There are uncountably many homeomorphism types of orbits in flows

by

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Abstract. We show the existence of uncountably many non-locally compact orbits of different homeomorphism type in dynamical systems with transformation group $\mathbb{R}$. This answers a question of J. M. Aarts. Until now only four different orbits were known, see [AF]. Using the characterization of non-locally compact orbits by J. M. Aarts and Z. Frolik we show that the $p$-frequency is an invariant which can be used to distinguish a continuum of orbits of distinct topological type.

All topological spaces under consideration are separable metric.

Introduction. Consider a flow $(X, \pi)$, i.e. $\pi: \mathbb{R} \to X$ is a continuous action of $\mathbb{R}$ on $X$.

There still is no satisfying classification of the homeomorphism types of the orbits $\Gamma(x) = \{\pi(x, t) | t \in \mathbb{R}\}$. If $\Gamma(x)$ is locally compact it is well known that $\Gamma(x)$ is homeomorphic to either $\mathbb{R}$, $S^1$ or a singleton, see [1]. However, if $\Gamma(x)$ is non-locally compact not much is known. An important difference between the two types of orbits is that non-locally compact orbits are recurrent but not periodic. Recall that $\Gamma(x)$ is recurrent if for every neighbourhood $V$ of $x$, the set $\{t | \pi(x, t) \in V\}$ is unbounded. $\Gamma(x)$ is positively recurrent if $\{t > 0 | \pi(x, t) \in V\}$ is unbounded.

In [AF] four different orbits in flows were distinguished. It is the purpose of this paper to show that the set of homeomorphism types has the power of the continuum.

Non-locally compact orbits were characterized by J. M. Aarts as suspensions $\Sigma(Q, h)$ of universally transitive homeomorphisms, see [A]. A homeomorphism $h: Q \to Q$ is called universally transitive if $Q = \{h^n(0) | n \in \mathbb{Z}\}$. One can think of a universally transitive homeomorphism as a cascade with only one orbit. Instead of universally transitive homeomorphisms we are going to use orbits in the two-sided shift $\sigma$ on the Cantor set $[0, 1]^\mathbb{Z}$. The homeomorphism $\sigma: [0, 1]^\mathbb{Z} \to [0, 1]^\mathbb{Z}$ is defined by $\sigma([x_n]) = ([x_n+1])$.
If $x \in \{0, 1\}^*$ is non-periodic and recurrent, then the orbit $O(x) = \{\sigma^n(x) \mid n \in \mathbb{Z}\}$ is homeomorphic to $Q$. In that case $\sigma$ is a universally transitive homeomorphism on $O(x)$.

J. M. Aarts proved that two orbits $\Sigma(Q, b)$ and $\Sigma(Q, k)$ are homeomorphic if there are non-empty clopen sets $A$ and $B$ of $Q$ such that the first-return maps $h_A$ and $h_B$ are conjugate. In this case $h$ and $k$ are called first-return equivalent. The first-return map $h_A : A \to A$ is defined by $h_A(x) = h^{\sigma^n(x)}(x)$ where $n(x) = \min\{n \in \mathbb{N} : \sigma^n(x) \in A\}$. Note that $h_A$ is a well-defined homeomorphism if $h$ is both positively and negatively recurrent. It is also to be observed that $A$ and $B$ are homeomorphic to $Q$. J. M. Aarts and Z. Frolik used this result to show the existence of four distinct orbits in flows. They showed that an almost periodic cascade and a "wild" cascade yield non-equivalent universally transitive homeomorphisms. Consider the two-sided shift $\sigma : \{0, 1\}^\mathbb{Z} \to \{0, 1\}^\mathbb{Z}$.

Let $x$ be an element of $\{0, 1\}^\mathbb{Z}$ such that $O(x)$ is dense in $\{0, 1\}^\mathbb{Z}$. In this paper $(O(x), \sigma)$ is called a wild cascade and $x$ is called a wild element of $\{0, 1\}^\mathbb{Z}$. Now J. M. Aarts and Z. Frolik took for both types of cascades a non-negatively recurrent and a both positively and negatively recurrent version to obtain four different orbits.

In this paper we concentrate on the second case and show that it is possible to distinguish a continuum of topologically distinct orbits in flows (suspensions over a wild orbit in $\{0, 1\}^\mathbb{Z}$). Hence in fact the existence of a continuum plus two topologically distinct orbits is shown. How can we show that there are in fact uncountably many different orbits? Translated into terms of recurrent elements in $\{0, 1\}^\mathbb{Z}$, first-return equivalence is an equivalence relation on $\{0, 1\}^\mathbb{Z}$. A tempting way to try and prove that there are uncountably many different orbits is to show that the equivalence classes are of the first category. This is difficult if not impossible. The difficulty in handling first-return equivalence is that there are continuum many clopen subsets of $Q$. We will take a different approach and construct elements of different types. The term type is made precise later. In general elements of different type need not be non-equivalent. However, it is possible to distinguish uncountably many non-equivalent types.

Before we start the construction we settle the notation.

**Definition 1.** A word $w$ is an element of $(0, 1)^{n+1}$ for some $n \in \mathbb{N}$. The integer $n + 1$ is called the length of $w$ and is denoted by $\lambda(w)$. The dictionary $d$ is the set of all words:

$$d = \bigcup \{w \mid \exists n \in \mathbb{N} \ w \in \{0, 1\}^{n+1}\}.$$  

In general an element $x \in \{0, 1\}^\mathbb{Z}$ is denoted sloppily by $x = \cdots w_{-2}w_{-1}w_0w_1\cdots \ w_i \in d$ for all $i \in \mathbb{Z}$.

Of course it has to be clear from the context what coordinates correspond to the words $w_i$. When this is not the case we use the following notation: $(x_0) = \cdots x_{-2}x_{-1}x_0x_1\cdots$ to mark where the negative coordinates end. We say that $x$ contains $w$ if $x = \cdots w_{-1}w_0w_1\cdots$. For $w = \xi_{n-1}w_n \in d$ the inverse word $\bar{w} = \xi_{n-1}\cdots \xi_{n-1}$ with $\xi_n = \xi_{n-1}$ for all $n \in \mathbb{N}$.

Note that an element $x$ of $(0, 1)^\mathbb{Z}$ has a dense orbit under $\sigma$ if $x$ contains all words. First we distinguish only three non-equivalent elements of $(0, 1)^\mathbb{Z}$ with the help of the frequency (of return). Intuitively, this is the clearest of the invariants we are going to use.

**Definition 2.** Let $x$ be an element of the Cantor set and let $V, W$ be clopen subsets of $O(x)$ such that $V \subset W$. For $n > 0$ the n-frequency of $V$ with respect to $W$ is defined by

$$f_n(V, \sigma^n, x) = \inf\{|[m \in N] \mid m \in N, \sigma^m(x) \in V|/|N| \}$$

The frequency of return of $V$ with respect to $W$ is defined as

$$f(V, \sigma^n, x) = \lim_{n \to \infty} f_n(V, \sigma^n, x).$$

Hence $f(V, \sigma^n, x)$ is the limit inferior of the sequence

$$|[m \in N] \mid m \in N, \sigma^m(x) \in V|/|N|.$$  

Similarly we can define $f_n(V, \sigma^{-n}, x)$ and $f(V, \sigma^{-n}, x)$.

This definition is inspired by the behaviour of almost periodic cascades. Recall that a point $x \in (0, 1)^\mathbb{Z}$ is almost periodic if for every open subset $V \subset O(x)$ the set $\{m \mid \sigma^m(x) \in V\}$ has bounded gaps. Hence for every clopen $V \subset W$ the frequency $f(V, \sigma^n, x)$ is positive. This is in general not true for wild elements.

Let $x$ be an element of $(0, 1)^\mathbb{Z}$ and let $\{V_n \mid n \in \mathbb{N}\}$ be a neighbourhood basis at $x$ of clopen sets in $O(x)$ such that for every $n \in \mathbb{N}$, $V_{n+1} \subset V_n$. The return map to $V_n$ is denoted by $\sigma_n$ to save indices. We distinguish the following three types of orbits:

**Type 1.** For every neighbourhood $V$ of $x$ both $f(V, \sigma^{-1}, x)$ and $f(V, \sigma, x)$ are positive.

**Type 2.** For every neighbourhood $V$ of $x$, $f(V, \sigma^{-1}, x)$ is positive and $f(V_{n+1}, \sigma_n, x) = 0$ for all $n \in \mathbb{N}$.

**Type 3.** For every $n \in \mathbb{N}$, $f(V_{n+1}, \sigma_n^{-1}, x) = f(V_{n+1}, \sigma_n, x) = 0$.

It is not hard to show that elements of different type are non-equivalent. Let $A \subset B \subset C \subset D$ be a chain of clopen subsets and let $x$ be an element of $A$. From the definition of frequency it is clear that

$$f(A, \sigma^n, x) \leq f(B, \sigma^n, x) \leq f(A, \sigma^n, x).$$

The following lemma is an easy exercise in these inequalities.

**Lemma 1.** Elements of different type are not first-return equivalent.

**Proof.** Suppose $x$ is of type 1 and $y$ of type 2 for some $x, y \in (0, 1)^\mathbb{Z}$. Let $\{V_n \mid n \in \mathbb{N}\}$ be a neighbourhood basis of clopen sets at $y$ such that $f(V_{n+1}, \sigma_n, x) = 0$ for every $n \in \mathbb{N}$. If $O(x, \sigma)$ and $O(y, \sigma)$ are first-return equivalent there are clopen sets $A \subset O(x)$ and $B \subset O(y)$ such that $\sigma_A$ is conjugate to $\sigma_B$. Without loss of generality we may assume
that \( x \in A, y \in B \subset V_0 \) and that the conjugating homeomorphism \( \gamma: A \to B \) maps \( x \) onto \( y \). There is an \( n \in \mathbb{N} \) such that \( V_0 \subset B \). Since \( y \) is of type 2 the frequency \( f(V_0, \sigma, y) = 0 \). On the other hand,

\[
f(V_{n+1}, \sigma, y) = f(\gamma^{-1}(V_{n+1}), \gamma^{-1} \circ \sigma \circ \gamma, x) = f(\gamma^{-1}(V_{n+1}), \sigma, x) > 0
\]

since \( x \) is of type 1.

The proof in the other cases is similar.

Using the fact that \( [0, 1]^2 \) is a topological group we define, for a clopen neighbourhood \( V \) of \( 0, n > 0 \) and \( \epsilon > 0 \), the set

\[
F_n(V, \sigma, \epsilon) = \{ x \mid f_n(x + V, \sigma, x) \geq \epsilon \}.
\]

This is a closed set with empty interior. Let \( \{ V_n \mid n \in \mathbb{N} \} \) be a neighbourhood basis of clopen sets at \( 0 \).

The set of elements of type 1 equals

\[
\bigcap_{n=1}^{\infty} \biggl( \bigcap_{m=n}^{\infty} F_n \left( V_n, \sigma, \frac{1}{m} \right) \biggr)^c \biggl( \bigcap_{n=1}^{\infty} \bigcap_{m=n}^{\infty} F_n \left( V_n, \sigma^{-1}, \frac{1}{m} \right) \biggr).
\]

This is a set of first category. It is our task to show that it has a non-empty intersection with the \( G \) of wild elements:

\[
\bigcap_{n=1}^{\infty} \{ x \in [0, 1]^2 \mid x \text{ contains } w \}.
\]

Observe that the set of elements of type 1 is not empty since it contains almost all periodic elements.

**Construction 1.** We want to construct a wild element \( x \in (0, 1]^2 \) of type 1. Hence \( x \) has to satisfy the following conditions:

(a) \( x \) contains all words.

(b) There is a neighbourhood basis \( \{ V_n \mid n \in \mathbb{N} \} \) of clopen sets at \( x \) in \( (0, 1]^2 \) such that \( f(V_n, \sigma, x) \) and \( f(V_n, \sigma^{-1}, x) \) are positive for all \( n \in \mathbb{N} \).

Note that condition (b) implies that \( x \) is both positively and negatively recurrent. In fact the constructed \( x \) will be symmetric, so we only have to worry about \( f(V_n, \sigma, x) \).

Let \( \pi: N \to A \) be an enumeration of the dictionary such that \( \pi(0) = 0 \). We will approximate \( x \) with periodic points: a sequence \((x_n)\) is constructed with \( \lim_{n \to \infty} x_n = x \) and every \( x_n \) is periodic. At step \( n \) we make sure that \( \pi(n) \) is inserted in \( x_n \).

Simultaneously we construct a descending sequence \( \{ V_n \mid n \in \mathbb{N} \} \) of clopen neighbourhoods of \( x \) such that \( V_{n+1} \subset V_n \) for all \( n > m \) and \( x_m \in V_n \). We have to make sure that \( f(V_n, \sigma, x) > 0 \). It is certainly true that \( f(V_n, \sigma, x) > 0 \) since \( x_n \) is periodic. This property must not be lost in the limit, so we choose integers \( i_n \) and positive real \( e_n \) such that \( f(V_n, \sigma, x) \geq e_n \) for every \( m \geq n \).

Let us sketch the first steps: choose \( x_0 = 0 = \ldots 0000 \ldots \). Instead of zeros we use a's since there are more than two letters in the alphabet; \( x_0 = \ldots aaaaaaaaaa \ldots \) and \( V_0 = \{ x_0 \mid x_0 = 0 \} \). Observe that \( x_0 \) contains \( \pi(0) = 0 \). The frequency of the zeros in \( x_0 \) is equal to 1. During the inductive process it must not drop below \( e_0 = \frac{1}{2} \). Now we have to insert \( i(1) \) in \( x_0 \). Let \( b \) be the word \( b = \pi(1) aaaa \ldots aaaa \pi(1) \). The periodic element \( x_1 = \ldots bbbbb \ldots \) contains \( \pi(1) \) and \( \pi(0) \) and the b's are placed in such a way that \( x_1 \) is symmetric and the zero coordinate is 0. The neighbourhood \( V_1 = \{ x \mid x = \ldots b \ldots \} \). The frequency of zeros in \( x_1 \) is larger than \( \frac{1}{2} \) for a suitable choice of \( b \) and the frequency of \( b \)'s is larger than some \( e_1 \). At the following step the word \( \pi(2) \) has to be inserted in \( x_1 \) in such a way that the frequency of the zeros does not drop below \( \frac{1}{4} \) and the frequency of \( b \)'s does not drop below \( e_1 \). To this end take \( c = \pi(2) bbb \ldots bbb \pi(2) \) and define \( x_2 = \ldotscccc \ldots \).

In this way we proceed inductively to define \( x_n, V_n, \sigma_n \) and \( i_n \), satisfying the following conditions:

(i) \( x_n \) contains \( \pi(0), \pi(1), \ldots, \pi(n) \).

(ii) \( x_n \in V_m \) for all \( m \geq n \).

(iii) \( x_n \) is symmetric and periodic.

(iv) \( V_n \subset C_n \) for all \( n \geq 0 \).

(v) \( f_n(x_n, \sigma_n) = f_n(V_n, \sigma^{-1}, x_n) \geq e_n \) for all \( n \geq 0 \).

Suppose we have defined \( x_n, V_n, \sigma_n \) and \( i_n \); now we have to construct \( x_{n+1}, V_{n+1} \), \( \sigma_{n+1} \), and \( i_{n+1} \). The non-precise expression “very long” is used to indicate that we want \( x_{n+1} \) to satisfy condition (v).

Let \( x_n \) be the element \( \ldots v \ldots \) and let \( V_n \) be the neighbourhood \( \{ x \mid x = \ldots v \ldots \} \) (it is obvious where \( v \) has to be placed: symmetric with respect to 0). Now \( w = \pi(n+1) \ldots \pi(n+1) \ldots \) is a very long sequence of \( v \)'s. The periodic point \( x_{n+1} = \ldots v\ldots \ldots \) is an element of \( V_n \) such that \( f_n(V_n, \sigma, x_{n+1}) \geq e_n \) for all \( n \geq 0 \). We choose \( V_{n+1} = \{ x \mid x = \ldots v \ldots \} \) such that \( x_{n+1} \in V_{n+1} \) and \( x_{n+1} \in V_n \). Obviously it is possible to choose \( \sigma_{n+1} \) and \( i_{n+1} \) such that \( f_{n+1}(V_{n+1}, \sigma_{n+1}, x_{n+1}) \geq e_{n+1} \).

In this way we construct a sequence \( (x_n) \) such that \( \lim_{n \to \infty} x_n \) is a wild element of type 1.

**Remark.** We could have dropped the condition that \( x \) contains all words. Then it would suffice to construct an almost periodic element of \( (0, 1]^2 \). A well known method, reminiscent of Construction 1, is to construct such an element by substitution, see [MH].

The proof that elements of type 1 exist is satisfactory in the sense that we have actually constructed such an element. The proof that elements of type 2 and 3 exist can be given constructively as we will see below. Here is a non-constructive proof that elements of type 3 exist:

Let \( \{ V_n \mid n \in \mathbb{N} \} \) be a neighbourhood basis of clopen sets at \( 0 \). Consider \( F_m(V_{n+1}, \sigma, x) = \{ x \in (0, 1]^2 \mid f_m(x + V_{n+1}, \sigma, x) \geq e \} \). This is a closed set with
empty interior in \((0, 1)^2\). The complement of elements of type 3 is equal to:
\[
\bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} F_3\left(V_{n+1}, \sigma_{n+1}, \frac{1}{m}\right) \cup \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} F_3\left(V_{n+1}, \sigma_n, \frac{1}{m}\right).
\]
This is a set of first category. Hence it is possible to distinguish three non-equivalent elements of \([0, 1]^2\).

Is it possible to refine the notation of frequency in order to obtain more orbits? The first thing that comes to mind is to measure the number of returns against a function depending on the time. For instance, we may consider the limit superior of the sequence
\[
\left\{ \left( m \in \mathbb{N} \mid \sigma(x) \in V \right) \right\}/\log N.
\]
It is necessary to impose a few restrictions on the function of the time. As usual the integer part of \(e^R\) is denoted by \([e]\).

**Definition 3.**  \(S\) is a subset of the set of all functions from \(\mathbb{N} \setminus \{0\}\) to \(\mathbb{N} \setminus \{0\}\). \(\varphi \in S\) if
(a) \(\varphi\) is increasing and surjective;
(b) For every \(e \in R_\varphi\) there is a \(C \in R_\varphi\) such that \(\varphi([cN]) \leq C\varphi(n)\) for all \(n \in \mathbb{N} \setminus \{0\}\).

Note that from these conditions a third can be derived:
(c) For every \(e \in R_\varphi\) there is a \(D \in R_\varphi\) such that \(\varphi([dN]) \geq D\varphi(n)\) for all \(n \in \mathbb{N} \setminus \{0\}\).

According to condition (b) there is a constant \(C\) such that \(\varphi([d^{-1}N]) \leq C\varphi(n)\). Hence \(\varphi(n) \leq \varphi([d^{-1}N]) + d^{-1} \leq C\varphi([dN]) + d^{-1}\). From this inequality it is possible to derive condition (c). We now are able to generalize the definition of frequency.

**Definition 4.** Let \(x\) be an element of the Cantor set and let \(V, W\) be clopen subsets of \(O(x)\) such that \(V \subset W\). For \(n > 0\) the \(n\)-\(\varphi\)-frequency of \(V\) with respect to \(W\) is defined by
\[
f_\varphi(V, \sigma_W, x) = \inf \{ [m \in \mathbb{N} \mid m \leq N, (\sigma_W)^n(x) \in V] \mid \varphi(N) \geq n \}.
\]

The \(\varphi\)-frequency of \(V\) is equal to \(\lim_{n \to \infty} f_\varphi(V, \sigma_W, x)\).

In general the \(\varphi\)-frequency is harder to handle than the frequency. The main difference is that the \(\varphi\)-frequency can be infinite. We say that the \(\varphi\)-frequency is positive if it is neither 0 nor \(\infty\). Before we are going to imitate Construction 1 we must decide what type of element has to be constructed.

**Definition 5.** An element \(x\) of \([0, 1]^2\) is of type \(\varphi\) if there exists a neighbourhood basis of clopen sets \(\{V_n \mid n \in \mathbb{N}\}\) at \(x\) such that both \(f_\varphi(V_{n+1}, \sigma_n, x)\) and \(f_\varphi(V_{n+1}, (\sigma^{-1})_n, x)\) are positive for every \(n \in \mathbb{N}\).

Is it worthwhile to construct elements of different type? For example: is the \(\varphi\)-frequency \(f_\varphi(V_{n+1}, \sigma_n, x)\) of an element of type \([\log]\) positive? It is our first task to prove that the answer to both questions is yes.

The \(k\)-fold iteration \(\varphi \circ \varphi \circ \ldots \circ \varphi\) of \(\varphi\) is denoted by \(\varphi^k\). Note that if \(\varphi \in S\), then \(\varphi^k \in S\).

**Lemma 2.** Suppose \(x\) is an element of type \(\varphi\) with respect to the neighbourhood basis \(\{V_n \mid n \in \mathbb{N}\}\). For all \(n, k \in \mathbb{N}\) the \(\varphi^k\)-frequency \(f_\varphi(V_{n+1}, \sigma_n, x)\) is positive.
We have convinced ourselves that it is worthwhile to construct elements of type \( \phi \).
So all that remains is an application of Construction 1. This is straightforward except for one technical detail. As pointed out before we must be extra careful since the \( \phi \)-frequency may be infinite. Whereas we had to make sure in Construction 1 that the frequency of return of \( x_{n-1} \) to \( x_n \) was not zero, now we have to make sure as well that it is not infinite. The element \( x_{n-1} \) is obtained from \( x_n \) by inserting words. In order to deal with the problem of inserting too many words we label them. This is done by using the maximal number of consecutive zeros. Let \( x = \lim_{n \to \infty} x_n \) be a wild element of type \( \phi \). At first sight the construction may seem different from Construction 1, but in fact it is a straightforward imitation.

Since \( x(0) = 0 \) we want \( x_0 \) to be a basis that the \( \phi \)-frequency of the zeros in \( x_0 \) is positive. Define \( x_0 = (\xi_0) \) with \( \xi_0 = 0 \) if \( \phi(0) \), \( \phi(0) = 0 \) for \( n > 0 \), \( \xi_0 = 0 \) and \( \xi_0 = 1 \). Observe that in the case that \( \phi = \text{id} \) is allowed we get almost the same \( x_0 \) as in Construction 1. As in Construction 1 we denote the zeros by \( a \). Now we have to insert \( \pi(1) \) in \( x_0 \). Take a very long block \( (\xi_0, \xi_1, \ldots, \xi_{n-1}, \xi_n) \) and insert \( \pi(1) \) between \( \xi_{n-1} \) and \( \xi_n \), insert \( \pi(1) \) between \( \xi_{n-1} \) and \( \xi_n \). Recall that \( \pi \) keeps the zero coordinate. The obtained word is called \( b \). Important words in \( x_0 \) are the \( a \)'s, the \( 1 \)'s are there to get the right frequency. In Construction 1 we replaced every \( a \) by \( b \). In this case that would be too much, just like \( \ldots, 0 \) contains many zeros to be the right \( x_0 \). That is why we replace the \( n \)th \( a \) by \( b \) if \( \phi(a) = \phi(a) = 1 \).

We must convince ourselves that the \( \phi \)-frequency of \( a \)'s in \( x_1 \) is at least \( 1 \) as in Construction 1. It comes as no surprise that this holds true; because \( \phi = \text{id} \), the fraction of altered \( a \)'s is negligible. We choose

\[
V_1 = \{ x \mid x = \ldots | 1 \mid 1 \ldots | 1 \ldots | 1 \ldots \},
\]

the \( \phi \)-frequency of the \( b \)'s with respect to the \( a \)'s is obviously greater than 0.

We construct \( x_n, V_n, \xi_n, M_n \) satisfying the following five conditions:

(i) \( x_n \) contains \( \pi(0), \pi(1), \ldots, \pi(n) \).

(ii) \( x_n \subseteq V_n \) for all \( m \leq n \).

(iii) \( x_n \) is symmetric and does not contain \( n+2 \) consecutive zeros.

(iv) \( V_n \subseteq V_n \) for all \( m \leq n \).

(v) \( \xi_n = f_n(x_{n+1}, \sigma_n, x_n) = f_n(x_{n+1}, \sigma_n, x_n) \leq M_n \) for all \( m \leq n \).

Suppose we have defined \( x_n, V_n, \xi_n, M_n \) now we have to construct \( x_{n+1}, \sigma_n, x_n, \xi_n, M_{n+1} \). It follows from the inductive procedure that \( V_n = \{ x \mid x = \ldots \} \) for some \( n \), just as in Construction 1 \( \phi \) is symmetric and contains the 0th coordinate. Unlike the \( x_n \) from Construction 1 the \( x_n \) in Construction 2 is not periodic. Between two subsequent \( a \)'s there are gaps in order to obtain the right frequency. Let \( (\xi_n^m) \) be a very long word in \( x_n \) such that \( (\xi_n^m) = 1 \ldots 1 \) (\( \xi \) contains the 0th coordinate). We insert \( n+m \) behind \( x_n \) and in \( n+m+1 \) front of \( x_n \).

In this way we obtain a word \( w \). Observe that the word \( w \) does not occur in \( x_n \) since \( w \) contains \( n+2 \) consecutive zeros and \( x_n \) does not. Therefore if we let every \( w \) into a \( w \) iff \( \phi(0) = \phi(0) = 1 \), then the \( \phi \)-frequency of \( w \) with respect to \( v \) becomes positive. We must take care that we do not spoil the \( \phi \)-frequencies of the previous words. Let \( \alpha \) be an integer, the fraction \( (\phi(a) = \phi(a)) \) can be made arbitrarily small (for all \( n \in \mathbb{N} \)) since \( \phi = \text{id} \). So if we take \( (\xi_n^m) \) sufficiently long and \( K \) sufficiently large, then we can insert \( n+m \) behind \( x_n \) and \( x_n \) in front of \( x_n \) and we can alter every nth word \( v \) into \( w \) iff \( |m| > K \) and \( \phi(0) = \phi(0) = 1 \) all without violating the conditions (i) through (v). If \( x_{n+1} = \{ x \mid x = \ldots \} \) \( (w \) contains 0th coordinate), \( \xi_n+1 \) is chosen smaller than the \( \phi \)-frequency of \( w \) with respect to \( v \), then we can find a suitable \( \xi_n+1 \).

The reader might argue that two different \( v \)'s can overlap. This is no serious obstruction to Construction 2, but we silently assumed that it was not possible. To repair the argument we may use a more sophisticated label; there is only one block row of \( n+1 \) consecutive zeros in \( x_n \). Since we have chosen \( n \) such that \( (\pi(0) \land n \in \mathbb{N}) \) contains all words \( \lim_{n \to \infty} x_n = x \) is a wild element. Also \( x_n \subseteq \bigcup_{n=1}^{n} (V_n, \sigma_n, x_n) \) \( \subseteq M_n \) since this is true for every \( x_n \) with \( n \geq m+1 \). We conclude that \( x \) is a wild element of type \( \phi \).

The last step we have to take is to show that there are continuum many non equivalent elements of \( \{0, 1\}^\infty \). This is an easy exercise in set theory. We first prove that \( (\mathcal{F}, \ll) \) contains a set of order type \( \mathcal{Q} \). It suffices to prove the following lemma.

**Lemma 4.** Let \( \phi, \psi \) be elements of \( \mathcal{F} \). If \( \phi < \psi \), then there is a \( \gamma \in \mathcal{F} \) such that \( \phi < \gamma < \psi \).

**Proof.** This depends upon a fairly standard diagonal argument. For technical reasons we want \( \psi \) to satisfy the following property.

(*) If \( \gamma \in \mathcal{F} \) and for every \( k \in \mathbb{N}, \gamma(n) < \psi(n) \) for all sufficiently large \( n \), then \( \gamma \ll \psi \).

The identity does not satisfy (a), but the equivalent \( \psi(n) = |n| \) does!

Therefore, if \( \psi \) does not satisfy (a) then we replace it by \( \psi(n) = |n| \). Since \( \phi < \psi \) we can choose a sequence of integers \( N_1 < N_2 < \ldots \) such that if \( n > N_k \), then \( \psi(n) > \frac{2^k}{k} \). We define \( \gamma(n) = \max (\gamma(n-1), \psi(n)) \) if \( N_{k+1} < n < N_{k+1} \), obviously \( \gamma(n) < \psi(n) \) for sufficiently large \( n \). By property (a), \( \gamma \ll \psi \).

Also for \( n > N_1 \) the following inequalities hold:

\[
\psi(n) > \frac{2^n}{n} \quad \text{if} \quad N_{m+1} < n < N_{m+2},
\]

and \( (\psi(n)) = \psi(n) > 2^n \psi(n) \).

This implies that \( \phi < \gamma \). It is clear from the definition of \( \gamma \) that this function satisfies condition (a) and (b) on elements of \( \mathcal{F} \).

Next we show that it is in fact possible to embed a set of order type \( R \) in \( (\mathcal{F}, \ll) \). Lemma 5 implies that there exists a continuously ordered subset of \( (\mathcal{F}, \ll) \) which contains the rationals. Therefore \( (\mathcal{F}, \ll) \) contains the ordered set \( R \), see [K], p. 84.

**Lemma 5.** Let \( \phi_1 < \phi_2 < \phi_3 < \ldots \) and \( \psi_1 < \psi_2 < \psi_3 < \ldots \) be a pair of sequences in \( \mathcal{F} \) such that \( \phi_n < \psi_n \) for every \( n, m \in \mathbb{N} \). There exists a \( \gamma \in \mathcal{F} \) such that \( \phi_n < \gamma < \psi_n \) for every \( n, m \in \mathbb{N} \).
Natural continuity space structures on dual Heyting algebras

by

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Abstract. Every dual Heyting algebra carries three intrinsic “generalized quasi-metrics”: $d(x,y) = x \rightarrow y$, $d'(x,y) = y \rightarrow x$, and $d''(x,y) = (x \rightarrow y) + (y \rightarrow x)$, where $\rightarrow$ denotes the relative dual pseudocomplement. Formally, these are “continuity functions” satisfying the triangle inequality. In a dual Heyting algebra, a set of positives is a dual ideal $P$ whose meet is 0. We investigate properties of the topologies, $T(A_0)$, $T(A_4)$, $T(A_4)$, which arise from the continuity spaces so defined. For example, $T(A_0)$ and $T(A_4)$ are completely distributive, and $T(A_4)$ is a zero-dimensional Hausdorff topology. Furthermore, we show that for any coframe, that is, for any complete dual Heyting algebra $A$:

1. $T(A_0)$ is the Scott topology iff $P$ satisfies the ascending chain condition.
2. $T(A_4)$ is the dual Scott topology iff $P$ satisfies the local descending chain condition.
3. $T(A_4)$ is the order topology (Lawson topology, interval topology) iff $P$ is locally finite iff $T(A_4)$ is compact.

1. Motivation and preliminaries. Continuity spaces are among the many generalizations of metric spaces found in the literature. In [18] it is shown that all topologies arise in a natural way from continuity spaces. More to the point for us, it was shown in [12] that the hull-kernel topology long studied on spaces of prime ideals (see, e.g., [10], [11], [13], [16], [25]) arises from a continuity space in which the distance between two ideals $I, J$ is their set-theoretic difference $d(I,J) = I \cap J$. The “reverse” continuity space, in which the distance $I \cap J$ is used in place of $J \cap I$, gives rise to the Scott topology on the power set of the underlying ring. Further, its “symmetrization”, in which $I \cap J$ is replaced by $(I \cap J) \cup (J \cap I)$, gives rise to the patch topology, which here agrees with the Lawson topology (see [9], [13]).

The above construction can be generalized from power sets to arbitrary dual Heyting algebras, alias (dual) Brouwerian lattices (cf. [1], [22], [24]). We shall carry out this construction below and describe just when it actually does give rise to Scott, dual Scott and Lawson topologies, respectively. The “structure spaces” of [12] are special cases of the “lattice continuity spaces” studied in the sequel.

References


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