A^{*}_{P}.*
(h) *Each monotone decreasing net in A is Cauchy (convergent) in A^{s}.*
(i) *Each monotone decreasing net in A converges to its infimum in A^{*}_{P}.*
(j) *$(A,\ T(A^{*}_{P})^{\vee})$ is a Priestley space.*

The proof of this theorem is quite similar (but not dual!) to that of 4.7 and refers to 3.4 (2) and 4.6 (2).

Using the fact that $C\tilde{C}_{A}$ is always a dual ideal (see 4.3), we arrive at

4.10. Corollary. *The following statements on a coframe A are equivalent*:

(a) *$\tilde{A}$ is an algebraic lattice whose compact elements form an ideal.*
(b) $\bigwedge C\tilde{C}_{A} = 0$.
(c) *$C\tilde{C}_{A}$ is the unique set P of positives with $T(A^{*}_{P}) = \sigma(\tilde{A})$.*
(d) *There is a set P of positives with $T(A^{*}_{P}) = \sigma(\tilde{A})$.*

A combination of 3.4 (2), 4.3 and 4.6 (4) yields:

4.11. Theorem. *A coframe A has a set P of positives with $T(A^{*}_{P}) = \upsilon(\tilde{A})$ iff its hypercocompact elements form a meet-dense dual ideal, and in this case $P = C\tilde{H}_{A}$ is the only such set of positives.*

If A is a well-partially ordered coframe then we certainly have $A = C\tilde{H}_{A}$, and consequently $T(A^{*}_{A}) = \upsilon(\tilde{A})$. However, this equation may also hold in the absence of a well-partial ordering, as Example 4.4 (4) shows.

Now we can prove:

4.12. Theorem. *For a set P of positives in a coframe A, the following conditions are equivalent*:

(a) $P = F\tilde{C}_{A}$.
(b) *$T(A^{*}_{P}) = \upsilon(\tilde{A})$, and A is algebraic.*
(c) *$T(A^{*}_{P}) = \sigma(\tilde{A})$, and A is algebraic.*

Furthermore, in (b) *and* (c), *"algebraic" can be replaced by "a frame", to obtain other equivalent statements.*

Proof. (a)$\Rightarrow$(b): Since $F\tilde{C}_{A} \subseteq C\tilde{H}_{A} \subseteq CC^{\circ}_{A} \subseteq P$ for any set P of positives, (a) implies $T(A^{*}_{P}) = \upsilon(\tilde{A})$, by 4.11. In order to show that A is algebraic, we shall use 1.2 twice. For each $p \in P = F\tilde{C}_{A}$, $[p)$ is a frame satisfying the DCC, hence dually algebraic. Now we infer from 1.2 that each element of $[p)$ is a meet of completely meet-irreducibles in $[p)$ (thus in A). Since P is meet-dense in A, we see that each element of A is a meet of completely meet-irreducibles, and again by 1.2, it follows that A is algebraic.

(b)$\Rightarrow$(c): Use the inclusion $\upsilon(\tilde{A}) \subseteq \sigma(\tilde{A}) \subseteq T(A^{*}_{P})$.

(c)$\Rightarrow$(a): An algebraic coframe is also a frame, by 1.2. If A is a frame with $T(A^{*}_{P}) = \sigma(\tilde{A})$ then for each $p \in P$, $[p)$ is strongly embedded in A, thus a frame, too, and by 4.9, $[p)$ satisfies the DCC. Hence $P \subseteq F\tilde{C}_{A}$, and the other inclusion is always true. ∎

4.13. Corollary. *A coframe A is algebraic and admits a set P of positives with $T(A^{*}_{P}) = \upsilon(\tilde{A})$ iff $\bigwedge F\tilde{C}_{A} = 0$ (in which case $F\tilde{C}_{A}$ is the only such set of positives).*

Example 4.4 (2) gives a non-algebraic coframe A for which $T(A^{*}_{A}) = \upsilon(\tilde{A}) = \alpha(\tilde{A})$.

Finally, we gather a list of topological and convergence-theoretic equivalents of the equation $P = FD_{A}$.

4.14. Theorem. *For a set P of positives in a coframe A, the following statements are equivalent*:

(a) *P is locally finite.*
(b) $P = FD_{A}$.
(c) *$T(A_{P}) = \upsilon(A)$ and $T(A^{*}_{P}) = \upsilon(\tilde{A})$.*
(d) *$T(A_{P}) = \sigma(A)$ and $T(A^{*}_{P}) = \sigma(\tilde{A})$.*
(e) $T(A^{s}_{P}) = \iota(A)$.
(f) *$T(A^{s}_{P})$ is compact.*
(g) *O-convergence on A coincides with convergence in A^{s}_{P}.*
(h) *Each monotone increasing or decreasing net in A is Cauchy (convergent) in A^{s}_{P}.*
(i) *$U(A^{s}_{P})$ is totally bounded.*
(j) *$(A,\ T(A^{s}_{P}))$ is a Priestley space.*

Moreover, in (e), *the interval topology $\iota(A)$ may be replaced by any other topology between $\iota(A)$ and $\Omega(A)$, e.g. by $\lambda(A)$ or $B(A)$.*

Each of these conditions implies that A is algebraic, and that $T(A^{s}_{P})$ is the patch topology of $T(A_{P})$ and also of $T(A^{*}_{P})$.

Proof. (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c)$\Leftrightarrow$(d)$\Leftrightarrow$(h): Combine 4.7 with 4.9 and 4.12, observing that P is locally finite iff it satisfies both local chain conditions.

By 3.4 (3), (e) implies $T(A^{s}_{P}) = \Omega(A)$, which in turn implies (c), by 4.6.

(d)$\Rightarrow$(g): Use the implications (c)$\Rightarrow$(g) in 4.7 and 4.9, observing that a filter econverges to x in A^{s}_{P} iff it converges to x in A_{P} and in A^{*}_{P}.

(g)$\Rightarrow$(h): Every monotone net in a complete lattice is O-convergent.

(c)$\Rightarrow$(e): This is immediate from the equation $T(A^{s}_{P}) = T(A_{P}) \vee T(A^{*}_{P})$.

(e)$\Rightarrow$(f): The interval topology is compact on any complete lattice.

(f)$\Leftrightarrow$(i): $T(A^{s}_{P})$ is the topology induced by $U(A^{s}_{P})$, and any uniformity inducing a compact topology is totally bounded.

(j)$\Leftrightarrow$(f): By 3.4 (3), $(A,\ T(A^{s}_{P}))$ is totally order-disconnected.

(i)$\Rightarrow$(a): Total boundedness of $U(A^{s}_{P})$ means that for each $p \in P$ ther is a finite $F \subseteq A$ such that $A = \bigcup\{[x-p,\ x+p]:\ x \in F\}$ (see 3.1). It follows that $[p) = \{x+p:\ x \in F\}$ is finite.

If condition (c) is fulfilled then by 4.7, A is algebraic, and from 3.4 (1), (2) and 2.3, we infer that $T(A^{s}_{P}) = T(A_{P}) \vee \upsilon(\tilde{A}) = \upsilon(A) \vee T(A^{*}_{P})$ is the patch topology of $T(A_{P})$ and of $T(A^{*}_{P})$. ∎

4.15. COROLLARY. *The following statements on a coframe A are equivalent*:

(a) $\bigwedge \boldsymbol{FD}_A = 0$.

(b) $P = \boldsymbol{FD}_A$ *is the unique set P of positives in A such that the equivalent conditions in* 4.14 *are fulfilled.*

(c) *There is a set P of positives with* $T(A_P^s) = \iota(A)(\lambda(A), B(A), \Omega(A))$.

These conditions hold whenever A is a product of finite distributive lattices, e.g., a power set. Hence our results apply to the situation discussed in [12].

4.16. COROLLARY. *Let A be a product of an arbitrary number of finite distributive lattices. Then* $\boldsymbol{FD}_A = \{p \in A\colon [p) \text{ is finite}\}$ *is the only set of positives such that* $T(A_P) = \sigma(A)$, $T(A_P^*) = \sigma(\tilde{A})$, *and* $T(A_P^s) = \iota(A) = \Omega(A)$.

References

[1] G. Birkhoff, *Lattice Theory*, 3rd edition, AMS Colloq. Publ., Providence, R.I., 1967.

[2] P. Crawley and R. P. Dilworth, *Algebraic Theory of Lattices*, Prentice-Hall, Englewood Cliffs, N.J., 1973.

[3] M. Erné, *Order-topological lattices*, Glasgow Math. J. 21 (1980), 57–68.

[4] — *Scott convergence and Scott topology in partially ordered sets II*, in: *Continuous Lattices*, Proc. Bremen 1979 (ed. B. Banaschewski and R.-E. Hoffmann), Lecture Notes in Math. 871, Springer-Verlag, Berlin–Heidelberg–New York 1981, 61–96.

[5] M. Erné and H. Gatzke, *Convergence and continuity in partially ordered sets and semilattices*, in: *Continuous Lattices and their Applications* (ed. R.-E. Hoffmann and K. H. Hofmann), Lecture Notes in Pure and Applied Math. 101, Marcel Dekker, Basel–New York 1985, 9–40.

[6] M. Erné, *Einführung in die Ordnungstheorie*, B. I. Wissenschaftsverlag, Mannheim 1982.

[7] J. Flachsmeyer, *Einige topologische Fragen in der Theorie der Booleschen Algebren*, Arch. Math. 16 (1965), 25–33.

[8] P. Fletcher and W. Lindgren, *Quasi-Uniform Spaces*, Lecture Notes in Pure and Applied Math. 77, Marcel Dekker, Basel–New York 1982.

[9] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove and D. S. Scott, *A Compendium of Continuous Lattices*, Springer-Verlag, Berlin–Heidelberg–New York 1980.

[10] L. Gillman and M. Jerison, *Rings of Continuous Functions*, van Nostrand, Princeton, N.J., 1960.

[11] M. Henriksen and M. Jerison, *The space of minimal prime ideals of a commutative ring*, in: *General Topology and its Relations to Modern Analysis and Algebra*, Prague 1962, and Trans. Amer. Math. Soc. 115 (1965), 110–130.

[12] M. Henriksen and R. D. Kopperman, *A general theory of structure spaces with applications to spaces of prime ideals*, preprint.

[13] M. Hochster, *Prime ideal structure in commutative rings*, Trans. Amer. Math. Soc. 142 (1969), 43–60.

[14] R.-E. Hoffmann, *Continuous posets, prime spectra of completely distributive lattices, and Hausdorff compactifications*, in: *Continuous Lattices*, Proc. Bremen 1979 (ed. B. Banaschewski and R.-E. Hoffmann), Lecture Notes in Math. 871, Springer-Verlag, Berlin–Heidelberg–New York 1981, 159–208.

[15] — *The Hausdorff limit in continuous lattice theory*, preprint, Bremen 1987.

[16] P. T. Johnstone, *Stone Spaces*, Cambridge University Press, Cambridge 1982.

[17] R. Kopperman, *Lengths on semigroups and groups*, Semigroup Forum 25 (1984), 345–360.

[18] — *All topologies come from generalized metrics*, Amer. Math. Monthly 95 (1988), 89–97.

[19] J. D. Lawson, *The versatile continuous order*, in: *Mathematical Foundations of Programming Language Semantics*, 3 Workshop, Tulane (ed. M. Mislove), Lecture Notes in Comput. Sci. 298, Berlin–Heidelberg–New York 1989, 134–160.

[20] H. A. Priestley, *Algebraic lattices as dual spaces of distributive lattices*, in: *Continuous Lattices and their Applications* (ed. R.-E. Hoffmann and K. H. Hofmann), Lecture Notes in Pure and Applied Math. 101, Marcel Dekker, Basel–New York 1985, 237–250.

[21] — *Ordered sets ad duality for distributive lattices*, in: Proc. Conf. Ens. Ordonnés Appl., Lyon 1982 (ed. M. Pouzet and D. Richard), North-Holland, Amsterdam 1986.

[22] H. Rasiowa and R. Sikorski, *The Mathematics of Metamathematics*, Monograf. Mat. 41, PWN, Warszawa 1963.

[23] M. H. Stone, *The theory of representations for Boolean algebras*, Trans. Amer. Math. Soc. 40 (1936), 37–111.

[24] — *Topological representation of distributive lattices and Brouwerian logics*, Cas. Mat. Fys. 67 (1937), 1–25.

[25] O. Zariski and P. Samuel, *Commutative Algebra*, Vol. I, van Nostrand, New York 1958.

[26] O. Wyler, *Dedekind complete posets and Scott topologies*, in: *Continuous Lattices*, Proc. Bremen 1979, Lecture Notes in Math. 871, Springer-Verlag, Berlin–Heidelberg–New York 1982, 384–389.

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