There are uncountably many homeomorphism types of orbits in flows

by

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Abstract. We show the existence of uncountably many non-locally compact orbits of different homeomorphism type in dynamical systems with transformation group $\boldsymbol{R}$. This answers a question of J. M. Aarts. Untill now only four different orbits were known, see [AF]. Using the characterization of non-locally compact orbits by J. M. Aarts and Z. Frolík we show that the φ-frequency is an invariant which can be used to distinguish a continuum of orbits of distinct topological type.

All topological spaces under consideration are separable metric.

Introduction. Consider a flow (X, π), i.e. π: $\boldsymbol{R} \to X$ is a continuous action of $\boldsymbol{R}$ on X. There still is no satisfying classification of the homeomorphism types of the orbits $\Gamma(x) = \{\pi(x, t) \mid t \in \boldsymbol{R}\}$. If $\Gamma(x)$ is locally compact it is well known that $\Gamma(x)$ is homeomorphic to either $\boldsymbol{R}$, S^1 or a singleton, see [I]. However, if $\Gamma(x)$ is non-locally compact not much is known. An important difference between the two types of orbits is that non-locally compact orbits are recurrent but not periodic. Recall that $\Gamma(x)$ is *recurrent* if for every neighbourhood V of x, the set $\{t \mid \pi(x, t) \in V\}$ is unbounded. $\Gamma(x)$ is *positively recurrent* if $\{t > 0 \mid \pi(x, t) \in V\}$ is unbounded.

In [AF] four different orbits in flows were distinguished. It is the purpose of this paper to show that the set of homeomorphism types has the power of the continuum.

Non-locally compact orbits were characterized by J. M. Aarts as suspensions $\Sigma(Q, h)$ of universally transitive homeomorphisms, see [A]. A homeomorphism h: $Q \to Q$ is called *universally transitive* if $Q = \{h^n(0) \mid n \in \boldsymbol{Z}\}$. One can think of a universally transitive homeomorphism as a cascade with only one orbit. Instead of universally transitive homeomorphisms we are going to use orbits in the two-sided shift σ on the Cantor set $\{0, 1\}^{\boldsymbol{Z}}$. The homeomorphism σ: $\{0, 1\}^{\boldsymbol{Z}} \to \{0, 1\}^{\boldsymbol{Z}}$ is defined by $\sigma((x_n)_n) = (x_{n+1})_n$.

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If $x \in \{0, 1\}^{\mathbf{Z}}$ is non-periodic and recurrent, then the orbit $O(x) = \{\sigma^n(x) \mid n \in \mathbf{Z}\}$ is homeomorphic to $\mathbf{Q}$. In that case σ is a universally transitive homeomorphism on $O(x)$.

J. M. Aarts proved that two orbits $\Sigma(\mathbf{Q}, h)$ and $\Sigma(\mathbf{Q}, k)$ are homeomorphic iff there are non-empty clopen sets A and B of $\mathbf{Q}$ such that the first-return maps h_A and k_B are conjugate. In this case h and k are called *first-return equivalent*. The *first-return map* $h_A\colon A \to A$ is defined by $h_A(x) = h^{n(x)}(x)$ where $n(x) = \min\{n \in N \mid n > 0, h^n(x) \in A\}$. Note that h_A is a well-defined homeomorphism if h is both positively and negatively recurrent. It is also to be observed that A and B are homeomorphic to $\mathbf{Q}$. J. M. Aarts and Z. Frolík used this result to show the existence of four different orbits in flows. They showed that an almost periodic cascade and a "wild" cascade yield non-equivalent universally transitive homeomorphisms. Consider the two-sided shift $\sigma\colon \{0, 1\}^{\mathbf{Z}} \to \{0, 1\}^{\mathbf{Z}}$. Let x be an element of $\{0, 1\}^{\mathbf{Z}}$ such that $O(x)$ is dense in $\{0, 1\}^{\mathbf{Z}}$. In this paper $(O(x), \sigma)$ is called a *wild cascade* and x is called a *wild element* of $\{0, 1\}^{\mathbf{Z}}$. Now J. M. Aarts and Z. Frolík took for both types of cascades a non-negatively recurrent and a both positively and negatively recurrent version to obtain four different orbits.

In this paper we concentrate on the second case and show that it is possible to distinguish a continuum of topologically distinct wild orbits in flows (suspensions over a wild orbit in $\{0, 1\}^{\mathbf{Z}}$). Hence in fact the existence of a continuum plus two topologically distinct orbits is shown. How can we show that there are in fact uncountably many different orbits? Translated into terms of recurrent elements in $(\{0, 1\}^{\mathbf{Z}}, \sigma)$, first-return equivalence is an equivalence relation on $\{0, 1\}^{\mathbf{Z}}$. A tempting way to try and prove that there are uncountably many different orbits is to show that the equivalence classes are of the first category. This is difficult if not impossible. The difficulty in handling first-return equivalence is that there are continuum many clopen subsets of $\mathbf{Q}$. We will take a different approach and construct elements of different types. The term type is made precise later. In general elements of different type need not be non-equivalent. However, it is possible to distinguish uncountably many non-equivalent types.

Before we start the construction we settle the notation.

Definition 1. A *word* w is an element of $\{0, 1\}^{n+1}$ for some $n \in N$. The integer $n+1$ is called the *length* of w and is denoted by $\lambda(w)$. The *dictionary* Δ is the set of all words:

$$\Delta = \bigcup \{w \mid \exists n \in N \;\; w \in \{0, 1\}^{n+1}\}.$$

In general an element x of $\{0, 1\}^{\mathbf{Z}}$ is denoted sloppily by

$$x = \ldots w_m \ldots w_{-1} w_0 w_1 \ldots w_n \ldots \quad \text{with } w_i \in \Delta \text{ for all } i \in \mathbf{Z}.$$

Of course it has to be clear from the context what coordinates correspond to the words w_i. When this is not the case we use the following notation: $(x_n) = \ldots x_{-2} x_{-1} \mid x_0 x_1 \ldots$ to mark where the negative coordinates end. We say that x *contains* w if $x = \ldots w \ldots$ For $w = (\xi_n)_{n=0}^m \in \Delta$ the *inverse word* $\bar{w} = (\xi_{m-n})_{n=0}^m$. An element x of $\{0, 1\}^{\mathbf{Z}}$ is *symmetric* if $x = (\xi_n)$ with $\xi_n = \xi_{-n}$ for all $n \in N$.

Note that an element x of $\{0, 1\}^{\mathbf{Z}}$ has a dense orbit under σ iff x contains all words. First we distinguish only three non-equivalent elements of $\{0, 1\}^{\mathbf{Z}}$ with the help of the frequency (of return). Intuitively, this is the clearest of the invariants we are going to use.

Definition 2. Let x be an element of the Cantor set and let V, W be clopen subsets of $O(x)$ such that $V \subset W$. For $n > 0$ the *n-frequency* of V with respect to W is defined by

$$f_n(V, \sigma_W, x) = \inf\{|\{m \in N \mid m \leqslant N, (\sigma_W)^m(x) \in V\}|/N \mid N \geqslant n\}.$$

The *frequency of return* of V with respect to W is defined as

$$f(V, \sigma_W, x) = \lim_{n\to\infty} f_n(V, \sigma_W, x).$$

Hence $f(V, \sigma_W, x)$ is the limes inferior of the sequence

$$(|\{m \in N \mid m \leqslant N, (\sigma_W)^m(x) \in V\}|/n)_n.$$

Similarly we can define $f_n(V, (\sigma^{-1})_W, x)$ and $f(V, (\sigma^{-1})_W, x)$.

This definition is inspired by the behaviour of almost periodic cascades. Recall that a point $x \in \{0, 1\}^{\mathbf{Z}}$ is almost periodic if for every open subset $V \subset O(x)$ the set $\{m \mid \sigma^m(x) \in V\}$ has bounded gaps. Hence for every clopen $V \subset W$ the frequency $f(V, \sigma_W, x)$ is positive. This is in general not true for wild elements.

Let x be an element of $\{0, 1\}^{\mathbf{Z}}$ and let $\{V_n \mid n \in N\}$ be a neighbourhood basis at x of clopen sets in $O(x)$, such that for every $n \in N$, $V_{n+1} \subset V_n$. The return map to V_n is denoted by σ_n to save indices. We distinguish the following three types of orbits:

Type 1. For every neighbourhood V of x both $f(V, \sigma^{-1}, x)$ and $f(V, \sigma, x)$ are positive.

Type 2. For every neighbourhood V of x, $f(V, \sigma^{-1}, x)$ is positive and $f(V_{n+1}, \sigma_n, x) = 0$ for all $n \in N$.

Type 3. For every $n \in N$, $f(V_{n+1}, \sigma_n^{-1}, x) = f(V_{n+1}, \sigma_n, x) = 0$.

It is not hard to show that elements of different type are non-equivalent. Let $A \subset B \subset C \subset D$ be a chain of clopen sets and let x be an element of A. From the definition of frequency it is clear that

$$f(A, \sigma_D, x) \leqslant \begin{Bmatrix} f(B, \sigma_D, x) \\ f(A, \sigma_C, x) \end{Bmatrix} \leqslant f(B, \sigma_C, x).$$

The following lemma is an easy exercise in these inequalities.

Lemma 1. *Elements of different type are not first-return equivalent.*

Proof. Suppose x is of type 1 and y of type 2 for some $x, y \in \{0, 1\}^{\mathbf{Z}}$. Let $\{V_n \mid n \in N\}$ be a neighbourhood basis of clopen sets at y such that $f(V_{n+1}, \sigma_n, y) = 0$ for every $n \in N$. If $(O(x), \sigma)$ and $(O(y), \sigma)$ are first-return equivalent there are clopen sets $A \subset O(x)$ and $B \subset O(y)$ such that σ_A is conjugate to σ_B. Without loss of generality we may assume that $x \in A$, $y \in B \subset V_0$ and that the conjugating homeomorphism $\gamma\colon A \to B$ maps x onto y. There is an $n \in N$ such that $V_n \subset B$. Since y is of type 2 the frequency $f(V_{n+1}, \sigma_B, y) = 0$. On the other hand,

$$f(V_{n+1}, \sigma_B, y) = f\big(\gamma^{-1}(V_{n+1}), \gamma^{-1} \circ \sigma_B \circ \gamma, x\big) = f\big(\gamma^{-1}(V_{n+1}), \sigma_A, x\big) > 0$$

since x is of type 1.

The proof in the other cases is similar.

Using the fact that $\{0, 1\}^{\mathbf{Z}}$ is a topological group we define, for a clopen neighbourhood V of 0, $n > 0$ and $\varepsilon > 0$, the set

$$F_n(V, \sigma, \varepsilon) = \{x \mid f_n(x+V, \sigma, x) \geqslant \varepsilon\}.$$

This is a closed set with empty interior. Let $\{V_n \mid n \in N\}$ be a neighbourhood basis of clopen sets at 0.

The set of elements of type 1 equals

$$\Big(\bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} F_k\Big(V_n, \sigma, \frac{1}{m}\Big)\Big) \cap \Big(\bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} F_k\Big(V_n, \sigma^{-1}, \frac{1}{m}\Big)\Big).$$

This is a set of first category. It is our task to show that it has non-empty intersection with the G_δ of wild elements:

$$\bigcap_{w \in \Delta} \{x \in \{0, 1\}^{\mathbf{Z}} \mid x \text{ contains } w\}.$$

Observe that the set of elements of type 1 is not empty since it contains all almost periodic elements.

Construction 1. We want to construct a wild element $x \in \{0, 1\}^{\mathbf{Z}}$ of type 1. Hence x has to satisfy the following conditions:

(a) x contains all words.

(b) There is a neighbourhood basis $\{V_n \mid n \in N\}$ of clopen sets at x in $\{0, 1\}^{\mathbf{Z}}$ such that $f(V_n, \sigma, x)$ and $f(V_n, \sigma^{-1}, x)$ are positive for all $n \in N$.

Note that condition (b) implies that x is both positively and negatively recurrent. In fact the constructed x will be symmetric, so we only have to worry about $f(V_n, \sigma, x)$!

Let $\pi\colon N \to \Delta$ be an enumeration of the dictionary such that $\pi(0) = 0$. We will approximate x with periodic points: a sequence (x_n) is constructed with $\lim_{n\to\infty} x_n = x$ and every x_n is periodic. At step n we make sure that $\pi(n)$ is inserted in x_n. Simultaneously we construct a descending sequence $\{V_n \mid n \in N\}$ of clopen neighbourhoods of x such that $V_n \subset V_m$ for all $n > m$ and $x_n \in V_n$. We have to make sure that $f(V_n, \sigma, x) > 0$. It is certainly true that $f(V_n, \sigma, x_n) > 0$ since x_n is periodic. This property must not be lost in the limit, so we choose integers i_n and positive real ε_n such that $f_{i_n}(V_n, \sigma, x_m) \geqslant \varepsilon_n$ for every $m \geqslant n$.

Let us sketch the first steps: choose $x_0 = 0 = \ldots 0000 \ldots$ Instead of zeros we use a's since there are more than two letters in the alphabet; $x_0 = \ldots aaaa \ldots$ and $V_0 = \{(\xi_n) \mid \xi_0 = 0\}$. Observe that x_0 contains $\pi(0) = 0$. The frequency of the zeros in x_0 is equal to 1. During the inductive proces it must not drop below $\varepsilon_0 = \frac{1}{2}$. Now we have to insert $i(1)$ in x_0. Let b be the word $b = \overline{\pi(1)}aaa \ldots aaa\, \pi(1)$. The periodic element $x_1 = \ldots bbbb \ldots$ contains $\pi(1)$ and $\pi(0)$ and the b's are placed in such a way that x_1 is symmetric and the zeroth coordinate is 0. The neighbourhood $V_1 = \{x \mid x = \ldots b \ldots\}$. The frequency of zeros in x_1 is larger than $\frac{1}{2}$ for a suitable choice of b and the frequency of b's is larger than some $\varepsilon_1 > 0$. At the following step the word $\pi(2)$ has to be inserted in x_1 in such a way that the frequency of the zeros does not drop below $\frac{1}{2}$ and the frequency of b's does not drop below ε_1. To this end take $c = \overline{\pi(2)}\, bbb \ldots bbb\pi(2)$ and define $x_2 = \ldots cccc \ldots$, etc.

In this way we proceed inductively to define x_n, V_n, ε_n and i_n, satisfying the following conditions:

(i) x_n contains $\pi(0), \pi(1), \ldots, \pi(n)$.

(ii) $x_n \in V_m$ for all $m \leqslant n$.

(iii) x_n is symmetric and periodic.

(iv) $V_n \subset V_m$ for all $m \leqslant n$.

(v) $f_{i_m}(V_m, \sigma, x_n) = f_{i_m}(V_m, \sigma^{-1}, x_n) \geqslant \varepsilon_m$ for all $m \leqslant n$.

Suppose we have defined x_n, V_n, ε_n and i_n; now we have to construct x_{n+1}, V_{n+1}, ε_{n+1} and i_{n+1}. The non-precise expression "very long" is used to indicate that we want x_{n+1} to satisfy condition (v).

Let x_n be the element $\ldots vvvv \ldots$ and let V_n be the neighbourhood $\{x \mid x = \ldots v \ldots\}$ (it is obvious where v has to be placed: symmetric with respect to 0). Now $w = \overline{\pi(n+1)}vvv \ldots vvv\pi(n+1)$ for a very long sequence of v's. The periodic point $x_{n+1} = \ldots wwww \ldots$ is an element of V_n such that $f_{i_m}(V_m, \sigma, x_{n+1}) \geqslant \varepsilon_m$ for all $m \leqslant n$. We choose $V_{n+1} = \{x \mid x = \ldots w \ldots\}$ such that $x_{n+1} \in V_{n+1}$ and $V_{n+1} \subset V_n$. Obviously it is possible to choose ε_{n+1} and i_{n+1} such that $f_{i_{n+1}}(V_{n+1}, \sigma, x_{n+1}) \geqslant \varepsilon_{n+1}$.

In this way we construct a sequence (x_n) such that $\lim_{n\to\infty} x_n$ is a wild element of type 1. ∎

Remark. We could have dropped the condition that x contains all words. Then it would suffice to construct an almost periodic element of $(\{0, 1\}^{\mathbf{Z}}, \sigma)$. A well known method, reminiscent of Construction 1, is to construct such an element by substitution, see [MH].

The proof that elements of type 1 exist is satisfactory in the sense that we have actually constructed such an element. The proof that elements of type 2 and 3 exist can be given constructively as we will see below. Here is a non-constructive proof that elements of type 3 exist:

Let $\{V_n \mid n \in N\}$ be a neighbourhood basis of clopen sets at 0. Consider $F_m(V_{n+1}, \sigma_n, \varepsilon) = \{x \in \{0, 1\}^{\mathbf{Z}} \mid f_m(x+V_{n+1}, \sigma_{x+V_n}, x) \geqslant \varepsilon\}$. This is a closed set with empty interior in $\{0, 1\}^{\mathbf{Z}}$. The complement of elements of type 3 is equal to:

$$\Big(\bigcap_{n=1}^{\infty}\bigcup_{m=1}^{\infty}\bigcup_{N=1}^{\infty}\bigcap_{k=N}^{\infty} F_k\Big(V_{n+1}, \sigma_n, \frac{1}{m}\Big)\Big)\cup\Big(\bigcap_{n=1}^{\infty}\bigcup_{m=1}^{\infty}\bigcup_{N=1}^{\infty}\bigcap_{k=N}^{\infty} F_k\Big(V_{n+1}, \sigma_n^{-1}, \frac{1}{m}\Big)\Big).$$

This is a set of first category. Hence it is possible to distinguish three non-equivalent elements of $\{0, 1\}^{\mathbf{Z}}$. ∎

Is it possible to refine the notation of frequency in order to obtain more orbits? The first things that comes to mind is to measure the number of returns against a function depending on the time. For instance, we may consider the limes superior of the sequence

$$|\{m \leqslant N|\ \sigma(x) \in V\}|/\log N.$$

It is necessary to impose a few restrictions on the function of the time. As usual the integer part of $c \in \mathbf{R}$ is denoted by $[c]$.

Definition 3. $\mathcal{F}$ is a subset of the set of all functions from $N\backslash\{0\}$ to $N\backslash\{0\}$. $\varphi \in \mathcal{F}$ iff

(a) φ is increasing and surjective;

(b) For every $c \in \mathbf{R}_{>0}$ there is a $C \in \mathbf{R}_{>0}$ such that $\varphi([cn]) \leqslant C\varphi(n)$ for all $n \in N\backslash\{0\}$.

Note that from these conditions a third can be derived:

(c) For every $d \in \mathbf{R}_{>0}$ there is a $D \in \mathbf{R}_{>0}$ such that $\varphi([dn]) \geqslant D\varphi(n)$ for all $n \in N\backslash\{0\}$.

According to condition (b) there is a constant C such that $\varphi([d^{-1}n]) \leqslant C\varphi(n)$. Hence $\varphi(n) \leqslant \varphi([d^{-1}[dn]]) + d^{-1} \leqslant C\varphi([dn]) + d^{-1}$. From this inequality it is possible to derive condition (c). We now are able to generalize the definition of frequency.

Definition 4. Let x be an element of the Cantor set and let V, W be clopen subsets of $O(x)$ such that $V \subset W$. For $n > 0$ the *n-φ-frequency* of V with respect to W is defined by

$$f_{n,\varphi}(V, \sigma_W, x) = \inf\{|\{m \in N|\ m \leqslant N, (\sigma_W)^m(x) \in V\}|/\varphi(N)|\ N \geqslant n\}.$$

The *φ-frequency* (*of return*) $f_\varphi(V, \sigma_W, x)$ is equal to $\lim_{n\to\infty} f_{n,\varphi}(V, \sigma_W, x)$.

In general the φ-frequency is harder to handle than the frequency. The main difference is that the φ-frequency can be infinite. We say that the φ-frequency is *positive* if it is neither 0 nor ∞. Before we are going to imitate Construction 1 we must decide what type of element has to be constructed.

Definition 5. An element x of $\{0, 1\}^{\mathbf{Z}}$ is of *type* φ if there exists a neighbourhood basis of clopen sets $\{V_n | n \in N\}$ at x such that both $f_\varphi(V_{n+1}, \sigma_n, x)$ and $f_\varphi(V_{n+1}, (\sigma^{-1})_n, x)$ are positive for every $n \in N$.

Is it worthwhile to construct elements of different types? For example: is an element of type $[\sqrt{\ }]$ non-equivalent to an element of type [log]? It is our first task to prove that the answer to both questions is yes.

The k-fold iteration $\varphi\circ\varphi\circ\ldots\circ\varphi$ of φ is denoted by φ^k. Note that if $\varphi \in \mathcal{F}$, then $\varphi^k \in \mathcal{F}$.

Lemma 2. *Suppose x is an element of type φ with respect to the neighbourhood basis $\{V_n|\ n \in N\}$. For all $n, k \in N$ the φ^k-frequency $f_{\varphi^k}(V_{n+k}, \sigma_n, x)$ is positive.*

Proof. We prove this for $k = 2$, the lemma follows by induction. Since x is of type φ there are $\varepsilon, M \in \mathbf{R}_{>0}$ such that

$$\varepsilon \leqslant |\{m \in N|\ m \leqslant N, (\sigma_n)^m(x) \in V_{n+1}\}| \cdot \varphi(N)^{-1} \leqslant M,$$

$$\varepsilon \leqslant |\{m \in N|\ m \leqslant N, (\sigma_{n+1})^m(x) \in V_{n+2}\}| \cdot \varphi(N)^{-1} \leqslant M$$

for all sufficiently large N. The inequalities imply that

$$\varepsilon \cdot \varphi([\varepsilon\varphi(N)]) \leqslant |\{m \in N|\ m \leqslant N, (\sigma_n)^m(x) \in V_{n+2}\}| \leqslant M\varphi([M\varphi(N)] + 1)$$

for all sufficiently large N.

By condition (b) and (c) on the function φ there exist constants $\delta, K \in \mathbf{R}_{>0}$ such that

$$\delta \leqslant |\{m \in N|\ m \leqslant N, (\sigma_n)^m(x) \in V_{n+2}\}| \cdot (\varphi^2(N))^{-1} \leqslant K$$

for all sufficiently large N. This proves Lemma 2.

The lemma implies that there is no difference between elements of type φ and of type φ^2. However, it also implies that an element of type φ and an element of type ψ are non-equivalent if ψ is considerably smaller than φ^n for every $n \in N$.

Definition 6. Let φ, ψ be elements of $\mathcal{F}$. φ is called *smaller* than ψ, denoted by $\varphi \ll \psi$, if

$$\lim_{n\to\infty} \frac{\varphi(n)}{\psi^k(n)} = 0 \quad \text{for all } k \in \mathbf{N}.$$

Now we can generalize Lemma 1.

Lemma 3. *Let x be an element of type φ and let y be an element of type ψ. If $\psi \ll \varphi$ then $(O(x), \sigma)$ and $(O(y), \sigma)$ are not first-return equivalent.*

The proof of Lemma 3 is almost an exact copy of the proof of Lemma 1.

Proof. Let x be of type φ with respect to $\{V_n|\ n \in N\}$, a neighbourhood basis of clopen sets at x. Similarly, let y be of type ψ with respect to $\{W_n|\ n \in N\}$. Suppose that $(O(x), \sigma)$ and $(O(y), \sigma)$ are first-return equivalent. Then there are clopen subsets $A \subset O(x)$ and $B \subset O(y)$ such that σ_A is conjugate to σ_B. Without loss of generality we may assume that $x \in A \subset V_0$ and $y \in B \subset W_0$ and that the conjugating homeomorphism γ maps x onto y.

Choose $n \in N$ such that $V_n \subset A$. There is an $m \in N$ such that $W_m \subset \gamma(V_{n+1})$. Hence

$$0 < f_{\psi^m}(W_m, \sigma_0, y) \leqslant f_{\psi^m}(\gamma(V_{n+1}), \sigma_0, y) \leqslant f_{\psi^m}(V_{n+1}, \sigma_n, x),$$

from which it follows that

$$\lim_{N\to\infty} |\{k|\ k \leqslant N, (\sigma_n)^k(x) \in V_{n+1}\}|/\psi^m(N) > 0.$$

This contradicts the fact that

$$\lim_{N\to\infty} \varphi(N)/\psi^m(N) = 0.$$

We have convinced ourselves that it is worthwhile to construct elements of type φ. So all that remains is an imitation of Construction 1. This is straightforward except for one technical detail. As pointed out before we must be extra careful since the φ-frequency may be infinite. Whereas we had to make sure in Construction 1 that the frequency of return of x_{i+1} to V_{i+1} was not zero, now we have to make sure as well that it is not infinite. The element x_{i+1} is obtained from x_i by inserting words. In order to deal with the problem of inserting too many words we label them. This is done by using the maximal number of consecutive zeros. Let $\pi\colon N\to \Delta$ be an injection such that the maximal number of consecutive zeros in $\pi(n)$ is equal to $n+1$. It is possible to choose π in such a way that for every $w\in\Delta$ there is an $i\in N$ such that w is contained in $\pi(i)$. Also $\pi(0)=0$. We assume that if $\varphi(n)-\varphi(n-1)=1$, then $\varphi(n+1)-\varphi(n)=0$ and that $\varphi\ll \mathrm{id}$.

Construction 2. Again we are going to construct a sequence (x_n) such that $x=\lim_{n\to\infty}x_n$ is a wild element of type φ. At first sight the construction may seem different from Construction 1, but in fact it is a straightforward imitation.

Since $\pi(0)=0$ we want x_0 to be an element such that the φ-frequency of the zeros in x_0 is positive. Define $x_0=(\xi_n)_n$ with $\xi_n=0$ iff $\varphi(|n|)-\varphi(|n|-1)=1$ for $|n|>1$, $\xi_0=0$ and $\xi_{-1}=\xi_1=1$. Observe that in the case that $\varphi=\mathrm{id}$ is allowed we get almost the same x_0 as in Construction 1. As in Construction 1 we denote the zeros by a's. Now we have to insert $\pi(1)$ in x_0. Take a very long block $(\xi_n)_{-N}^{N}=11\ldots1|01\ldots11$ and insert $\overline{\pi(1)}$ between ξ_{-N} and ξ_{-N+1}, insert $\pi(1)$ between ξ_{N-1} and ξ_N. Recall that the $|$ marks the zeroth coordinate. The obtained word is called b. The important words in x_0 are the a's, the 1's are there to get the right frequency. In Construction 1 we replaced every a by b. In this case that would be too much, just like $\ldots0000\ldots$ contains too many zeros to be the right x_0. That is why we replace the nth a by b iff $\varphi(n)-\varphi(n-1)=1$. We must convince ourselves that the φ-frequency of a's in x_1 is at least $\frac{1}{2}$, as in Construction 1. It comes as no surprise that this holds true; because $\varphi\ll\mathrm{id}$, the fraction of altered a's is negligible. We choose

$$V_1=\{x\mid x=\ldots1\overline{\pi(1)}1\ldots1|01\ldots1\pi(1)1\ldots\},$$

the φ-frequency of the b's with respect to the a's is obviously greater than 0.

We construct x_n, V_n, $i_n\varepsilon_n$ and M_n satisfying the following five conditions:

(i) x_i contains $\pi(0),\pi(1),\ldots,\pi(n)$.
(ii) $x_n\in V_m$ for all $m\leqslant n$.
(iii) x_n is symmetric and does not contain $n+2$ consecutive zeros.
(iv) $V_n\subset V_m$ for all $m\leqslant n$.
(v) $\varepsilon_m\leqslant f_{i_m,\varphi}(V_{m+1},\sigma_m,x_n)=f_{i_m,\varphi}(V_{m+1},\sigma_m,x_n)\leqslant M_m$ for all $m\leqslant n-1$.

Suppose we have defined x_n, V_n, ε_n, i_n and M_n; now we have to construct x_{n+1}, V_{n+1}, ε_{n+1}, i_{n+1} and M_{n+1}. It follows from the inductive procedure that $V_n=\{x\mid x=\ldots v\ldots\}$ for some word v, just as in Construction 1 (v is symmetric and contains the 0th coordinate. Unlike the x_n from Construction 1 the x_n in Construction 2 is not periodic. Between two subsequent v's there are gaps in order to obtain the right frequency. Let $(\xi_n)_{-N}^{N}$ be a very long word in x_n such that $(\xi_n)_{-N}^{N}=11\ldots v\ldots11$ (v contains the 0th coordinate). We insert $\overline{\pi(n+1)}$ behind ξ_{-N} and in $\pi(n+1)$ front of ξ_N. In this way we obtain a word w. Observe that the word w does not occur in x_n since w contains $n+2$ consecutive zeros and x_n does not. Therefore if we alter every mth v into a w iff $\varphi(|m|)-\varphi(|m|-1)=1$, then the φ-frequency of w with respect to v becomes positive. We must take care that we do not spoil the φ-frequencies of the previous words. Let K be an integer, the fraction $(\varphi^{(n)}-\varphi^{(K)})/n$ can be made arbitrarily small (for all $n\in N$) since $\varphi\ll\mathrm{id}$. So if we take $(\xi_n)_{-N}^{N}$ sufficiently long and K sufficiently large, then we can insert $\overline{\pi(n+1)}$ behind ξ_{-N}, $\pi(n+1)$ in front of ξ_N and we can alter every mth word v into w if $|m|>K$ and $\varphi(|m|)-\varphi(|m|-1)=1$ all without violating the conditions (i) through (v). If $V_{n+1}=\{x\mid x=\ldots w\ldots\}$ (w contains 0th coordinate), ε_{n+1} is chosen smaller than the φ-frequency of w with respect to v, then we can find a suitable i_{n+1}.

The reader might argue that two different v's can overlap. This is no serious obstruction to Construction 2, but we silently assumed that it was not possible. To repair the argument we may use a more sophisticated label: there is only one block row of $n+1$ consecutive zeros in $\pi(n)$. Since we have chosen π such that $\{\pi(n)\mid n\in N\}$ contains all words $\lim_{n\to\infty}x_n=x$ is a wild element. Also $\varepsilon_m\leqslant f_{i_m,\varphi}(V_{m+1},\sigma_m,x)\leqslant M_m$ since this is true for every x_n with $n\geqslant m+1$. We conclude that x is a wild element of type φ. ∎

The last step we have to take is to show that there are continuum many non equivalent elements of $\{0,1\}^{\boldsymbol{Z}}$. This is an easy exercise in set theory. We first prove that $(\mathscr{F},\ll)$ contains a set of order type $\boldsymbol{Q}$. It suffices to prove the following lemma.

LEMMA 4. *Let φ, ψ be elements of $\mathscr{F}$. If $\varphi\ll\psi$, then there is a $\gamma\in\mathscr{F}$ such that $\varphi\ll\gamma\ll\psi$.*

Proof. This depends upon a fairly standard diagonal argument. For technical reasons we want ψ to satisfy the following property.

($*$) If $\gamma\in\mathscr{F}$ and for every $k\in N$, $\gamma(n)\leqslant\psi^k(n)$ for all sufficiently large n, then $\gamma\ll\psi$.

The identity does not satisfy ($*$), but the equivalent $\psi(n)=[\frac{1}{2}n]$ does!

Therefore, if ψ does not satisfy ($*$) then we replace it by $\tilde{\psi}(n)=[\frac{1}{2}\cdot\psi(n)]$. Since $\varphi\ll\psi$ we can choose a sequence of integers $N_1<N_2<\ldots$ such that if $n>N_k$, then $\psi^k(n)\geqslant 2^k\varphi(n)$. We define γ by $\gamma(n)=\max\{\gamma(n-1),\psi^k(n)\}$ if $N_{k^2}\leqslant n<N_{(k+1)^2}$. Obviously $\gamma(n)\leqslant\psi^k(n)$ for sufficiently large n. By property ($*$), $\gamma\ll\psi$.

Also for $n>N_{k^2}$ the following inequalities hold:

$$\gamma^k(n)\geqslant(\psi^m)^k(n)\quad\text{if } N_{m^2}\leqslant n<N_{(m+1)^2}\quad\text{and}\quad(\psi^m)^k(n)=\psi^{mk}(n)\geqslant 2^{m^2}\varphi(n).$$

This implies that $\varphi\ll\gamma$. It is clear from the definition of γ that this function satisfies condition (a) and (b) on elements of $\mathscr{F}$. ∎

Next we show that it is in fact possible to embed a set of order type $\boldsymbol{R}$ in $(\mathscr{F},\ll)$. Lemma 5 implies that there exists a continuously ordered subset of $(\mathscr{F},\ll)$ which contains the rationals. Therefore $(\mathscr{F},\ll)$ contains the ordered set $\boldsymbol{R}$, see [K], p. 84.

LEMMA 5. *Let $\varphi_1\ll\varphi_2\ll\varphi_3\ll\ldots$ and $\psi_1\gg\psi_2\gg\varphi_3\gg\ldots$ be a pair of sequences in $\mathscr{F}$ such that $\varphi_n\ll\psi_m$ for every n, $m\in N$. There exists a $\gamma\in\mathscr{F}$ such that $\varphi_n\ll\gamma\ll\psi_m$ for every n, $m\in N$.*

Proof. The proof is almost identical to the proof of Lemma 4. Without loss of generality every φ_i, ψ_j satisfies $(*)$ and also for all $i<j$ and all n we assume that $\varphi_i(n) \leqslant \varphi_j(n)$ and $\psi_i(n) \geqslant \psi_j(n)$. Again we choose an ascending sequence of integers $N_1 < N_2 < \ldots$ such that if $n > N_k$ then $(\psi_k)^k(n) > 2^k \varphi_k(n)$. The function γ is defined by $\gamma(n) = \max\{\gamma(n-1), \psi_k^k(n)\}$ if $N_{k^2} \leqslant n < N_{(k+1)^2}$. Obviously $\gamma \ll \psi_i$ for every $i \in \mathbf{N}$. Also if $n > N_{k^2}$, then

$$\gamma^k(n) \geqslant (\psi_m)^{mk}(n) \quad \text{if } N_{m^2} < n < N_{(m+1)^2}.$$

Therefore

$$\gamma^k(n) \geqslant (\psi_m)^{m^2}(n) \geqslant (\psi_{m^2})^{m^2}(n) \geqslant 2^{m^2} \varphi_{m^2}(n) \geqslant 2^k \varphi_k(n),$$

which implies that $\varphi_n \ll \gamma$ for all $n \in \mathbf{N}$. ■

The main theorem now follows as a corollary of all the work we have done.

THEOREM. *There exists a continuum of topologically distinct orbits.*

Proof. According to Lemmas 4 and 5 the space $(\mathscr{F}, \ll)$ contains $\mathbf{R}$ as an ordered subset. Any two different elements of $\mathbf{R}$ correspond to non-equivalent orbits according to Lemma 3.

We conclude with an unsolved problem. Consider an irrational rotation ϱ_α on the circle S^1. The flow $\Sigma(S^1, \varrho_\alpha)$ is called an irrational flow on the torus. It is well known when $\Sigma(S^1, \varrho_\alpha)$ and $\Sigma(S^1, \varrho_\beta)$ are equivalent. Consider one orbit Γ_α from $\Sigma(S^1, \varrho_\alpha)$ and Γ_β from $\Sigma(S^1, \varrho_\beta)$. It is highly unsatisfactory that the following question has not been answered yet:

QUESTION. Are there α and β such that Γ_α and Γ_β are not homeomorphic?

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Natural continuity space structures on dual Heyting algebras

by

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Abstract. Every dual Heyting algebra carries three intrinsic "generalized quasi-metrics": $d(x, y) = x-y$, $d^*(x, y) = y-x$, and $d^s(x, y) = (x-y)+(y-x)$, where $x-y$ denotes the relative dual pseudocomplement. Formally, these are "continuity functions" satisfying the triangle inequality. In a dual Heyting algebra, a set of positives is a dual ideal P whose meet is 0. We investigate properties of the topologies, $T(A_P)$, $T(A_P^*)$, $T(A_P^s)$, which arise from the continuity spaces so defined. For example, $T(A_P)$ and $T(A_P^*)$ are completely distributive, and $T(A_P^s)$ is a zero-dimensional Hausdorff topology. Furthermore, we show that for any coframe, that is, for any complete dual Heyting algebra A:

(1) $T(A_P)$ is the Scott topology iff P satisfies the ascending chain condition.
(2) $T(A_P^*)$ is the dual Scott topology iff P satisfies the local descending chain condition.
(3) $T(A_P^s)$ is the order topology (Lawson topology, interval topology) iff P is locally finite iff $T(A_P^s)$ is compact.

1. Motivation and preliminaries. Continuity spaces are among the many generalizations of metric spaces found in the literature. In [18] it is shown that all topologies arise in a natural way from continuity spaces. More to the point for us, it was shown in [12] that the hull-kernel topology long studied on spaces of prime ideals (see, e.g., [10], [11], [13], [16], [25]) arises from a continuity space in which the distance between two ideals I, J is their set-theoretic difference $d(I, J) = J \setminus I$. The "converse" continuity space, in which the distance $I \setminus J$ is used in place of $J \setminus I$, gives rise to the Scott topology on the power set of the underlying ring. Further, its "symmetrization", in which $J \setminus I$ is replaced by $(J \setminus I) \cup (I \setminus J)$, gives rise to the patch topology, which here agrees with the Lawson topology (see [9], [13]).

The above construction can be generalized from power sets to arbitrary dual Heyting algebras, alias (dual) Brouwerian lattices (cf. [1], [22], [24]). We shall carry out this construction below and describe just when it actually does give rise to Scott, dual Scott and Lawson topologies, respectively. The "structure spaces" of [12] are special cases of the "lattice continuity spaces" studied in the sequel.

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